Combinatorial Sums

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1 Common Techniques

- **Change order of summation:** If the summation is a double summation, you should almost always try switching the order of summation. If a summation $S_n = \sum_{k=0}^n a_{n,k}$ only has one sum, you can try transforming it to a double sum by considering a generating function

$$f(z) = \sum_{n=0}^{\infty} S_n z^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_{n,k} z^n = \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} a_{n,k} z^k.$$

  If you can find a nice form for $f(z)$, then the coefficient of z^n will be the desired S_n.

- **Interpret combinatorially:** Sometimes it will be possible to interpret an expression as counting some quantity or computing the probability of some event. In this case, it is often useful to replace the expression by a simpler way of doing the same computation.

- **Induction:** If the series S_n depends on some parameter n, you might be able to write a recurrence for S_n, guess a general form, and then prove it using the recurrence. In particular, the difference $S_{n+1} - S_n$ might itself be a simpler summation.

- **Manipulation:** Techniques for other sums also apply here. You can try to force telescoping or to factor the expression, keeping in mind the binomial expansion.

- **Take derivatives or integrals:** If you have a expression $f(z) = \sum_{n=0}^{\infty} a_n z^n$ for the series $\sum_{n=0}^{\infty} a_n z^n$ with $f(z)$ an explicit function, it yields related sums

$$\int_0^z f(w)dw = \sum_{n=0}^{\infty} \frac{a_n}{n+1} z^{n+1} \qquad \text{and} \qquad f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}.$$

2 Useful Facts

Fact 1 (Multinomial Theorem)**.** For integer $n > 0$, we have

$$(x_1 + \cdots + x_k)^n = \sum_{a_1 + \cdots + a_k = n} \binom{n}{a_1, \ldots, a_k} x_1^{a_1} \cdots x_k^{a_k}.$$

Fact 2 (Hockeystick Identity)**.** For integer $k > 0$, we have that

$$\sum_{n=k}^{m} \binom{n}{k} = \binom{m+1}{k+1}.$$

Fact 3 (Negative binomial expansion)**.** For integer $k > 0$ and $|z| < 1$, we have

$$\sum_{n=k}^{\infty} \binom{n}{k} z^{n-k} = (1-z)^{-k-1}.$$

Fact 4 (Vandermonde convolution)**.** For integers $r, m, n > 0$, we have

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{m+n}{r}.$$

Fact 5. For integers $n, m > 0$, we have

$$\int_0^1 x^n(1-x)^m dx = \frac{1}{(n+m+1)\binom{n+m}{n}}.$$

3 Problems

Problem 1. Take r such that $1 \le r \le n$, and consider all subsets of r elements of the set $\{1, 2, \ldots, n\}$. Each subset has a smallest element. Let $F(n, r)$ be the arithmetic mean of these smallest elements. Prove that: $F(n, r) = \frac{n+1}{r+1}$.

Problem 2. Amy starts with a piece of paper with the number 0 on it. On each of $\binom{n+m}{m}$ days, she starts at the lower left corner of a neighborhood that consists of a $n \times m$ grid of blocks separated by roads, and travels along the roads to the upper right corner, only going right or up at each intersection. When she gets there, she counts the number of blocks, b, that lie above here path, and adds the monomial q^b to the value on her paper. If Amy never took the same path more than once, show that, by the end of the $\binom{n+m}{m}$ days, the sum on her paper is equal to the q-binomial coefficient

$$\binom{\mathbf{n+m}}{\mathbf{m}}_q = \frac{(\mathbf{n+m})_q!}{(\mathbf{n})_q!(\mathbf{m})_q!}.$$

(Here $(\mathbf{a})_q!$ denotes the q-factorial $(1+q)(1+q+q^2)\cdot \cdots \cdot (1+q+q^2+\cdots+q^{a-1})$.)

Problem 3. Let $S = \{1, 2, \ldots, n\}$ for some integer $n > 1$. Say a permutation π of S has a local maximum at $k \in S$ if

1. $\pi(k) > \pi(k+1)$ for $k = 1$;
2. $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ for $1 < k < n$;
3. $\pi(k-1) < \pi(k)$ for $k = n$

What is the average number of local maxima of a permutation of S, averaging over all permutations of S?

Problem 4. Define the Fibonacci numbers by $F_0 = 1$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$. Find

$$\sum_{n=0}^{\infty} F_n z^n.$$

Problem 5. Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ be the n^{th} Catalan number. Find

$$\sum_{n=0}^{\infty} C_n z^n.$$

Problem 6. Show that

$$\sum_{k=0}^{n} (-1)^k \frac{1}{k+1}\binom{n+k}{2k}\binom{2k}{k} = 0.$$

Problem 7. Show that

$$\sum_{t=0}^{n-k} \frac{(-1)^t}{t+k+1}\binom{n-k}{t} = \frac{k!(n-k)!}{(n+1)!}.$$

Problem 8. Let $p_n(k)$ be the number of permutations of the set $\{1,2,3,\ldots,n\}$ which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k p_n(k) = n!.$$

Problem 9 (Euler's pentagonal number theorem)**.** Prove the identity

$$\prod_{n\geq 1}(1-z^n) = \sum_{k\in\mathbb{Z}}(-1)^k x^{k(3k-1)/2} = 1+\sum_{k=1}^{\infty}(-1)^k\left(x^{k(3k+1)/2}+x^{k(3k-1)/2}\right).$$

Problem 10. Let $n\geq 3$ be a fixed integer. Each side and each diagonal of a regular n-gon is labeled with a number from the set $\{1,2,\ldots,r\}$ in a way such that the following two conditions are fulfilled:

1. Each number from the set $\{1,2,\ldots,r\}$ occurs at least once as a label.
2. In each triangle formed by three vertices of the n-gon, two of the sides are labeled with the same number, and this number is greater than the label of the third side.

(a) Find the maximal r for which such a labeling is possible.
(b) For this maximal value of r, how many such labellings are there?

Problem 11. Express

$$\sum_{k=0}^{n}(-1)^k(n-k)!(n+k)!$$

in closed form.

Problem 12. Let n be a positive integer. Prove that

$$\binom{n}{0}^{-1}+\binom{n}{1}^{-1}+\cdots+\binom{n}{n}^{-1}=\frac{n+1}{2^{n+1}}\left(\frac{2}{1}+\frac{2^2}{2}+\cdots+\frac{2^{n+1}}{n+1}\right).$$

Problem 13. Let $n>1$ be a given integer, and let $S=\{z_1,\ldots,z_n\}$ and $T=\{w_1,\ldots,w_n\}$ be two sets of complex numbers. Prove that

$$\sum_{k=1}^{n}\frac{\prod_{i=1}^{n}(z_k+w_i)}{\prod_{i\neq k}(z_k-z_i)}=\sum_{k=1}^{n}\frac{\prod_{i=1}^{n}(w_k+z_i)}{\prod_{i\neq k}(w_k-w_i)}.$$

Problem 14. Let n be a given integer with n greater than 7, and let $\mathcal{P}$ be a convex polygon with n sides. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a triangulation of $\mathcal{P}$ into $n-2$ triangles. A triangle in the triangulation of $\mathcal{P}$ is an interior triangle if all of its sides are diagonals of $\mathcal{P}$. Express, in terms of n, the number of triangulations of $\mathcal{P}$ with exactly two interior triangles, in closed form.

Problem 15. Let m,n be positive integers with $m\geq n$, and let S be the set of all ordered n-tuples $(a_1,a_2,\ldots,a_n)$ of positive integers such that $a_1+a_2+\cdots+a_n=m$. Show that

$$\sum_S 1^{a_1}2^{a_2}\cdot\cdots\cdot n^{a_n}=\binom{n}{n}n^m-\binom{n}{n-1}(n-1)^m+\cdots+(-1)^{n-2}\binom{n}{2}2^m+(-1)^{n-1}\binom{n}{1}.$$

Problem 16. For a triple (m,n,r) of integers with $0\leq r\leq n\leq m-2$, define

$$p(m,n,r)=\sum_{k=0}^{r}(-1)^k\binom{n+m-2(k+1)}{n}\binom{r}{k}.$$

Prove that

$$\sum_{r=0}^{n}p(m,n,r)=\binom{m+n}{n}.$$