$\mathcal{D}$-Modules and Representation Theory

Part III Essay Topic 96

$\mathcal{D}$-MODULES AND REPRESENTATION THEORY

Contents

1. Introduction 1
1.1. Motivation 1
1.2. Organization and references 2
1.3. Conventions and notation 3
1.4. Acknowledgments 3
2. $\mathcal{D}$-modules on smooth algebraic varieties 3
2.1. Definition and examples 3
2.2. Left and right $\mathcal{D}$-modules 6
2.3. Pullback and pushforward of $\mathcal{D}$-modules 8
2.4. The singular support 14
2.5. Holonomic $\mathcal{D}$-modules and duality 17
2.6. D-modules on $\mathbb{P}^1$ 21
3. The geometric setting 23
3.1. Preliminaries on semisimple algebraic groups 24
3.2. Flag and Schubert varieties 24
3.3. Equivariant vector bundles on the flag variety 24
3.4. Preliminaries on Lie algebras 26
3.5. Lie algebra actions on G-equivariant sheaves 26
3.6. Chevalley and Harish-Chandra isomorphisms 27
4. Beilinson-Bernstein localization 30
4.1. Twisted differential operators 30
4.2. Lie algebroids and enveloping algebras on the flag variety 31
4.3. A family of twisted differential operators 32
4.4. The localization functor and the localization theorem 33
4.5. The map $U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X^\lambda)$ 35
4.6. The geometry of the nilpotent cone and Kostant's Theorem 36
4.7. The global sections functor on the flag variety 41
References 44

1. Introduction

1.1. **Motivation.** Let X be a smooth algebraic variety over a field of characteristic 0. This essay will be primarily concerned with $\mathcal{D}$-modules, or modules over a sheaf $\mathcal{D}$ of differential operators defined on X. On general varieties, $\mathcal{D}$-modules behave in a much more structured manner than $\mathcal{O}$-modules, and this often allows them to carry more global information. For instance, as we shall point out, it is possible to recover the de Rham cohomology of a complex algebraic variety in terms of some $\mathcal{D}$-modules on it. Our primary focus in this essay, however, will be on the relationship between the theory of these differential operators and representation theory.

Let now G be a semisimple algebraic group over an algebraically closed field. Associated to any such G is a semisimple Lie algebra $\mathfrak{g}$ which we often think of as representing the infinitesimal behavior of G. Indeed, $\mathfrak{g}$ can be realized as the space of invariant vector fields on G, and we can attempt to study its representation theory from this perspective. Broadly speaking, however, this approach has two drawbacks. First, this characterization is too global, as the invariance condition does not allow us to understand different aspects of the representation theory of $\mathfrak{g}$ at once.[1] We might consider modifying this approach to use instead the *sheaf* of vector fields on G, discarding the invariance condition.

Date: May 6, 2011.

[1]Our precise meaning here is that it does not allow us to study all the Borel subalgebras of $\mathfrak{g}$ at once.

This is much closer to the correct notion. However, we must make one further modification; it turns out that we would like to consider only the behavior of the Borel subalgebras of $\mathfrak{g}$ simultaneously. Thus, we should consider only the space of vector fields invariant under the action of a Borel subgroup $B \subset G$. The resulting sheaf of rings generated by these vector fields is exactly the sheaf of differential operators $\mathcal{D}_X$ on the flag variety $X := G/B$ of G. The primary result presented in this essay, the Beilinson-Bernstein localization theorem, states "approximately" that we have the following correspondence

$$\boxed{\mathcal{D}_X\text{-modules}} \leftrightarrow \boxed{U(\mathfrak{g})\text{-representations}}$$

between $\mathcal{D}_X$-modules on the flag variety and representations of the corresponding Lie algebra. We say "approximately," however, because to make this correspondence rigorous (or true!) requires us to specify refinements on both sides.

These twists will result from the essentially different nature of the two objects. As a sheaf, a $\mathcal{D}_X$-module contains a significant amount of local information, which in this case will correspond to the action of the Borel subalgebras of our Lie algebra. On the other hand, $U(\mathfrak{g})$-representations seem are quite global objects, as the choice of such a representation often requires a global choice of Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. It is remarkable, therefore, that such a correspondence can exist. Indeed, the way that it avoids this seeming contradiction is by on the one hand studying non-trivial cases where the local behavior of all Borel subalgebras coincide (on the $\mathcal{D}$-module case), and on the other hand considering a global invariant that is independent of the choice of Borel (on the $U(\mathfrak{g})$-representation side). The relevant objects in each case will be the characters of the abstract Cartan and the central characters of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$, leaving us with the following refined correspondence, known as the Beilinson-Bernstein correspondence.

$$\boxed{\text{“}\mathcal{D}_X\text{-modules” with specified } \mathfrak{b}\text{-action for } \textit{all } \mathfrak{b}} \leftrightarrow \boxed{U(\mathfrak{g})\text{-representations with specific } Z(\mathfrak{g})\text{-action}}$$

The purpose of this essay is to give an introduction to the theory of $\mathcal{D}$-modules and then to rigorously formulate and prove this correspondence.

1.2. **Organization and references.** Let us now discuss the specific structure of this essay. In Section 2, we give an introduction to the theory of $\mathcal{D}$-modules, highlighting the central results of Kashiwara's Theorem (Theorem 2.19), Bernstein's Inequality (Theorem 2.32), and the b-function lemma (Lemma 2.43).

In Section 3, we describe the flag variety of a semisimple algebraic group and characterize its relationship with the corresponding Lie algebra. Of particular importance will be the statements and consequences of the Chevalley and Harish-Chandra isomorphisms (Theorems 3.13 and 3.16, respectively), as they will feature prominently in the statement of the Beilinson-Bernstein correspondence.

In Section 4, our main work takes place; we formulate and prove the Beilinson-Bernstein correspondence. After formulating the statement in terms of a family of sheaves $\mathcal{D}_X^\lambda$ of twisted differential operators, there are two main steps. First, we show that the global sections $\Gamma(X, \mathcal{D}_X^\lambda)$ correspond to quotients of the universal enveloping algebra $U(\mathfrak{g})$ by the kernels of central characters. The key input for this step will be a close study of the geometry of the nilpotent cone via its Springer resolution. To finish, we show that the global sections functor defines the desired equivalence of categories. The approach we follow, which is the original one taken by Beilinson and Bernstein, proceeds by using the additional information provided by the differential structure to relate any given $\mathcal{D}_X^\lambda$-module to one where the desired result holds.

The material we present here is not original, but rather drawn from a number of different sources, detailed as follows. For the theory of $\mathcal{D}$-modules, we are generally indebted to [Ber83] and [HTT08], which we have referenced throughout, and were extremely important in our study of the subject. We also consulted [BGK+87] and [Gai05] to much lesser extents. For the proof of the equivalence of categories in the Beilinson-Bernstein correspondence, we present essentially the original proof given in [BB81], though our treatment benefited greatly from some other viewpoints on the proof, particularly those of [Kas89], [HTT08], and [Gai05]. We also referred to [BB93], [BB83], [Beĭ83], and [BK81] for some context about the ideas involved. For the geometry of the nilpotent cone, we followed mainly the treatment of [Gai05], consulting also [HTT08] and the original papers of [Kos59] and [BL96]. We have assumed background knowledge from algebraic geometry, algebraic groups, and Lie algebras throughout. We attempt, however, to provide relevant citations (to a perhaps excessive degree). The sources for these are mentioned in text. Finally, we have attempted to be thorough in our exposition, but we have throughout omitted the proofs of some facts which would have taken us too far afield.

1.3. **Conventions and notation.** We collect here some notations which we will use throughout this essay. We work over a field k of characteristic 0, which beginning in Section 3 we will assume to be algebraically closed. Denote by $\mathrm{Sch}_{/k}$ the category of schemes over k. Let X be a scheme, possibly equipped with the action of an algebraic group G. Denote by $\mathrm{QCoh}(X)$ the category of quasi-coherent sheaves on X and by $\mathrm{QCoh}(X)^G$ the category of G-equivariant quasi-coherent sheaves on X. By a vector bundle $\mathcal{M}$ on X, we mean a locally free sheaf on X, and we denote by $\mathrm{Sym}_X \mathcal{M}$ the geometric realization of $\mathcal{M}$, which has structure sheaf $\mathrm{Sym}_{\mathcal{O}_X} \mathcal{M}^*$. By $\mathcal{H}om$, we mean the sheaf Hom.

Let X now be a smooth algebraic variety. By T_X and T_X^*, we mean its tangent and cotangent bundles, respectively. By $\Omega_X = \det(T_X^*)$, we mean the line bundle of top differential forms on X.

1.4. **Acknowledgments.** This essay was written for Part III of the Mathematical Tripos in academic year 2010-2011. I would like to thank Professor Ian Grojnowski for setting this essay and for several helpful conversations. I would also like to thank George Boxer for useful discussions in the course of learning the basic theory of $\mathcal{D}$-modules and Jonathan Wang for several useful discussions relating to derived categories in algebraic geometry.

2. $\mathcal{D}$-modules on smooth algebraic varieties

In this section, we introduce and develop the theory of $\mathcal{D}$-modules on smooth algebraic varieties. We emphasize the portions of the theory which provide some relevant geometric context for the topic of the second half of this essay, the Beilinson-Bernstein localization theorem. The main source for this section is the classical notes of Bernstein [Ber83] and the book of [HTT08], supplemented by the notes [Gai05] and the book [BGK$^+$87].

2.1. **Definition and examples.** Let X be a smooth algebraic variety over a field k of characteristic 0. If $X = \mathrm{Spec}(A)$ is affine, define a *differential operator* d of order at most n to be a k-linear map $d : A \to A$ such that for all $f_0, \ldots, f_k \in A$, we have

$$[f_n, [f_{n-1}, [\cdots, [f_0, d]]]] = 0$$

as a k-linear map $A \to A$, where we view an element $f \in A$ as a k-linear map $A \to A$ via multiplication. Such operators form a ring $\mathcal{D}(A)$ with multiplication given by composition. We call this the *ring of differential operators* on $\mathrm{Spec}(A)$. Define a filtration $F_n\mathcal{D}(A) \subset \mathcal{D}(A)$ on $\mathcal{D}(A)$ by letting $F_n\mathcal{D}(A)$ consist of the differential operators of order at most n. We call this filtration on $\mathcal{D}(A)$ the *order filtration*.

Example 2.1. If $A = k[t_1, \ldots, t_n]$, then $\mathcal{D}(A) = k[t_1, \ldots, t_n, \partial_1, \ldots, \partial_n]$ is generated over k by t_i and the formal partial derivatives ∂_i subject to the commutation relations

$$[t_i, t_j] = 0, [\partial_i, \partial_j] = 0, \text{ and } [\partial_i, t_j] = \delta_{ij},$$

where δ_{ij} denotes the Kronecker delta function.

We may glue this notion to a notion of differential operators on the entire variety X. For this, we say that a sheaf $\mathcal{A}$ of non-commutative $\mathcal{O}_X$-algebras is *quasi-coherent* (with respect to the left $\mathcal{O}_X$-structure) if for any affine U and $f \in \mathcal{O}_U$, we have

$$\Gamma(U_f, \mathcal{A}) = \mathcal{O}_{U_f} \underset{\mathcal{O}_U}{\otimes} \Gamma(U, \mathcal{A}).$$

Of course, when $\mathcal{A}$ is commutative, this restricts to the usual notion of quasi-coherence. We say $\mathcal{A}$ is quasi-coherent with respect to the right $\mathcal{O}_X$-structure if we have

$$\Gamma(U_f, \mathcal{A}) = \Gamma(U, \mathcal{A}) \underset{\mathcal{O}_U}{\otimes} \mathcal{O}_{U_f}$$

for all affine U and $f \in \mathcal{O}_U$. We will always mean quasi-coherence with respect to the left structure.

Lemma 2.2. *There exists a sheaf $\mathcal{D}_X$ of $\mathcal{O}_X$-algebras quasi-coherent with respect to both the left and right $\mathcal{O}_X$-actions such that*

$$\Gamma(\mathrm{Spec}(A), \mathcal{D}_X) = \mathcal{D}(A).$$

Proof. To check that $\mathcal{D}_X$ patches correctly into a quasi-coherent sheaf on X, we must check that

$$\mathcal{D}(A_f) \simeq \mathcal{D}(A) \underset{A}{\otimes} A_f \simeq A_f \underset{A}{\otimes} \mathcal{D}(A)$$

for any non-nilpotent $f \in A$. We will only give a sketch of one case here and refer the reader to [Gai05, Proposition 5.5] for the complete proof. Construct the map

$$\phi : \mathcal{D}(A) \to \mathcal{D}(A_f)$$

inductively on $F_i\mathcal{D}(A)$ as follows. On $F_0\mathcal{D}(A) = A$, it is just the identity map. Suppose now that ϕ is defined on $F_i\mathcal{D}(A)$; for $d \in F_{i+1}\mathcal{D}(A)$ and $\frac{g}{f^n} \in A_f$, we have $[f^n, d] \in F_i\mathcal{D}(A)$, motivating the definition

$$\phi(d)\left(\frac{g}{f^n}\right) := \frac{d(g)}{f^n} + \frac{\phi([f^n, d])(g/f^n)}{f^n}.$$

It is clear that this yields a well-defined map ϕ that is compatible with the filtrations. To check that it is an isomorphism, it remains only to check that it is an isomorphism on the associated graded level, which will follow from Proposition 2.5 below.[2] $\square$

We call $\mathcal{D}_X$ the *sheaf of differential operators* on X. It naturally inherits an order filtration $F_n\mathcal{D}_X$. The following results show that $F_n\mathcal{D}_X$ gives $\mathcal{D}_X$ the structure of a sheaf of filtered rings and provide some basic properties.

Lemma 2.3. *For non-negative integers n_1, n_2, we have that*

$$F_{n_1}\mathcal{D}_X \cdot F_{n_2}\mathcal{D}_X \subseteq F_{n_1+n_2}\mathcal{D}_X.$$

Proof. Take $d_1 \in F_{n_1}\mathcal{D}_X$ and $d_2 \in F_{n_2}\mathcal{D}_X$. Then, for any $f_0, \dots, f_{n_1+n_2} \in \mathcal{O}_X$, we note that

$$[f_0, [f_1, \cdots [f_{n_1+n_2}, d_1 d_2]]]$$

is the sum of monomials of the form

$$[f_{i_1}, [f_{i_2}, \dots, [f_{i_k}, d_1]]] \cdot [f_{j_1}, [f_{j_2}, \dots, [f_{j_{n_1+n_2+1-k}}, d_2]]],$$

where $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_{n_1+n_2+1-k}\}$ form a partition of $\{0, \dots, n_1 + n_2\}$. By the definition of $F_{n_1}\mathcal{D}_X$ and $F_{n_2}\mathcal{D}_X$, this means that either $[f_{i_1}, [f_{i_2}, \dots, [f_{i_k}, d_1]]]$ or $[f_{j_1}, [f_{j_2}, \dots, [f_{j_{n_1+n_2+1-k}}, d_2]]]$ vanishes for each monomial, hence the entire sum vanishes and $d_1 d_2 \in F_{n_1+n_2}\mathcal{D}_X$ as desired. $\square$

Proposition 2.4. *The order filtration on $\mathcal{D}_X$ satisfies the following:*

(i) $F_0\mathcal{D}_X \simeq \mathcal{O}_X$, *and*
(ii) $F_1\mathcal{D}_X \simeq \mathcal{O}_X \oplus T_X$.

Proof. For (i), notice that local sections $\xi \in F_0\mathcal{D}_X$ satisfy $[f, \xi] = 0$ for all $f \in \mathcal{O}_X$, hence are $\mathcal{O}_X$-linear maps $\mathcal{O}_X \to \mathcal{O}_X$, that is, sections of $\mathcal{O}_X$.

For (ii), for any $\xi \in F_1\mathcal{D}_X$, we claim that $\xi' = \xi - \xi(1)$ is a derivation $\mathcal{O}_X \to \mathcal{O}_X$, hence a section of T_X. Indeed, for any two sections $f_0, f_1 \in \mathcal{O}_X$, we have

$$f_1 f_0 \xi(1) - f_1 \xi(f_0) - f_0 \xi(f_1) + \xi(f_0 f_1) = 0$$

and therefore

$$\xi'(f_0 f_1) = \xi(f_0 f_1) - \xi(1) f_0 f_1 = f_1 \xi(f_0) - f_0 d(f_1) - 2\xi(1) f_0 f_1 = f_0 \xi'(f_1) + f_1 \xi'(f_0).$$

The map $F_1\mathcal{D}_X \to \mathcal{O}_X \oplus T_X$ given by $\xi \mapsto (\xi(1), \xi - \xi(1))$ is then inverse to the evident addition map $\mathcal{O}_X \oplus T_X \to F_1\mathcal{D}_X$, providing the desired isomorphism. $\square$

Proposition 2.5. *The canonical map*

$$\operatorname{Sym}_{\mathcal{O}_X} T_X \to \operatorname{gr} \mathcal{D}_X$$

is well-defined and an isomorphism.

Proof. For this, we must first check that $\operatorname{gr} \mathcal{D}_X$ is commutative, for which it suffices to show that $[F_{n_1}\mathcal{D}_X, F_{n_2}\mathcal{D}_X] \subset F_{n_1+n_2-1}\mathcal{D}_X$. Indeed, for $d_1 \in F_{n_1}\mathcal{D}_X$, $d_2 \in F_{n_2}\mathcal{D}_X$, and $f_0, \dots, f_{n_1+n_2-1} \in \mathcal{O}_X$, repeatedly applying the Jacobi identity gives that

$$[f_0, [f_1, \cdots [f_{n_1+n_2-1}, [d_1, d_2]]]]$$

is the sum of monomials of the form

$$[[f_{i_1}, [f_{i_2}, \dots, [f_{i_k}, d_1]]], [f_{j_1}, [f_{j_2}, \dots, [f_{j_{n_1+n_2-k}}, d_2]]]],$$

where similarly to before $\{i_1, \dots, i_k\}$ and $\{j_1, \dots, j_{n_1+n_2-k}\}$ form a partition of $\{0, \dots, n_1 + n_2 + 1\}$. In this case, if $k > n_1$, then $[f_{i_1}, [f_{i_2}, \dots, [f_{i_k}, d_1]]] = 0$ and if $k < n_1$, then $[f_{j_1}, [f_{j_2}, \dots, [f_{j_{n_1+n_2-k}}, d_2]]] = 0$. On the other hand, if $k = n_1$, then both $[f_{i_1}, [f_{i_2}, \dots, [f_{i_k}, d_1]]]$ and $[f_{j_1}, [f_{j_2}, \dots, [f_{j_{n_1+n_2-k}}, d_2]]]$ are differential operators of order 0, hence they commute. Thus in all cases, the monomial vanishes, so the entire sum vanishes and we see that $[d_1, d_2] \in F_{n_1+n_2-1}\mathcal{D}_X$, as desired.

[2] Although this may seem to allow for the possibility of circular logic, our proof of Proposition 2.5 restricts to the affine case without reference to this lemma.

The decomposition of Proposition 2.4(ii) then yields a natural map of commutative $\mathcal{O}_X$-algebras $\psi : \mathrm{Sym}_{\mathcal{O}_X} T_X \to \mathrm{gr}\, \mathcal{D}_X$. To show that it is an isomorphism, we construct an inverse map $\phi^i : \mathrm{gr}^i \mathcal{D}_X \to \mathrm{Sym}^i_{\mathcal{O}_X} T_X$ inductively as follows. On $\mathrm{gr}^0 \mathcal{D}_X$, the map ϕ^0 is just the inverse of the isomorphism of Proposition 2.4(i). Suppose now that we have defined maps ϕ^j for $j < i$ that are isomorphisms on the j^{th} graded piece. Then, consider the map

$$\mathcal{O}_X \otimes_k F_i\mathcal{D}_X \to \mathrm{gr}^{i-1} \mathcal{D}$$

given by $f \otimes d \mapsto [d, f]$. This map kills $\mathcal{O}_X \otimes F_{i-1}\mathcal{D}_X$, hence defines a map

$$\mathcal{O}_X \otimes \mathrm{gr}^i \mathcal{D}_X \to \mathrm{gr}^{i-1} \mathcal{D}_X \overset{\phi^{i-1}}{\to} \mathrm{Sym}^{i-1}_{\mathcal{O}_X} T_X$$

which for each element of $\mathrm{gr}^i \mathcal{D}_X$ yields a derivation $\mathcal{O}_X \to \mathrm{gr}^{i-1} \mathcal{D}_X$. Therefore, we may take ϕ^i to be the induced composition

$$\phi^i : \mathrm{gr}^i \mathcal{D}_X \to \mathrm{Der}_k(\mathcal{O}_X, \mathrm{Sym}^{i-1}_{\mathcal{O}_X} T_X) \simeq \mathrm{Sym}^{i-1}_{\mathcal{O}_X} T_X \otimes T_X \to \mathrm{Sym}^i_{\mathcal{O}_X} T_X.$$

That $\phi^i \circ \psi^i = \mathrm{id}$ is obvious, so to check that ϕ^i is inverse to ψ^i, we note that the map

$$\mathrm{gr}^i \mathcal{D}_X \to \mathrm{gr}^{i-1} \mathcal{D}_X \otimes T_X$$

sends $d \in \mathrm{gr}^i \mathcal{D}_X$ to some $d' \otimes \xi \in T_X \otimes \mathrm{gr}^{i-1} \mathcal{D}_X$ for which $\xi(f)d' = [d, f]$. This means then that $d'(\xi(f)) = \xi(f)d'(1) = d(f) - fd(1)$, so $d'\xi = d - d(1)$, thus $d'\xi = d$ in $\mathrm{gr}^i \mathcal{D}_X$, completing the check. □

Proposition 2.6. *The sheaf of $\mathcal{O}_X$-algebras $\mathcal{D}_X$ is generated over $\mathcal{O}_X$ by T_X subject to the obvious commutation relation $[\xi, f] = \xi(f)$ for sections $\xi \in T_X$ and $f \in \mathcal{O}_X$.*

Proof. This is essentially a formal consequence of Proposition 2.5; we refer the reader to [Gai05, Proposition 5.3] for the complete proof. □

Recall now that we are working over a variety X which we assumed to be smooth, which means that we may locally trivialize the tangent bundle T_X of X. As a result, we recall that, locally on X, there exist sections $f_1, \ldots, f_n \in \mathcal{O}_X$ such that their images df_i in T^*_X form a free basis for T^*_X; we call such a system an *étale coordinate system.* This technique will be quite useful for us in the future, particularly because it allows us to perform explicit calculations using the presentation of Proposition 2.6.

Proposition 2.7. *Suppose that $f_1, \ldots, f_n \in \mathcal{O}_X$ form an étale coordinate system for X. Then, locally there exist vector fields $\xi_1, \ldots, \xi_n$ on X such that the sheaf of differential operators has the presentation*

$$\mathcal{D}_X \simeq \mathcal{O}_X[\xi_1, \ldots, \xi_n]/\langle [\xi_i, \xi_j] = 0, [\xi_i, f_j] = \delta_{ij}\rangle.$$

Proof. It suffices to take $\xi_i \in T_X$ to be sections forming a dual basis to the basis df_i for T^*_X. That $\mathcal{D}_X$ has the desired presentation follows from Proposition 2.6. □

We are now ready to define $\mathcal{D}$-modules, the fundamental geometric objects of study in this section.

Definition 2.8. A (left, resp. right) $\mathcal{D}_X$-*module* on X is a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ equipped with a (left, resp. right) $\mathcal{D}_X$-action compatible with the $\mathcal{O}_X$-action on $\mathcal{F}$. Denote by $\mathrm{DMod}(X)$ and $\mathrm{DMod}(X)^r$ the categories of left and right $\mathcal{D}_X$-modules.

Proposition 2.9. *Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Then, to give a left $\mathcal{D}_X$-module structure on $\mathcal{F}$ is to give a k-linear action of T_X on $\mathcal{F}$ such that for sections $f \in \mathcal{O}_X$, $\xi_1, \xi_2 \in T_X$, and $m \in \mathcal{F}$, we have*

- $(f\xi) \cdot m = f(\xi \cdot m)$,
- $\xi \cdot (fm) = \xi(f) \cdot m + f(\xi \cdot m)$, *and*
- $[\xi_1, \xi_2] \cdot m = \xi_1 \cdot (\xi_2 \cdot m) - \xi_2 \cdot (\xi_1 \cdot m)$.

To give a right $\mathcal{D}_X$-module structure on $\mathcal{F}$ is to give a k-linear action of T_X on $\mathcal{F}$ such that for sections $f \in \mathcal{O}_X$, $\xi_1, \xi_2 \in T_X$, and $m \in \mathcal{F}$, we have

- $m(f\xi) = f(m\xi)$,
- $(fm)\xi = \xi(f) \cdot m - f(m\xi)$, *and*
- $m[\xi_1, \xi_2] = m\xi_1\xi_2 - m\xi_2\xi_1$.

Proof. This follows immediately from the presentation of $\mathcal{D}_X$ in Proposition 2.6. □

Using Proposition 2.9, we may now see some basic examples of $\mathcal{D}$-modules.

Example 2.10. We may equip $\mathcal{D}_X$ itself with both a left and right $\mathcal{D}_X$-module structure by left and right multiplication, as $\mathcal{D}_X$ is quasi-coherent with respect to both the left and right $\mathcal{O}_X$-module structures by Lemma 2.2.

Example 2.11. The structure sheaf $\mathcal{O}_X$ is a $\mathcal{D}_X$-module with respect to the action of T_X by derivations.

Example 2.12. This example provides some analytic motivation for the definition of a $\mathcal{D}$-module. For $j = 1, \ldots, k$ and $i = 1, \ldots, n$, choose differential operators $D_{ij} \in \mathcal{D}_X$ and consider the system

$$\sum_{i=1}^{n} D_{ij} f_i = 0 \tag{1}$$

of partial differential equations in the functions $f_1, \ldots, f_n$. Construct a $\mathcal{D}_X$-module

$$\mathcal{M} = \operatorname{coker}\left(\mathcal{D}_X^n \stackrel{D_{ij}}{\to} \mathcal{D}_X^k\right)$$

corresponding to this system. Then, we see that (global) solutions to (1) with $f_i \in \mathcal{O}_X$ correspond to morphisms of $\mathcal{D}_X$-modules $\mathcal{M} \to \mathcal{O}_X$. Further, any space of functions on X on which differential operators $\mathcal{D}_X$ act corresponds exactly to a $\mathcal{D}_X$-module $\mathcal{F}$; then (global) solutions to (1) in this space of functions are simply morphisms of $\mathcal{D}_X$-modules $\mathcal{M} \to \mathcal{F}$.

In general, we will show later that any $\mathcal{D}_X$-module $\mathcal{M}$ which is coherent as an $\mathcal{O}_X$-module will be locally free. In this case, the conditions for $\mathcal{M}$ to be a $\mathcal{D}$-module given in Proposition 2.9 translate exactly into the conditions for $\mathcal{M}$ to be endowed with a flat connection $\nabla : T_X \to \operatorname{End}_k \mathcal{M}$, where we define ∇ simply to be the action of T_X given by the $\mathcal{D}_X$-module structure.

Example 2.13. For any closed point $x \in X$, define the right $\mathcal{D}_X$-module δ_x to be generated by a formal element 1_x subject to the single relation

$$1_x \cdot f = 1_x \cdot f(x)$$

for $f \in \mathcal{O}_X$. By this, we mean that δ_x is isomorphic to

$$(i_x)_* k_x \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$$

as an $\mathcal{O}_X$-module, where the right $\mathcal{D}_X$-action is by right multiplication. Here the nomenclature δ_x reflects the fact that elements of $\mathcal{O}_X$ act on 1_x by evaluation at x. In particular, in the case where $X = \mathbb{A}^1 = \operatorname{Spec}(k[t])$ and $x = 0 \in X$, then we see that

$$\delta_0 \simeq 1_0 \cdot k[\partial_t],$$

where the action of ∂_t is by multiplication and the action of t is by $\partial_t^n \cdot t = n\partial_t^{n-1}$. In particular, we note that the action of t annihilates δ_0 as a right $\mathcal{O}_X$-module. We will see a similar construction to this one later when we prove Kashiwara's Theorem (Theorem 2.19).

2.2. **Left and right $\mathcal{D}$-modules.**

The goal of this subsection is to introduce an operation which will allow us to convert between left and right $\mathcal{D}$-modules. We will see that this operation gives an equivalence between the categories of left and right $\mathcal{D}_X$-modules on a smooth variety X. Therefore, we will focus our attention on left $\mathcal{D}$-modules. In particular, in the sequel, by a "$\mathcal{D}$-module" we will always mean a "left $\mathcal{D}$-module" unless we specify that it is a right $\mathcal{D}$-module.

To give this construction, we must use some basic Hom and tensor operations on $\mathcal{D}$-modules. Let us first see that these make sense.

Proposition 2.14. *Let $\mathcal{F}, \mathcal{F}'$ be left $\mathcal{D}_X$-modules and $\mathcal{G}, \mathcal{G}'$ be right $\mathcal{D}_X$-modules. Then, the following $\mathcal{O}_X$-modules acquire a left $\mathcal{D}_X$-module structure via the indicated action of T_X:*

- *$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}'$ with action given by*
$$\xi \cdot (m \otimes m') = \xi(m) \otimes m' + m \otimes \xi(m');$$
- *$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}')$ with action given by*
$$(\xi \cdot \phi)(m) = \xi(\phi(m)) - \phi(\xi(m));$$
- *$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{G}')$ with action given by*
$$(\xi \cdot \phi)(n) = -\phi(n)\xi + \phi(n\xi).$$

The following $\mathcal{O}_X$-modules acquire a right $\mathcal{D}_X$-module structure via the indicated action of T_X:

- *$\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{F}$ with action given by*
$$(n \otimes m)\xi = n\xi \otimes m - n \otimes \xi(m);$$
- *$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ with action given by*
$$(\phi\xi)(m) = \phi(m)\xi + \phi(\xi(m)).$$

Here $\xi \in T_X, m \in \mathcal{F}, m' \in \mathcal{F}', n \in \mathcal{G}, n' \in \mathcal{G}'$ are sections of the corresponding sheaves.

Proof. In each case, we must check that the indicated action is well-defined and satisfies the relevant conditions of Proposition 2.9. We do this in the first case, leaving the rest of the verifications to the reader. Take a section $f \in \mathcal{O}_X$ and $\xi_1, \xi_2 \in T_X$. First, to show that the action is well-defined, it suffices to note that

$$\begin{aligned} \xi \cdot (fm \otimes m') = \xi(fm) \otimes m' + fm \otimes \xi(m') &= \xi(f)m \otimes m' + f\xi(m) \otimes m' + fm \otimes \xi(m') \\ &= f\xi(m) \otimes m' + m \otimes \xi(fm') = \xi \cdot (m \otimes fm'). \end{aligned}$$

We now check each condition in Proposition 2.9 in turn. The first is evident, the second follows from

$$\xi\cdot(f\cdot m\otimes m') = \xi(fm)\otimes m' + fm\otimes\xi(m') = \xi(f)m\otimes m' + f\xi(m)\otimes m' + fm\otimes\xi(m') = \xi(f)m\otimes m' + f\xi(m\otimes m'),$$

and the third from

$$\begin{aligned} [\xi_1, \xi_2] \cdot (m_1 \otimes m_2) &= [\xi_1, \xi_2]m_1 \otimes m_2 + m_1 \otimes [\xi_1, \xi_2]m_2 \\ &= (\xi_1\xi_2 - \xi_2\xi_1)m_1 \otimes m_2 + m_1 \otimes (\xi_1\xi_2 - \xi_2\xi_1)m_2 \\ &= (\xi_1\xi_2 m_1 \otimes m_2 + \xi_1 m_1 \otimes \xi_2 m_2 + \xi_2 m_1 \otimes \xi_1 m_2 + m_1 \otimes \xi_1\xi_2 m_2) \\ &\quad - (\xi_2\xi_1 m_1 \otimes m_2 + \xi_1 m_1 \otimes \xi_2 m_2 + \xi_2 m_1 \otimes \xi_1 m_2 + m_1 \otimes \xi_2\xi_1 m_2) \\ &= (\xi_1\xi_2 - \xi_2\xi_1) \cdot (m_1 \otimes m_2). \end{aligned}$$

□

We may now give the correspondence between left and right $\mathcal{D}$-modules. Let $\Omega_X := \det(T_X^*)$ denote the sheaf of top dimensional differentials on X. Recall the natural action of T_X on Ω_X via the *Lie derivative* Lie_ξ; it is given explicitly by

$$\mathrm{Lie}_\xi(\omega) := \Big(\xi_1 \wedge \cdots \wedge \xi_n \mapsto \xi(\omega(\xi_1 \wedge \cdots \wedge \xi_n)) - \sum_{i=1}^n \omega(\xi_1, \ldots, \xi_{i-1}, [\xi, \xi_{i-1}], \xi_{i+1}, \ldots, \xi_n)\Big).$$

Lemma 2.15. *The action $\omega \cdot \xi = -Lie_\xi(\omega)$ gives Ω_X the structure of a right $\mathcal{D}_X$-module.*

Proof. We must check the conditions of Proposition 2.9 using the explicit form of Lie_ξ. Having performed one such check in the proof of Proposition 2.14, we leave the gory details to the reader. □

Consider now the $\mathcal{O}_X$-module

$$\Omega_{\mathcal{D}_X} := \Omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X.$$

Note that $\Omega_{\mathcal{D}_X}$ is endowed with two commuting structures of right $\mathcal{D}_X$-module via right multiplication by $\mathcal{D}_X$ and the construction of Lemma 2.14. Therefore, for any left $\mathcal{D}_X$-module $\mathcal{F}$, the $\mathcal{O}_X$-module

$$\Omega_{\mathcal{D}_X} \underset{\mathcal{D}_X}{\otimes} \mathcal{F} \simeq \Omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{F}$$

acquires a right $\mathcal{D}_X$-module structure in two equivalent ways, either by applying Proposition 2.14 to the second expression or by considering the right $\mathcal{D}_X$-module structure on $\Omega_{\mathcal{D}_X}$ in the first expression. These two descriptions are equivalent, giving a functor

$$\Omega_{\mathcal{D}_X} \otimes_{\mathcal{D}_X} - \simeq \Omega_X \otimes_{\mathcal{O}_X} -$$

from left $\mathcal{D}_X$-modules to right $\mathcal{D}_X$-modules. Similarly, we obtain a functor

$$\mathcal{H}om_{\mathcal{D}_X^r}(\Omega_{\mathcal{D}_X}, -) \simeq \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, -)$$

from right $\mathcal{D}_X$-modules to left $\mathcal{D}_X$-modules, where we note that for a right $\mathcal{D}_X$-module $\mathcal{G}$, $\mathcal{H}om_{\mathcal{D}_X^r}(\Omega_{\mathcal{D}_X}, \mathcal{G})$ has two commuting structures of left $\mathcal{D}_X$-module coming from the two commuting structures of right $\mathcal{D}_X$-module on $\Omega_{\mathcal{D}_X}$. These two functors will provide the claimed equivalence between left and right $\mathcal{D}_X$-modules.

Proposition 2.16. *The functors $\Omega_{\mathcal{D}_X} \otimes_{\mathcal{D}_X} -$ and $\mathcal{H}om_{\mathcal{D}_X^r}(\Omega_{\mathcal{D}_X}, -)$ define an equivalence between the categories of left and right $\mathcal{D}_X$-modules.*

Proof. We first check that the functors are adjoint. For this, we claim that the standard Hom-tensor adjunction map for $\mathcal{O}_X$-modules restricts to a map

$$\mathrm{Hom}_{\mathcal{D}_X^r}(\Omega_X \underset{\mathcal{O}_X}{\otimes} -, -) \to \mathrm{Hom}_{\mathcal{D}_X}(-, \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, -)).$$

For this, it suffices to check that the adjunction map $\Psi(\phi) = \Big(m \mapsto (\omega \mapsto \phi(\omega \otimes m))\Big)$ restricts to an isomorphism between maps of right $\mathcal{D}_X$-modules and maps of left $\mathcal{D}_X$-modules. Notice that

$$\xi \cdot \Psi(\phi)(m) = \xi \cdot \Big(\omega \mapsto \phi(\omega \otimes m)\Big) = \Big(\omega \mapsto -\phi(\omega \otimes m)\xi + \phi(\omega\xi \otimes m)\Big)$$

and

$$\Psi(\phi)(\xi m) = \Big(\omega \mapsto \phi(\omega \otimes \xi m)\Big),$$

thus $\Psi(\phi)$ is a map of left $\mathcal{D}_X$-modules if and only if

$$\phi(\omega \otimes m)\xi = \phi(\omega\xi \otimes m) - \phi(\omega \otimes \xi m) = \phi((\omega \otimes m)\xi)$$

for all $\omega \otimes m \in \Omega_X \otimes_{\mathcal{O}_X} \mathcal{F}$. This occurs if and only if ϕ is a map of right $\mathcal{D}_X$-modules, giving the desired adjunction.

It now remains to check that the unit and co-unit maps

$$\Omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, -) \simeq \Omega_{\mathcal{D}_X} \underset{\mathcal{D}_X}{\otimes} \mathcal{H}om_{\mathcal{D}_X^r}(\Omega_{\mathcal{D}_X}, -) \to \mathrm{id}$$

and

$$\mathrm{id} \to \mathcal{H}om_{\mathcal{D}_X^r}(\Omega_{\mathcal{D}_X}, \Omega_{\mathcal{D}_X} \underset{\mathcal{D}_X}{\otimes} -) \simeq \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \Omega_X \underset{\mathcal{O}_X}{\otimes} -)$$

are isomorphisms. On the level of $\mathcal{O}_X$-modules, this follows from the statement that $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ is the dual line bundle to Ω_X, thus it holds on the level of $\mathcal{D}_X$-modules as well. $\square$

In general, we find the $\mathcal{D}$-module actions given by Proposition 2.14 on tensor products to be a great deal more intuitive than the ones on $\mathcal{H}om$'s. Therefore, let us reformulate the functor $\mathcal{H}om_{\mathcal{D}_X^r}(\Omega_{\mathcal{D}_X}, -)$ in these terms. Define $\mathcal{D}_X^{\Omega} := \mathcal{H}om_{\mathcal{D}_X^r}(\Omega_{\mathcal{D}_X}, \mathcal{D}_X)$, which carries two commuting left $\mathcal{D}_X$-module structures, one via the left action on $\mathcal{D}_X$ and one via Proposition 2.14 as $\mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{D}_X)$. Then, for a right $\mathcal{D}_X$-module $\mathcal{G}$, we see that

$$\mathcal{H}om_{\mathcal{D}_X^r}(\Omega_{\mathcal{D}_X}, \mathcal{G}) \simeq \mathcal{G} \underset{\mathcal{D}_X}{\otimes} \mathcal{H}om_{\mathcal{D}_X}(\Omega_X, \mathcal{D}_X) \simeq \mathcal{G} \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_X^{\Omega},$$

where the left $\mathcal{D}_X$-module structure on $\mathcal{G} \otimes_{\mathcal{D}_X} \mathcal{D}_X^{\Omega}$ is given by the left action on $\mathcal{D}_X$.[3]

2.3. **Pullback and pushforward of $\mathcal{D}$-modules.** Consider now a smooth map $\phi : X \to Y$ of smooth algebraic varieties. In this subsection, we describe a way to pullback and pushforward $\mathcal{D}$-modules along ϕ. We will first define "standard" functors for the pullback and pushfoward in this subsection. However, these functors will not be well-behaved, and the "correct" versions which we define later will exist only in the derived category. A complication, however, is that only the derived pullback coincides with the derived functor of the standard pullback; the derived pushforward will be a closely related but different functor, as the standard pushforward will be neither left nor right exact in general.

We now discuss some notational issues arising from this. We use ϕ_* and ϕ^* to denote the standard pushforward and pullback in the category of quasi-coherent $\mathcal{O}$-modules and $\phi_\cdot$ and $\phi^\cdot$ to denote the pushforward and pullback on the level of sheaves. We denote the standard pullback of $\mathcal{D}$-modules by ϕ^Δ and the standard pushforward by ϕ_+.[4] Finally, we denote the derived pullback by $\phi^!$ and the derived pushforward by $\phi_\star$.

---

[3]It is tempting here to consider the dual line bundle $\Omega_X^{-1} := \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X)$ to Ω_X and use the isomorphism

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{F}) \simeq \mathcal{H}om_{\mathcal{O}_X}(\Omega_X, \mathcal{O}_X) \underset{\mathcal{O}_X}{\otimes} \mathcal{F} \simeq \Omega_X^{-1} \underset{\mathcal{O}_X}{\otimes} \mathcal{F}$$

of $\mathcal{O}_X$-modules to express $\mathcal{H}om_{\mathcal{D}_X^r}(\Omega_{\mathcal{D}_X}, -)$ as a tensor product over $\mathcal{O}_X$. However, we caution the reader that there is in general *no $\mathcal{D}_X$-module structure on* Ω_X^{-1}. Indeed, for a smooth algebraic curve, this is a consequence of what is known as Oda's rule, which states that a line bundle on a curve of genus g has a left $\mathcal{D}$-module structure if and only if it has degree 0 and a right $\mathcal{D}$-module structure if and only if it has degree $2g - 2$. We refer the reader to [HTT08, Remark 1.2.10] for more details.

[4]We will see that ϕ^Δ coincides with ϕ^* on the level of $\mathcal{O}$-modules. However, it is important to note that ϕ_+ and $\phi_\star$ will generally be different.

2.3.1. *The definitions.* First, let $\mathcal{F}$ be a $\mathcal{D}_Y$-module. Define its *$\mathcal{D}$-module pullback* $\phi^\Delta(\mathcal{F})$ to be isomorphic to $\phi^*(\mathcal{F})$ as an $\mathcal{O}_X$-module with $\mathcal{D}_X$-action induced by the map $d\phi : T_X \to \phi^* T_Y$ corresponding to ϕ. Explicitly, this means that

$$\phi^\Delta(\mathcal{F}) \simeq \mathcal{O}_X \underset{\phi^\cdot(\mathcal{O}_Y)}{\otimes} \phi^\cdot(\mathcal{F})$$

with action of $\mathcal{D}_X$ given according to the Leibnitz rule by $\xi \cdot (f \otimes m) = \xi(f) \otimes m + f \cdot d\phi(\xi)(m)$. Defining the $(\mathcal{D}_X, \phi^\cdot(\mathcal{D}_Y))$-bimodule

$$\mathcal{D}_{X\to Y} := \phi^\Delta(\mathcal{D}_Y) = \mathcal{O}_X \underset{\phi^\cdot(\mathcal{O}_Y)}{\otimes} \phi^\cdot(\mathcal{D}_Y),$$

we may express the $\mathcal{D}$-module pullback of $\mathcal{F}$ in the form

$$\phi^*(\mathcal{F}) \simeq \mathcal{D}_{X\to Y} \underset{\phi^\cdot(\mathcal{D}_Y)}{\otimes} \phi^\cdot(\mathcal{F}),$$

where the $\mathcal{D}_X$-action on $\phi^\Delta(\mathcal{F})$ is simply the left $\mathcal{D}_X$-action on $\mathcal{D}_{X\to Y}$.

Lemma 2.17. *Let $\phi : Y \to Z$, $\psi : X \to Y$ be morphisms of smooth varieties. The $\mathcal{D}$-module pullback satisfies the composition rule*

$$\psi^\Delta \circ \phi^\Delta = (\phi \circ \psi)^\Delta.$$

Proof. It suffices simply to check that

$$\mathcal{D}_{X\to Z} \simeq \mathcal{D}_{X\to Y} \underset{\phi^\cdot \mathcal{D}_Y}{\otimes} \phi^\cdot \mathcal{D}_{Y\to Z}.$$

This is clear on the level of $\mathcal{O}$-modules, so it remains only to notice that the $\mathcal{D}$-module structure respects the $\mathcal{O}$-module isomorphism. □

Let us now try to imitate this construction to produce a $\mathcal{D}$-module pushforward. For a right $\mathcal{D}_X$-module $\mathcal{G}$, notice that $\phi_\cdot(\mathcal{G} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$ is naturally a right $\mathcal{D}_Y$-module as the sheaf-theoretic pushforward of a sheaf of $\phi^\cdot(\mathcal{D}_Y)$ modules. Apply the functors of Proposition 2.16 to translate this to a functor between left $\mathcal{D}$-modules, and call the result ϕ_+. We see then that for a left $\mathcal{D}_X$-module $\mathcal{F}$, we have

$$\begin{aligned}
\phi_+(\mathcal{F}) :&= \mathcal{H}om_{\mathcal{D}_Y}(\Omega_{\mathcal{D}_Y}, \phi_\cdot((\Omega_{\mathcal{D}_X} \underset{\mathcal{D}_X}{\otimes} \mathcal{F}) \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y})) \\
&\simeq \phi_\cdot((\Omega_{\mathcal{D}_X} \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y}) \underset{\mathcal{D}_X}{\otimes} \mathcal{F}) \underset{\mathcal{D}_Y}{\otimes} \mathcal{D}_Y^\Omega \\
&\simeq \phi_\cdot((\Omega_{\mathcal{D}_X} \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y} \underset{\phi^\cdot(\mathcal{D}_Y)}{\otimes} \phi^\cdot \mathcal{D}_Y^\Omega) \underset{\mathcal{D}_X}{\otimes} \mathcal{F}).
\end{aligned}$$

Therefore, defining the $(\phi^\cdot(\mathcal{D}_Y), \mathcal{D}_X)$-bimodule

$$\mathcal{D}_{Y\leftarrow X} := \Omega_{\mathcal{D}_X} \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y} \underset{\phi^\cdot(\mathcal{D}_Y)}{\otimes} \phi^\cdot \mathcal{D}_Y^\Omega,$$

we obtain the *$\mathcal{D}$-module pushforward*

$$\phi_+(\mathcal{F}) = \phi_\cdot(\mathcal{D}_{Y\leftarrow X} \underset{\mathcal{D}_X}{\otimes} \mathcal{F}).$$

Remark. In the manipulations we did to reduce $\phi_+(\mathcal{F})$ above, we made several moves which involved permuting the order in which we took tensor products involving $\Omega_{\mathcal{D}_X}$ and $\mathcal{D}_Y^\Omega$. We justify these for $\Omega_{\mathcal{D}_X}$ as follows; the argument for $\mathcal{D}_Y^\Omega$ will be entirely analogous. Note that the map $\tau : \Omega_{\mathcal{D}_X} \to \Omega_{\mathcal{D}_X}$ extending

$$\tau(\omega \otimes \eta) = -\mathrm{Lie}_\eta \omega \otimes 1 - \omega \otimes \eta$$

is an involution intertwining the two right $\mathcal{D}_X$-actions on $\Omega_{\mathcal{D}_X}$. Thus, any left $\mathcal{D}_X$-module $\mathcal{F}$, τ induces an isomorphism of $\mathcal{O}_X$-modules

$$\Omega_{\mathcal{D}_X} \underset{\mathcal{D}_X}{\otimes} \mathcal{F} \simeq \Omega'_{\mathcal{D}_X} \underset{\mathcal{D}_X}{\otimes} \mathcal{F},$$

where we denote by $\Omega_{\mathcal{D}_X}$ and $\Omega'_{\mathcal{D}_X}$ the object $\Omega_{\mathcal{D}_X}$ with its two different right $\mathcal{D}_X$-actions. For instance, in the manipulation above, the isomorphism

$$(\Omega_{\mathcal{D}_X} \underset{\mathcal{D}_X}{\otimes} \mathcal{F}) \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y} \simeq (\Omega_{\mathcal{D}_X} \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y}) \underset{\mathcal{D}_X}{\otimes} \mathcal{F}$$

simply exchanges the choice of which right $\mathcal{D}_X$-action on $\Omega_{\mathcal{D}_X}$ each of $\mathcal{F}$ and $\mathcal{D}_{X\to Y}$ are tensored over.

In the definition of the $\mathcal{D}$-module pushforward, observe that $\phi_+(\mathcal{F})$ is the composition of the left exact functor $\phi_\cdot$ and the right exact functor $\mathcal{D}_{Y\leftarrow X} \otimes_{\mathcal{D}_X} -$. As a result, ϕ_+ is not well-behaved in general. For instance, a composition rule analogous to Lemma 2.17 does not hold . However, when ϕ is affine, $\phi_\cdot$ is exact, and this pushforward is better behaved. We now examine a particular instance of this in detail.

2.3.2. *Closed embeddings and Kashiwara's theorem.* Let us now suppose that $\phi : X \to Y$ is a closed embedding. In particular, this means that ϕ is affine, hence $\phi_\cdot$ is exact and ϕ_+ is right exact. In this case, we may define a different pullback functor

$$\phi^+(\mathcal{F}) := \mathcal{H}om_{\phi^\cdot \mathcal{D}_Y}(\mathcal{D}_{Y\leftarrow X}, \phi^\cdot \mathcal{F}),$$

where the left $\mathcal{D}_X$-module structure on $\phi^+(\mathcal{F})$ comes from the right $\mathcal{D}_X$-module structure on $\mathcal{D}_{Y\leftarrow X}$. We will later see that ϕ^+ is related to the standard pullback functor in the derived category. For now, let us first relate it to ϕ_+.

Proposition 2.18. *If $\phi : X \to Y$ is a closed embedding, we have an adjoint pair of functors*

$$\phi_+ : \mathrm{DMod}(X) \rightleftharpoons \mathrm{DMod}(Y) : \phi^+.$$

Proof. The chain of natural isomorphisms

$$\mathrm{Hom}_{\mathcal{D}_Y}(\phi_\cdot(\mathcal{D}_{Y\leftarrow X} \underset{\mathcal{D}_X}{\otimes} -), -) \simeq \mathrm{Hom}_{\phi^\cdot \mathcal{D}_Y}(\mathcal{D}_{Y\leftarrow X} \underset{\mathcal{D}_X}{\otimes} -, \phi^\cdot -) \simeq \mathrm{Hom}_{\mathcal{D}_X}(-, \mathcal{H}om_{\phi^\cdot \mathcal{D}_Y}(\mathcal{D}_{Y\leftarrow X}, -))$$

gives the result, where the first follows from the fact that $\phi_\cdot(\mathcal{F})$ is supported on X for all $\mathcal{D}_Y$-modules $\mathcal{F}$, hence applying $\phi^\cdot$ is an isomorphism on Hom's out of $\phi_\cdot(\mathcal{F})$ and the second follows from (taking global sections of) the Hom-tensor adjunction

$$\mathcal{H}om_{\phi^\cdot \mathcal{D}_Y}(\mathcal{D}_{Y\leftarrow X} \underset{\mathcal{D}_X}{\otimes} -, \phi^\cdot -) \simeq \mathcal{H}om_{\mathcal{D}_X}(-, \mathcal{H}om_{\phi^\cdot \mathcal{D}_Y}(\mathcal{D}_{Y\leftarrow X}, -)).$$ ☐

As we saw in the proof of Proposition 2.18, any $\mathcal{D}_Y$-module in the image of ϕ_+ is supported on $X \subset Y$. Denote by the $\mathrm{DMod}(Y)_X$ the full subcategory of $\mathrm{DMod}(Y)$ generated by the $\mathcal{D}_Y$-modules with support lying in X. The following central result in the theory of $\mathcal{D}$-modules shows that ϕ_+ defines an equivalence of categories onto $\mathrm{DMod}(Y)_X$.[5]

Theorem 2.19 (Kashiwara's Theorem)**.** *If $\phi : X \to Y$ is a closed embedding, we have an equivalence of categories*

$$\phi_+ : \mathrm{DMod}(X) \rightleftharpoons \mathrm{DMod}(Y)_X : \phi^+.$$

Proof. Having shown that ϕ_+ and ϕ^+ form an adjoint pair, it suffices to check that the unit and counit are isomorphic to the identity. This is a local assertion, so we reduce to the case where $X = \mathrm{Spec}(A/I)$ and $Y = \mathrm{Spec}(A)$. Because X and Y are smooth varieties, we can find a regular sequence $I = (f_1, \ldots, f_n)$ generating the defining ideal I of X. To prove the assertion for $X \hookrightarrow Y$, it suffices to prove it for each embedding in the chain

$$X \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_{n-1} \hookrightarrow Y,$$

where $X_i = \mathrm{Spec}(A/(f_i, \ldots, f_n))$.

We have therefore reduced to the case where $X = \mathrm{Spec}(A/f)$ and $Y = \mathrm{Spec}(A)$ for some regular element $f \in A$. Because A is regular, after further localization we may find an étale coordinate system $f_1, \ldots, f_n$ where X is cut out locally by the vanishing of the last coordinate (by extending the equation defining X to a regular sequence in A). In this case, we see that

$$\mathcal{D}_X = \mathcal{O}_X[\xi_1, \ldots, \xi_{n-1}]/\langle [\xi_i, f_j] = \delta_{ij}\rangle$$

and

$$\mathcal{D}_Y = \mathcal{O}_Y[\xi_1, \ldots, \xi_{n-1}, \xi]/\langle [\xi_i, f_j] = \delta_{ij}\rangle,$$

where $\mathcal{O}_X = \mathcal{O}_Y/(f)$ for $f = f_n$ and $\xi = \xi_n$. Let us now compute $\mathcal{D}_{Y\leftarrow X}$ in these coordinates, keeping track of the $\mathcal{D}$-module structures. Notice that

$$\mathcal{D}_{X\to Y} = \mathcal{O}_X \underset{\phi^\cdot \mathcal{O}_Y}{\otimes} \phi^\cdot \mathcal{D}_Y \simeq \mathcal{O}_X[\xi_1, \ldots, \xi]/\langle [\xi_i, f_j] = \delta_{ij}\rangle \simeq \mathcal{D}_X[\xi]$$

with the left $\mathcal{D}_X$ and right $\phi^\cdot \mathcal{D}_Y$-actions given by left and right multiplication in the obvious way. Now, we see that

$$\mathcal{D}_{Y\leftarrow X} = \Omega_{\mathcal{D}_X} \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y} \underset{\phi^\cdot \mathcal{D}_Y}{\otimes} \phi^\cdot \mathcal{D}_Y^\Omega \simeq \mathcal{O}_X[\xi_1, \ldots, \xi_n]/\langle [\xi_i, f_j] = \delta_{ij}\rangle = [\xi]\mathcal{D}_X$$

with the left $\phi^\cdot \mathcal{D}_Y$ and right $\mathcal{D}_X$ actions given again by left and right multiplication. We recall here that ξ was defined to satisfy $[\xi, f] = 1$.

[5]This is quite different from the situation for $\mathcal{O}$-modules. If $X \subset Y$ is defined by the sheaf of ideals $\mathcal{I}_X \subset \mathcal{O}_Y$, it is possible that a $\mathcal{O}_Y$-module $\mathcal{F}$ supported on X only vanishes upon repeated multiplication by $\mathcal{I}_X$, hence does not lie in the image of the $\mathcal{O}$-module pushfoward. For a simple example, take $Y = \mathrm{Spec}(A)$ affine and $X = \mathrm{Spec}(A/f)$ defined by a single equation; then $\mathcal{F} := A/(f^n)$ is supported at X, but is not a pushforward of any quasicoherent $\mathcal{O}_X$-module.

Having set up our coordinates, we may proceed to checking the isomorphisms. For any $\mathcal{D}_X$-module $\mathcal{F}$, the map $\mathcal{F} \to \phi^+\phi_+\mathcal{F}$ is given by

$$\mathcal{F} \to \mathcal{H}om_{\phi^\cdot \mathcal{D}_Y}([\xi]\mathcal{D}_X, \phi^\cdot\phi_.([\xi]\mathcal{D}_X \underset{\mathcal{D}_X}{\otimes} \mathcal{F})) \simeq \mathcal{H}om_{\phi^\cdot \mathcal{D}_Y}([\xi]\mathcal{D}_X, [\xi]\mathcal{F}).$$

Because $\phi^\cdot\mathcal{D}_Y$ acts surjectively on $[\xi]\mathcal{D}_X$, a section of the right hand side is specified by a section $s = \sum_i \xi^i m_i$ of $[\xi]\mathcal{F}$ killed by f. But because $f \cdot m = 0$, we see that

$$f \cdot \sum_i \xi^i m_i = \sum_i i\xi^{i-1} m_i \neq 0$$

unless $m_i = 0$ for $i > 0$. This means that s must lie in the grade 0 component $\mathcal{F}$ of $[\xi]\mathcal{F}$ and hence that the map $\mathcal{F} \to \phi^+\phi_+\mathcal{F}$ is an isomorphism.

It remains then to check that for a $\mathcal{D}_Y$-module $\mathcal{F}$ supported on X, the evaluation map

$$\phi_+\phi^+(\mathcal{F}) \simeq \phi_.([\xi]\mathcal{D}_X \underset{\mathcal{D}_X}{\otimes} \mathcal{H}om_{\phi^\cdot \mathcal{D}_Y}([\xi]\mathcal{D}_X, \phi^\cdot\mathcal{F})) \to \mathcal{F}$$

is an isomorphism. For this, we must analyze the structure of $\mathcal{F}$ more carefully.

Consider the operator $T : \mathcal{F} \xrightarrow{f\xi} \mathcal{F}$ given by the action of $f\xi$ and define $\mathcal{F}^k := \ker(\mathcal{F} \xrightarrow{T-k} \mathcal{F})$. We see immediately that for $m \in \mathcal{F}^k$, we have

$$f\xi \cdot fm = (f^2\xi + f)m = (k+1)fm$$

and

$$f\xi \cdot \xi m = (\xi f\xi - \xi)m = (k-1)\xi m,$$

meaning that $fm \in \mathcal{F}^{k+1}$ and $\xi m \in \mathcal{F}^{k-1}$. Thus, we may consider the diagram

$$\cdots \underset{f}{\overset{\xi}{\rightleftarrows}} \mathcal{F}^0 \underset{f}{\overset{\xi}{\rightleftarrows}} \mathcal{F}^{-1} \underset{f}{\overset{\xi}{\rightleftarrows}} \mathcal{F}^{-2} \underset{f}{\overset{\xi}{\rightleftarrows}} \mathcal{F}^{-3} \underset{f}{\overset{\xi}{\rightleftarrows}} \cdots \tag{2}$$

where we have $\xi f - f\xi = 1$ as maps $\mathcal{F}^k \to \mathcal{F}^k$ for each k. By definition $f\xi$ acts by k on $\mathcal{F}^k$, so ξf acts by $k+1$ on $\mathcal{F}^k$. In particular, this implies that for $k \leq -1$, the map $\mathcal{F}^k \xrightarrow{\xi} \mathcal{F}^{k-1}$ is an isomorphism, and for $k \leq -2$, the map $\mathcal{F}^k \to \mathcal{F}^{k+1}$ is an isomorphism. We may therefore conclude that $\mathcal{F}^{-1} \simeq \mathcal{F}^{-2} \simeq \mathcal{F}^{-3} \simeq \cdots$.

It remains now to check that $\mathcal{F} \subset \mathcal{F}^{-1} \oplus \mathcal{F}^{-2} \oplus \cdots$. Because $\mathcal{F}$ is supported on X, we have a filtration

$$\mathcal{F} = \bigcup_{k=1}^{\infty} \mathcal{F}_k,$$

where $\mathcal{F}_k := \ker(\mathcal{F} \xrightarrow{f^k} \mathcal{F})$. Note that $\mathcal{F}_1$ is non-empty because the map $\mathcal{F} \xrightarrow{f} \mathcal{F}$ is not injective. We will now induct on k to show that $\mathcal{F}_k \subset \mathcal{F}^{-1} \oplus \cdots \oplus \mathcal{F}^{-k}$. For the base case $k = 1$, if $m \in \mathcal{F}_1$, then $Tm = f\xi m = \xi f m - m = -m$, hence $\mathcal{F}_1 \subset \mathcal{F}^{-1}$. Suppose now that $\mathcal{F}_{k-1} \subset \mathcal{F}^{-1} \oplus \cdots \oplus \mathcal{F}^{-k+1}$ for some k and take $m \in \mathcal{F}_k$. Then, notice that $fm \in \mathcal{F}_{k-1} \subset \mathcal{F}^{-1} \oplus \cdots \oplus \mathcal{F}^{-k+1}$, hence $\xi fm \in \mathcal{F}^{-1} \oplus \cdots \oplus \mathcal{F}^{-k}$. Similarly, we see that $\xi f^2 \xi m = \xi f \xi f m - \xi f m \in \mathcal{F}^{-1} \oplus \cdots \oplus \mathcal{F}^{-k}$ because $fm \in \mathcal{F}_{k-1}$, so by the induction hypothesis and (2), we see that $f\xi m \in \mathcal{F}^{-1} \oplus \cdots \oplus \mathcal{F}^{-k}$, meaning that

$$m = \xi f m - f\xi m \in \mathcal{F}^{-1} \oplus \cdots \oplus \mathcal{F}^{-k}.$$

Thus, we have obtained an isomorphism of $\mathcal{D}_Y$-modules

$$\mathcal{F} = \bigoplus_{k=1}^{\infty} \mathcal{F}^{-k} \simeq [\xi]\mathcal{F}^{-1}, \tag{3}$$

where f acts by 0 on $\mathcal{F}^{-1}$.[6]

[6]We note here the evident formal similarity between $\mathcal{F} \simeq [\xi]\mathcal{F}^{-1}$ and the $\mathcal{D}$-module δ_x of Example 2.13. Indeed, trivially generalizing the latter construction to apply to any closed embedding $i : Y \hookrightarrow X$, we obtain for any $\mathcal{D}_Y$-module $\mathcal{M}$ a right $\mathcal{D}_X$-module $i_*\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{D}_X$ (which is the pushforward of right $\mathcal{D}$-modules). In the situation of the present proof, if we take $\mathcal{M} = \mathcal{F}^{-1}$ and then apply the equivalence between left and right $\mathcal{D}_X$-modules, the result is exactly the $\mathcal{D}_X$-module $\mathcal{F}$, which shows that any $\mathcal{D}_X$-module supported on some closed smooth subvariety Y cut out by a single equation may be realized as the pushforward of a $\mathcal{D}_Y$-module. This is the key idea behind the proof we are giving of Kashiwara's Theorem.

Returning to our original objective, we see that

$$\begin{aligned}\phi_+\phi^+(\mathcal{F}) &\simeq \phi_\cdot([\xi]\mathcal{D}_X \underset{\mathcal{D}_X}{\otimes} \mathcal{H}om_{\phi^\cdot\mathcal{D}_Y}([\xi]\mathcal{D}_X, \phi^\cdot\mathcal{F}))\\ &\simeq \phi_\cdot([\xi]\mathcal{D}_X \underset{\mathcal{D}_X}{\otimes} \mathcal{H}om_{\phi^\cdot\mathcal{D}_Y}([\xi]\mathcal{D}_X, \phi^\cdot[\xi]\mathcal{F}^{-1}))\\ &\simeq \phi_\cdot([\xi]\mathcal{D}_X \underset{\mathcal{D}_X}{\otimes} \phi^\cdot\mathcal{F}^{-1}),\end{aligned}$$

where the final isomorphism follows because a section of $\mathcal{H}om_{\phi^\cdot\mathcal{D}_Y}([\xi]\mathcal{D}_X, \phi^\cdot[\xi]\mathcal{F}^{-1}))$ is specified by a section of $[\xi]\mathcal{F}^{-1}$ killed by f, which must lie in $\mathcal{F}^{-1}$ itself. Under this identification, the map $\phi_+\phi^+(\mathcal{F}) \to \mathcal{F}$ is simply multiplication, thus an isomorphism by (3). □

Remark. We have thus far only considered $\mathcal{D}$-modules on smooth algebraic varieties. While our definition of the sheaf of differential operators $\mathcal{D}_X$ did not depend on the fact that X was smooth, a similar definition on singular varieties is pathological. Instead, we sketch an approach suggested by Kashiwara's Theorem. We may locally realize any singular variety X as a closed subvariety $X \hookrightarrow Y$ of some smooth subvariety Y. Then, we define the category $\mathrm{DMod}(X) := \mathrm{DMod}(Y)_X$ locally to be the category of $\mathcal{D}_Y$-modules supported on X. To check that our definition did not depend on the choice of Y, for two such embeddings $X \hookrightarrow Y_1$ and $X \hookrightarrow Y_2$, take a third smooth variety Z such that the two embeddings fit into the commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ Y_2 \arrow[r] & Z. \end{tikzcd}$$

Then, we have equivalences of categories

$$\mathrm{DMod}(Y_1)_X \simeq (\mathrm{DMod}(Z)_{Y_1})_X \simeq \mathrm{DMod}(Z)_X \simeq (\mathrm{DMod}(Z)_{Y_2})_X \simeq \mathrm{DMod}(Y_2)_X$$

by Kashiwara's theorem on the embeddings of smooth varieties $Y_1 \hookrightarrow Z$ and $Y_2 \hookrightarrow Z$. It remains to glue these local constructions we have made into a global category on all of X and to check that the gluing procedure is independent of any choices we made. This will follow from some further abstract nonsense which we suppress. We do note here, however, that the category $\mathrm{DMod}(X)$ we have just defined is a priori unrelated to the category of $\mathcal{D}_X$-modules when X is singular, and there may not be functors in either direction.

Kashiwara's Theorem suggests that the behavior of $\mathcal{D}$-modules is much more strongly structured than the behavior of $\mathcal{O}_X$-modules. We illustrate this with the following immediate consequence.

Proposition 2.20. *If a $\mathcal{D}_X$-module $\mathcal{F}$ is coherent as an $\mathcal{O}_X$-module, then it is locally free as an $\mathcal{O}_X$-module.*

Proof. It suffices to show that $\mathcal{F}$ is flat, for which it suffices to check that the dimensions of the fibers $\mathcal{F}/\mathfrak{m}_x\mathcal{F}$ of $\mathcal{F}$ are locally constant. We claim it suffices to check this for the restriction of $\mathcal{F}$ to any nonsingular curve $i : C \hookrightarrow X$ embedded into X; such a restriction will be $\mathcal{O}_X$-coherent because i^Δ and i^* coincide in this case (as the set of points which may be connected to a given point by a smooth curve is open).

We have now reduced to the case where X is a curve. We claim that $\mathcal{F}$ is torsion-free on X; indeed, if $\mathcal{F}$ has torsion at a point x, then this means there is a non-zero $\mathcal{G} \subset \mathcal{F}$ such that $\mathfrak{m}_x^N$ kills $\mathcal{G}$. That is, $\mathcal{G}$ is supported on $\{x\}$, so by Kashiwara's Theorem we may write

$$\mathcal{G} = (i_x)_+(\mathcal{H}) = (i_x)_\cdot(\mathcal{D}_{X\leftarrow\{x\}} \underset{k}{\otimes} \mathcal{H})$$

for some $\mathcal{D}_{\{x\}}$-module $\mathcal{H}$ (which is nothing more than a vector space). But recall from the proof of Kashiwara's Theorem that $\mathcal{D}_{X\leftarrow\{x\}} \simeq k[\xi]$ as a vector space over k, hence $\mathcal{G} = (i_x)_\cdot(\mathcal{H}[\xi])$ is not coherent as an $\mathcal{O}_X$-module, a contradiction.

Because X was a curve, each stalk $\mathcal{F}_x$ is a module over $\mathcal{O}_{X,x}$, which is a regular local ring of dimension 1, hence a DVR. But a module over a DVR is free if and only if it is torsion free, meaning that $\mathcal{F}_x$ is free over $\mathcal{O}_{X,x}$ for each x. We conclude that $\mathcal{F}$ is locally free on X, so the dimensions of its fibers $\mathcal{F}/\mathfrak{m}_x\mathcal{F}$ on X are locally constant, as needed. □

Remark. We distinguish here between the notions of a coherent $\mathcal{D}_X$-module, which has not yet been defined and will appear in Subsection 2.4, and a $\mathcal{D}_X$-module which is coherent as an $\mathcal{O}_X$-module.

2.3.3. *The derived definitions.* Let us now return to our original goal of defining the pullback and pushforward of $\mathcal{D}$-modules. As we said before, these operations should take place in the (bounded) derived category $D(\mathrm{DMod}(X))$ of $\mathcal{D}_X$-modules. For this to make sense, we must make an additional assumption that all varieties we consider are *quasi-projective*, meaning they admit a locally closed embedding into some projective space. Then, we have the following technical result.

Proposition 2.21. *If X is quasi-projective, the category* $\mathrm{DMod}(X)$ *has enough injectives and local projectives and has finite homological dimension.*

Proof. This is a combination of [HTT08, Proposition 1.4.14] and [HTT08, Corollary 1.4.20]. □

By Proposition 2.21, we may expect the operations of derived Hom and tensor to be well-behaved. Let $D(\mathrm{DMod}(X)) := D(\mathrm{DMod}(X))$ be the bounded derived category of $\mathrm{DMod}(X)$. Then, we may define $\mathcal{RH}om_{\mathcal{D}_X}(-,-)$ and $-\overset{L}{\otimes}_{\mathcal{D}_X}-$ in the usual way. Now, let $\phi : X \to Y$ be a morphism. We define the *derived pullback* $\phi^!$ as

$$\phi^!(-) := L\phi^\Delta(-)[\dim X - \dim Y] = \mathcal{D}_{X\to Y} \overset{L}{\underset{\phi^\cdot \mathcal{D}_Y}{\otimes}} \phi^\cdot(-)[\dim X - \dim Y]$$

and the *derived pushforward* $\phi_\star$ as

$$\phi_\star(-) := R\phi_\cdot(\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\mathcal{D}_X}{\otimes}} -),$$

which will be well-defined as maps between the bounded derived categories. Let us see what these definitions mean in some special cases.

Example 2.22. If ϕ is an open embedding, then ϕ^Δ is exact, and

$$\mathcal{D}_{X\to Y} \simeq \mathcal{O}_X \underset{\phi^\cdot \mathcal{O}_Y}{\otimes} \phi^\cdot \mathcal{D}_Y \simeq \mathcal{O}_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \simeq \mathcal{D}_X$$

by quasicoherence of $\mathcal{D}_Y$. Thus, we see that $\phi^! \simeq \phi^\cdot[\dim X - \dim Y]$. Similarly, note that

$$\mathcal{D}_{Y\leftarrow X} \simeq \Omega_{\mathcal{D}_X} \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X\to Y} \underset{\phi^\cdot \mathcal{D}_Y}{\otimes} \phi^\cdot \mathcal{D}_Y^\Omega \simeq \Omega_{\mathcal{D}_X} \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_X^\Omega \simeq \mathcal{D}_X,$$

which implies that $\phi_+ \simeq \phi_\cdot \simeq \phi_*$ is the usual pushfoward and $\phi_\star \simeq R\phi_*$.

Example 2.23. If ϕ is a closed embedding, then taking locally a regular sequence in the sheaf of ideals defining X, we can consider the *Koszul resolution*

$$0 \to K^{\dim Y - \dim X} \underset{\phi^\cdot \mathcal{O}_Y}{\otimes} \phi^\cdot \mathcal{D}_Y \to \cdots \to K^0 \underset{\phi^\cdot \mathcal{O}_Y}{\otimes} \phi^\cdot \mathcal{D}_Y \to \mathcal{D}_{X\to Y} \to 0,$$

which gives a locally free resolution for $\mathcal{D}_{X\to Y}$ as a right $\phi^\cdot\mathcal{D}_Y$-module.[7] Thus, to compute $\phi^!$, we take

$$\phi^!(\mathcal{F}) \simeq \mathcal{D}_{X\to Y} \overset{L}{\underset{\phi^\cdot \mathcal{D}_Y}{\otimes}} \phi^\cdot \mathcal{F} \simeq K^\bullet \underset{\phi^\cdot \mathcal{O}_Y}{\otimes} \phi^\cdot \mathcal{D}_Y \overset{L}{\underset{\phi^\cdot \mathcal{D}_Y}{\otimes}} \phi^\cdot \mathcal{F} \simeq K^\bullet \underset{\phi^\cdot \mathcal{O}_Y}{\otimes} \phi^\cdot \mathcal{F}.$$

For $\phi_\star$, we saw that ϕ_+ was exact, and the proof of Kashiwara's Theorem showed that $\mathcal{D}_{Y\leftarrow X}$ is locally free as a right $\mathcal{D}_X$-module, hence we see that

$$\phi_\star(-) \simeq R\phi_\cdot(\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\mathcal{D}_X}{\otimes}} -) \simeq R\phi_\cdot(\mathcal{D}_{Y\leftarrow X} \overset{L}{\underset{\mathcal{D}_X}{\otimes}} -) \simeq R\phi_+ \simeq \phi_+.$$

Example 2.24. Suppose now that ϕ is the map $X \to \mathrm{pt}$. Then, we see that

$$\mathcal{D}_{X\to \mathrm{pt}} \simeq \mathcal{O}_X \text{ and } \mathcal{D}_{\mathrm{pt}\leftarrow X} \simeq \Omega_{\mathcal{D}_X} \underset{\mathcal{D}_X}{\otimes} \mathcal{O}_X \simeq \Omega_X,$$

where there is a resolution

$$0 \to \Omega_X^0 \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to \Omega_X^1 \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to \cdots \to \Omega_X^n \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to \Omega_X \to 0 \tag{4}$$

of locally free right $\mathcal{D}_X$-modules (see [HTT08, Lemma 1.5.27]), where $\Omega_X^1 := T_X^*$. The resolution (4) is a form of the *Spencer resolution*. Therefore, we see that

$$\phi_\star(-) \simeq R\phi_\cdot(\Omega_X \overset{L}{\underset{\mathcal{D}_X}{\otimes}} -) \simeq R\phi_\cdot(\Omega_X^\bullet \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \underset{\mathcal{D}_X}{\otimes} -) \simeq R\phi_\cdot(\Omega_X^\bullet \underset{\mathcal{O}_X}{\otimes} -).$$

Taking the special case of $\phi_\star(\mathcal{O}_X)$, we see that $\phi_\star(\mathcal{O}_X) \simeq R\phi_\cdot(\Omega_X^\bullet)$ and therefore that

$$R^i\Gamma(\phi_\star(\mathcal{O}_X)) \simeq R^i\phi_\cdot(\Omega_X^\bullet) \simeq \mathbf{H}^i(X, \Omega_X^\bullet),$$

[7]We note here that this particular instance is given by tensoring the ordinary Koszul resolution from commutative algebra by $\mathcal{D}_Y$.

which is the *algebraic de Rham cohomology* of X (where $\mathbf{H}^i$ denotes the hypercohomology). In particular, when X is defined over $\mathbb{C}$, then by Grothendieck's algebraic de Rham theorem [Gro66, Theorem 1], $R^i\Gamma(\phi_\star(\mathcal{O}_X))$ gives $H^i_{DR}(X, \mathbb{C})$, the de Rham cohomology of the associated complex algebraic variety.

2.4. **The singular support.** We now give a construction that allows us to consider a second notion of support for a class of $\mathcal{D}_X$-modules satisfying a certain local finiteness condition. The relevant notion arises from the observation of Proposition 2.5 that $\operatorname{gr} \mathcal{D}_X \simeq \operatorname{Sym}_{\mathcal{O}_X} T_X \simeq \mathcal{O}_{T^*_X}$, suggesting that we may wish to view a $\mathcal{D}_X$-module in terms of a related graded module over $\operatorname{Sym}_{\mathcal{O}_X} T_X$, where we note that $\operatorname{Sym}_{\mathcal{O}_X} T_X \simeq \mathcal{O}_{T^*_X}$. We may then consider its support on the cotangent bundle T^*_X of X. Let us now carry out these constructions more rigorously.

2.4.1. *The definition.* Recall the order filtration $F_0\mathcal{D}_X \subset F_1\mathcal{D}_X \subset \cdots \subset \mathcal{D}_X$ on $\mathcal{D}_X$ defined in Subsection 2.1. For a $\mathcal{D}_X$-module $\mathcal{F}$, we say that a filtration $F_i\mathcal{F}$ on $\mathcal{F}$ is compatible with $F_i\mathcal{D}_X$ if $F_i\mathcal{D}_X \cdot F_j\mathcal{F} \subset F_{i+j}\mathcal{F}$ for all i, j. For any $\mathcal{D}_X$-module $\mathcal{F}$ equipped with a filtration, we see that $\operatorname{gr}_F \mathcal{F}$ acquires the structure of a graded $\operatorname{gr} \mathcal{D}_X \simeq \mathcal{O}_{T^*_X}$-module. We would like to define a notion of support for $\operatorname{gr}_F \mathcal{F}$ which depends only on $\mathcal{F}$ and not on the choice of filtration; for this, we require the following key definition.

Definition 2.25. A filtration $F_0\mathcal{F} \subset F_1\mathcal{F} \subset \cdots$ of $\mathcal{F}$ is *good* if one of the following two equivalent conditions hold:

(i) $\operatorname{gr}_F \mathcal{F}$ is a coherent $\mathcal{O}_{T^*_X}$-module, or
(ii) $F_i\mathcal{F}$ is a coherent $\mathcal{O}_X$-module for all i and $F_1\mathcal{D}_X \cdot F_i\mathcal{F} = F_{i+1}\mathcal{F}$ for i large.

Lemma 2.26. *The two characterizations of good filtrations given in Definition 2.25 are equivalent.*

Proof. That (ii) implies (i) is obvious. To show (i) implies (ii), pick a set of generators $\{[f_i]\}$ for $\operatorname{gr}_F \mathcal{F}$ over $\mathcal{O}_{T^*_X}$, and let $f_i \in \mathcal{F}$ be a representative of each generator, where $f_i \in F_{k_i} - F_{k_i-1}$ for a unique k_i. Then, there is some N such that $f_i \in F_N\mathcal{F}$ for all i. Now, choose locally a set of generators $\{\xi_j\}$ for T_X over $\mathcal{O}_X$ and for $K \geq N$ consider the set of elements

$$S_K = \{\xi_{j_1} \cdots \xi_{j_k} \cdot f_i \mid k \leq K - k_i\}.$$

We claim that the obvious map $\mathcal{O}_X^{\oplus S_K} \to F_K\mathcal{F}$ is surjective; indeed, filtering $\mathcal{O}_X^{\oplus S_K}$ by the maximal order of a non-zero element corresponding to $s \in S_K$, the map respects filtrations and is surjective on the associated graded because $[f_i]$ generated $\operatorname{gr}_F \mathcal{F}$. This shows that $F_K\mathcal{F}$ is coherent for all $K \geq N$. Further, because $S_{K+1} \subset F_1\mathcal{D}_X \cdot S_K$, we obtain also that $F_1\mathcal{D}_X \cdot F_i\mathcal{F} = F_{i+1}\mathcal{F}$ for $i \geq N$, giving (ii). $\square$

Let now $\mathcal{F}$ be a $\mathcal{D}_X$-module equipped with a good filtration $F_i\mathcal{F}$. By definition, $\operatorname{gr}_F \mathcal{F}$ is then a coherent $\mathcal{O}_{T^*_X}$-module, so we may then define the *singular support of* $\mathcal{F}$ to be

$$\mathcal{SS}(\mathcal{F}) := \operatorname{supp}(\operatorname{gr}_F \mathcal{F}),$$

the support of $\operatorname{gr}_F \mathcal{F}$. As is implicit in the definition, we must check that this definition is independent of the choice of good filtration on $\mathcal{F}$.

Proposition 2.27. *The singular support of $\mathcal{F}$ does not depend on the choice of good filtration on $\mathcal{F}$.*

Proof. Take two good filtrations $F^1\mathcal{F}$ and $F^2\mathcal{F}$, and denote by $\mathcal{SS}^1(\mathcal{F})$ and $\mathcal{SS}^2(\mathcal{F})$ the (a priori different) singular supports associated to them. We now use an elegant trick which we believe is due to Bernstein. Say that the two filtrations are *neighbors* if

$$F^1_i \subset F^2_i \subset F^1_{i+1} \subset F^2_{i+1} \tag{5}$$

for all i. We first claim that if F^1 and F^2 are neighbors, then $\mathcal{SS}^1(\mathcal{F}) = \mathcal{SS}^2(\mathcal{F})$. Indeed, condition (5) means that the identity map $\mathcal{F} \to \mathcal{F}$ is compatible with the two filtrations, hence it induces a map $\operatorname{gr}^1 \mathcal{F} \to \operatorname{gr}^2 \mathcal{F}$. Let us place this map in an exact sequence

$$0 \to K \to \operatorname{gr}^1 \mathcal{F} \to \operatorname{gr}^2 \mathcal{F} \to C \to 0,$$

where we notice that $K = \bigoplus_i F^2_i\mathcal{F}/F^1_i\mathcal{F}$ and $C = \bigoplus_i F^2_{i+1}/F^2_{i+1}\mathcal{F}$. In particular, C and K are isomorphic up to a shift in grading. Therefore, if $x \notin \operatorname{supp}(\operatorname{gr}^1 \mathcal{F})$, then $K_x = 0$ and therefore $C_x = 0$, meaning that $(\operatorname{gr}^2 \mathcal{F})_x \simeq (\operatorname{gr}^2 \mathcal{F})_x = 0$. Similarly, if $x \notin \operatorname{supp}(\operatorname{gr}^2 \mathcal{F})$ then, we see also that $x \notin \operatorname{supp}(\operatorname{gr}^1 \mathcal{F})$, which implies that $\mathcal{SS}^1(\mathcal{F}) = \operatorname{supp}(\operatorname{gr}^1 \mathcal{F}) = \operatorname{supp}(\operatorname{gr}^2 \mathcal{F}) = \mathcal{SS}^2(\mathcal{F})$, as desired.

It remains now to check that any two good filtrations F^1 and F^2 on $\mathcal{F}$ are linked up to a shift by a chain of neighboring filtrations (as the singular support is obviously invariant under shifts of the filtration). Indeed, define the filtrations

$$G^k_i\mathcal{F} := F^1_i\mathcal{F} + F^2_{i+k}\mathcal{F}$$

for all $k \in \mathbb{Z}$. It is clear that G^k and G^{k+1} are good and are neighbors for all k. Now, choose K so that $F_1\mathcal{D}_X \cdot F_i^1\mathcal{F} = F_{i+1}^1\mathcal{F}$ and $F_1\mathcal{D}_X \cdot F_i^2\mathcal{F} = F_{i+1}^2\mathcal{F}$ for $i \geq K$. Take N so that $F_K^2\mathcal{F} \subset F_N^1\mathcal{F}$ and $F_K^2\mathcal{F} \subset F_N^2\mathcal{F}$. Then, we claim that $G^{-N} = F^1$ and that G^N is a shift of F^1. For the first claim, our choice of K and N implies that $F_{i-N}^2\mathcal{F} \subset F_{i-N+K}^2\mathcal{F} \subset F_i^1\mathcal{F}$, we see that

$$G_i^{-N}\mathcal{F} = F_i^1\mathcal{F} + F_{i-N}^2\mathcal{F} = F_i^1\mathcal{F}.$$

For the second claim, the choice of K and N gives $F_i^1\mathcal{F} \subset F_{i+K}^1\mathcal{F} \subset F_{i+N}^2\mathcal{F}$, hence

$$G_i^N\mathcal{F} = F_i^1\mathcal{F} + F_{i+N}^2\mathcal{F} = F_{i+N}^2\mathcal{F}.$$

Thus, we see that F^1 and a shifted version of F^2 are linked by neighboring good filtrations, completing the proof. □

By Proposition 2.27, when a good filtration exists, we have a well-defined notion of singular support. We would like now to characterize when this happens. We say that a $\mathcal{D}_X$-module is *coherent* if it is locally finitely generated (over $\mathcal{D}_X$). It will turn out that these are precisely the $\mathcal{D}_X$-modules for which a good filtration exists. We give some basic properties of coherent $\mathcal{D}_X$-modules below.

Lemma 2.28. *Any coherent $\mathcal{D}_X$-module $\mathcal{F}$ on X is generated by a submodule which is coherent as an $\mathcal{O}_X$-module.*

Proof. Recall that X was assumed to be quasi-projective, hence quasi-compact, so we may cover X by a finite collection of affines $X = \bigcup_i U_i$ such that $\mathcal{F}|_{U_i}$ is finitely generated as a $\mathcal{D}_{U_i}$-module over U_i. On each U_i, pick a set of generators for $\mathcal{F}$ over $\mathcal{D}_{U_i}$ and let $\mathcal{G}_i$ be the $\mathcal{O}_{U_i}$-submodule generated by them. Then there exists a $\mathcal{O}_X$-submodule $\mathcal{F}_i$ of $\mathcal{F}$ such that $\mathcal{F}_i|_{U_i} = \mathcal{G}_i$. Set $\mathcal{F}' = \sum_i \mathcal{F}_i \subset \mathcal{F}$; notice that $\mathcal{F}'$ is coherent as the finite sum of coherent submodules of $\mathcal{F}$. It is clear that $\mathcal{F}'$ generates $\mathcal{F}$ as a $\mathcal{D}_X$-module, as needed. □

Lemma 2.29. *Let $U \subset X$ be an open set and $\mathcal{F}$ a $\mathcal{D}_X$-module. Then, if $\mathcal{F}|_U$ is coherent as a $\mathcal{D}_U$-module, there exists a coherent submodule $\mathcal{F}' \subset \mathcal{F}$ such that $\mathcal{F}'|_U = \mathcal{F}|_U$.*

Proof. Take a $\mathcal{O}_U$-coherent submodule $\mathcal{G} \subset \mathcal{F}|_U$ that generates it as a $\mathcal{D}_U$-module. Then, we may find a $\mathcal{O}_X$-coherent submodule $\mathcal{G}'$ of $\mathcal{F}$ such that $\mathcal{G}'|_U \simeq \mathcal{G}$. Let $\mathcal{F}'$ be the $\mathcal{D}_X$-submodule of $\mathcal{F}$ generated by $\mathcal{G}'$; it is clearly coherent and satisfies $\mathcal{F}'|_U \simeq \mathcal{F}|_U$, as needed. □

Using these, we can now show that good filtrations exist exactly for coherent $\mathcal{D}_X$-modules.

Proposition 2.30. *A $\mathcal{D}_X$-module $\mathcal{F}$ admits a good filtration if and only if it is coherent.*

Proof. If $\mathcal{F}$ admits a good filtration $F_i\mathcal{F}$, then picking K so that $F_1\mathcal{D}_X \cdot F_i\mathcal{F} = F_{i+1}\mathcal{F}$ for $i \geq K$, a set of $\mathcal{O}_X$-generators for $F_K\mathcal{F}$ will be a set of $\mathcal{D}_X$-generators for $\mathcal{F}$, showing that $\mathcal{F}$ is coherent. Conversely, if $\mathcal{F}$ is coherent, take by Lemma 2.28 an $\mathcal{O}_X$-coherent submodule $\mathcal{F}' \subset \mathcal{F}$ that generates it and consider the filtration

$$F_i\mathcal{F} := F_i\mathcal{D}_X \cdot \mathcal{F}'.$$

Because $F_1\mathcal{D}_X \cdot F_i\mathcal{D}_X = F_{i+1}\mathcal{D}_X$ for all i by Proposition 2.5, we see that $F_1\mathcal{D}_X \cdot F_i\mathcal{F} = F_{i+1}\mathcal{F}$ for all i.[8] Now, because $F_0\mathcal{F} = \mathcal{F}'$ is coherent, it follows that $F_i\mathcal{F} = (F_1\mathcal{D}_X)^i \cdot F_0\mathcal{F}$ is coherent for each i, showing that $F_i\mathcal{F}$ is a good filtration. □

2.4.2. *Bounding the singular support.* Having now defined the singular support for coherent $\mathcal{D}$-modules, let us characterize its behavior. In general, it is somewhat difficult to compute, but the following special case is simple and important.

Example 2.31. Suppose that $\mathcal{F}$ is $\mathcal{O}_X$-coherent. Then the trivial filtration on $\mathcal{F}$ with $F_i\mathcal{F} = \mathcal{F}$ is good, meaning that $\mathrm{gr}_F\,\mathcal{F} \simeq \mathcal{F}$, where the action of T_X is trivial. This implies that $\mathcal{SS}(\mathcal{F})$ is contained in the image of the zero section $X \hookrightarrow T_X^*$.

As we see from Example 2.31, $\mathcal{D}_X$-modules which are coherent as $\mathcal{O}_X$-modules have singular support contained in X. According to the following result of Bernstein, this is "smallest possible" in the following sense.

Theorem 2.32 (Bernstein's Inequality)**.** *For any non-zero coherent $\mathcal{D}_X$-module $\mathcal{F}$, we have $\mathcal{SS}(\mathcal{F}) \geq \dim(X)$.*

[8]In fact, Proposition 2.5 shows that $F_i\mathcal{D}_X$ is a good filtration.

We defer the proof of Theorem 2.32 for a moment to present and motivate a key result used in it. The basic idea of the proof will be to reduce by induction to the case where $\dim \operatorname{supp}(\mathcal{F}) = \dim(X)$ by restricting the base of $\mathcal{F}$ to (an open subset) of its support using Kashiwara's Theorem. To accomplish this, we need to understand how the singular support transforms under pullback and pushforward. Unfortunately, this is meaningless in general, as the (derived) pushforward and pullback do not always preserve coherence. However, it holds in the following special cases, which will be enough for us to prove Bernstein's inequality.

Proposition 2.33. *Let $\phi : X \to Y$ be a morphism of varieties. Then, we have the following:*

(i) *if ϕ is a closed embedding, then a $\mathcal{D}_X$-module $\mathcal{F}$ is coherent if and only if $\phi_+(\mathcal{F})$ is, and in this case we have*
$$\dim \mathcal{SS}(\phi_+(\mathcal{F})) - \dim(Y) = \dim \mathcal{SS}(\mathcal{F}) - \dim(X);$$

(ii) *if ϕ is an open embedding, then for a coherent $\mathcal{D}_Y$-module $\mathcal{F}$, $\phi^\Delta(\mathcal{F})$ is coherent, and in this case we have*
$$\dim \mathcal{SS}(\phi^\Delta(\mathcal{F})) = \dim \mathcal{SS}(\mathcal{F}).$$

Proof. For (i), as in the proof of Kashiwara's Theorem, we may reduce to the situation where Y is affine with a étale coordinate system $f_1, \ldots, f_n$ with corresponding dual vector fields $\xi_1, \ldots, \xi_n$ and X is defined by the vanishing of the last coordinate $f = f_n$. As before, set $\xi = \xi_n$. We computed previously that in this case $\phi_+(\mathcal{F}) \simeq [\xi]\mathcal{F}$, thus any good filtration on $\phi_+(\mathcal{F})$ restricts to a good filtration on $\mathcal{F}$ and for a good filtration $F_i\mathcal{F}$, we may construct the good filtration
$$F_n \phi_+(\mathcal{F}) := \sum_{i+j=n} \xi^i \cdot F_j \mathcal{F}$$
on $\phi_+(\mathcal{F})$ such that
$$\operatorname{gr}^n \phi_+(\mathcal{F}) = \bigoplus_{i+j=n} \xi^i \cdot \operatorname{gr}^j \mathcal{F}.$$
This shows that $\phi_+(\mathcal{F})$ is coherent if and only if $\mathcal{F}$ is. To show that ϕ_+ preserves $\dim \mathcal{SS}(\mathcal{F}) - \dim(X)$, note that f kills $\operatorname{gr} \phi_+(\mathcal{F})$ because $f \cdot \xi^i = \xi^{i-1} f$ and f kills $\operatorname{gr} \mathcal{F}$. Because the action of all other generators of $\operatorname{Sym}_{\mathcal{O}_Y} T_Y$ commute with ξ, we find that
$$\operatorname{Ann}_Y(\operatorname{gr} \phi_+(\mathcal{F}))/(f) \simeq \operatorname{Ann}_X(\mathcal{F}),$$
where f is regular in $\operatorname{Ann}_Y(\operatorname{gr} \phi_+(\mathcal{F}))$. This shows as needed that
$$\begin{aligned} \dim \mathcal{SS}(\phi_+ \mathcal{F}) - \dim(Y) &= \dim(Y) - \dim \operatorname{Ann}_Y(\operatorname{gr} \phi_+(\mathcal{F})) \\ &= \dim(X) + 1 - \dim \operatorname{Ann}_X(\mathcal{F}) - 1 \\ &= \dim \mathcal{SS}(\mathcal{F}) - \dim(X). \end{aligned}$$

For (ii), because ϕ^Δ coincides with the $\mathcal{O}$-module pullback, it preserves coherence, as the pullback of a good filtration will be a good filtration. Further, ϕ^Δ induces an open embedding $\phi_\flat : T^*_X \to T^*_Y$, hence by exactness of ϕ^Δ we have an isomorphism $\operatorname{gr}^i \phi^\Delta \mathcal{F} \simeq \phi_\flat^* \operatorname{gr}^i \mathcal{F}$. This shows that $\mathcal{SS}(\phi^\Delta \mathcal{F}) \simeq \mathcal{SS}(\mathcal{F}) \cap \phi_\flat(T^*_X)$, giving the claim.[9] □

With Proposition 2.33 in hand, we are now ready to give the proof of Theorem 2.32.

Proof of Theorem 2.32. Given Kashiwara's Theorem and Proposition 2.33, the proof is relatively simple. Proceed by induction on $\dim(X)$; if $\dim(X) = 0$, the statement is vacuous. Now, for $\dim(X) > 0$, suppose that the claim holds in smaller dimensions, and restrict to an open subset of X so that $\operatorname{supp}(\mathcal{F})$ consists of a single irreducible component. By Proposition 2.33(ii), this preserves the hypotheses and desired conclusions of the claim.

Now, if $\operatorname{supp}(\mathcal{F}) = X$, then we see that $X \subset \mathcal{SS}(\mathcal{F})$ (under the embedding of $X \hookrightarrow T^*_X$ via the zero section), meaning that $\dim \mathcal{SS}(\mathcal{F}) \geq \dim(X)$. Otherwise, $Z = \operatorname{supp}(\mathcal{F})$ is a proper closed subset of X of codimension at least 1. Pick some open set U of X so that $U \cap Z$ is non-empty and smooth, and write $j : U \hookrightarrow X$ and $i : U \cap U \cap Z \hookrightarrow U$ for the inclusions. Then, $j^\Delta \mathcal{F}$ is supported on $U \cap Z$, so by Kashiwara's Theorem we may find some $\mathcal{D}_{U \cap Z}$-module $\mathcal{G}$ such that $i_+ \mathcal{G} = j^\Delta \mathcal{F}$. By Proposition 2.33, we see that $j^\Delta \mathcal{F}$ and $\mathcal{G}$ are both coherent and thus that
$$\dim \mathcal{SS}(\mathcal{F}) - \dim(X) = \dim \mathcal{SS}(j^\Delta \mathcal{F}) - \dim(U) = \dim \mathcal{SS}(\mathcal{G}) - \dim(U \cap Z) \geq 0$$

[9]We note that (ii) may be generalized much further (with appropriate modifications to the conclusion); in particular, we may consider any smooth morphism ϕ. We will only use the simpler case where ϕ is an open embedding, however, so we restrict to it here.

by the induction hypothesis. □

By Bernstein's inequality, we see that $\mathcal{D}_X$-modules which are coherent as $\mathcal{O}_X$-modules must have singular support $\mathcal{SS}(\mathcal{F}) \subset X$ of dimension exactly X, meaning that their singular supports consist of some of the irreducible components of X. Decomposing the singular support into irreducible components in this manner is a fruitful general technique, as we demonstrate in the following proposition.

Proposition 2.34. *Let $\mathcal{G} \subset \mathcal{F}$ be coherent $\mathcal{D}_X$-modules. Then, we have*

$$\mathcal{SS}(\mathcal{F}) = \mathcal{SS}(\mathcal{G}) \cup \mathcal{SS}(\mathcal{F}/\mathcal{G}),$$

and further this decomposition respects the multiplicities of the irreducible components on both sides.

Proof. A good filtration on $\mathcal{F}$ induces good filtrations on $\mathcal{G}$ and $\mathcal{G}/\mathcal{F}$ by restriction and projection such that the short exact sequence

$$0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{G}/\mathcal{F} \to 0$$

is compatible with filtrations. Passing to the associated graded, we obtain a short exact sequence

$$0 \to \operatorname{gr} \mathcal{G} \to \operatorname{gr} \mathcal{F} \to \operatorname{gr} \mathcal{G}/\mathcal{F} \to 0,$$

from which it follows that $\mathcal{SS}(\mathcal{F}) = \mathcal{SS}(\mathcal{G}) \cup \mathcal{SS}(\mathcal{F}/\mathcal{G})$.

Let us now see a refinement of this statement. Observe that this decomposition respects the irreducible components of $\mathcal{SS}(\mathcal{F})$. That is, for every irreducible component I_i of $\mathcal{SS}(\mathcal{F})$ with generic point $\mathfrak{p}_i$, we obtain a short exact sequence

$$0 \to (\operatorname{gr} \mathcal{G})_{\mathfrak{p}_i} \to (\operatorname{gr} \mathcal{F})_{\mathfrak{p}_i} \to (\operatorname{gr} \mathcal{G}/\mathcal{F})_{\mathfrak{p}_i} \to 0$$

of Artinian modules of finite length, where the length of $(\operatorname{gr} \mathcal{F})_{\mathfrak{p}_i}$ is by definition the *multiplicity* $m_i(\mathcal{F})$ of I_i in $\operatorname{gr} \mathcal{F}$. Then this short exact sequence implies that

$$m_i(\mathcal{F}) = m_i(\mathcal{G}) + m_i(\mathcal{G}/\mathcal{F})$$

for each irreducible component I_i of $\mathcal{SS}(\mathcal{F})$.[10] □

Corollary 2.35. *Any holonomic $\mathcal{D}$-module $\mathcal{F}$ has finite length.*

Proof. This is immediate from the result about irreducible components in Proposition 2.34, as the length of $\mathcal{F}$ is bounded by the sum of the multiplicities of the irreducible components of $\mathcal{SS}(\mathcal{F})$. □

2.5. **Holonomic $\mathcal{D}$-modules and duality.** In view of Bernstein's Inequality from the previous section, it is natural to make the following definition of $\mathcal{D}_X$-modules which are of minimal size in some sense.

Definition 2.36. A coherent $\mathcal{D}_X$-module $\mathcal{F}$ is *holonomic* if and only if $\dim \mathcal{SS}(\mathcal{F}) = \dim(X)$.

We shall see that the class of holonomic $\mathcal{D}$-modules is particularly well-behaved. The following proposition shows that they are very close to $\mathcal{O}$-coherent $\mathcal{D}$-modules.

Proposition 2.37. *Let $\mathcal{F}$ be a holonomic $\mathcal{D}_X$ module. Then, there exists an open dense subset $U \subset X$ such that $\mathcal{F}|_U$ is $\mathcal{O}_U$-coherent.*

Proof. Set $\mathcal{M} := \operatorname{gr} \mathcal{F}$, and let $\mathcal{M}^0 \subset \mathcal{M}$ be the submodule of $\mathcal{M}$ supported away from $X \subset T^*_X$, where we interpret X as the zero section; that is, $\mathcal{M}^0$ is the submodule annihilated by the ideal of elements of positive grade. Now, notice that $T^*_X - X$ carries a $\mathbb{G}_m$-action by multiplication on fibers and further that the quotient Y of $T^*_X - X$ under this action exists. Observe now that $\mathcal{M}^0$ is a graded module over $\operatorname{Sym}_{\mathcal{O}_X} T_X$ supported on $T^*_X - X$; thus, it defines a sheaf on Y, and its support $\operatorname{supp}(\mathcal{M}^0)$ is invariant under the $\mathbb{G}_m$-action on fibers. This means that the fibers of $\operatorname{supp}(\mathcal{M}^0)$ have dimension at least 1 under the projection $\pi : \operatorname{supp}(\mathcal{M}^0) - X \to X$, implying that

$$\dim(X) \geq \dim \operatorname{supp}(\mathcal{M}) \geq \dim \operatorname{supp}(\mathcal{M}^0) > \dim \pi(\operatorname{supp}(\mathcal{M}^0)).$$

Then, letting $U = X - \pi(\operatorname{supp}(\mathcal{M}^0))$, we see that U is open dense and that $\operatorname{supp}(\mathcal{M}^0)$ is disjoint from $\pi^{-1}(U)$, thus $\operatorname{gr} \mathcal{F}|_U$ is supported on $T^*_U \cap (U \cup \pi^{-1}(U)) \subset U$. But this means that the good filtration on $\mathcal{F}|_U$ is eventually constant, hence $\mathcal{F}|_U$ is $\mathcal{O}_U$-coherent. □

[10] Technically speaking, we should have showed that $m_i(\mathcal{F})$ was well-defined independent of the choice of good filtration on $\mathcal{F}$. To show this, the argument we used to prove Proposition 2.27 works verbatim with every instance of $x \notin \operatorname{supp}(\operatorname{gr} \mathcal{F})$ replaced by $(\operatorname{gr} \mathcal{F})_{\mathfrak{p}_i} = 0$.

Let us now see that holonomic $\mathcal{D}$-modules behave well under the various operations on $\mathcal{D}$-modules we have defined so far. By Proposition 2.34, it is clear that quotients and extensions of holonomic $\mathcal{D}$-modules remain holonomic. For the rest of our compatibility properties, we must pass to the derived category of holonomic $\mathcal{D}$-modules, which we define on a variety X to be the subcategory $D(\mathrm{DMod}(X)^{\mathrm{hol}})$ of $D(\mathrm{DMod}(X))$ consisting of complexes whose cohomologies are holonomic. Similarly, we may define the derive category of coherent $\mathcal{D}$-modules, which we denote by $D(\mathrm{DMod}(X)^c)$ The main compatibility result on holonomic $\mathcal{D}$-modules is that this category is well behaved under the derived pushforward and pullback. Presenting a complete proof of Theorem 2.38 is beyond the ken of this essay, but we will give some indication of the ingredients.

Theorem 2.38 ([HTT08, Theorem 3.2.3])**.** *Let $\phi : X \to Y$ be a morphism of varieties. Then, $\phi^!$ maps $D(\mathrm{DMod}(Y)^{hol})$ to $D(\mathrm{DMod}(X)^{hol})$ and $\phi_\star$ maps $D(\mathrm{DMod}(X)^{hol})$ to $D(\mathrm{DMod}(Y)^{hol})$.*

2.5.1. *The duality functor.* We must first take a detour toward a duality functor which is defined for all coherent $\mathcal{D}$-modules, which is itself of independent interest. Define the *duality functor* $\mathbb{D}_X : D(\mathrm{DMod}(X)^c)^r \to D(\mathrm{DMod}(X)^c)$ by

$$\mathbb{D}_X(\mathcal{F}^\bullet) := \mathcal{RH}om_{\mathcal{D}_X}(\mathcal{F}^\bullet, \mathcal{D}_X^\Omega)[\dim X].$$

To compute $\mathbb{D}_X$, we may take a locally projective resolution $\mathcal{P}^\bullet \to \mathcal{F}^\bullet$, for which we have

$$\mathbb{D}_X(\mathcal{F}^\bullet) = \mathbb{D}_X(\mathcal{P}^\bullet) = \mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}^\bullet, \mathcal{D}_X^\Omega)[\dim X].$$

Therefore, we see that

$$\begin{aligned}\mathbb{D}_X^2(\mathcal{F}^\bullet) = \mathbb{D}_X^2(\mathcal{P}^\bullet) &\simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}^\bullet, \mathcal{D}_X^\Omega)[\dim X], \mathcal{D}_X^\Omega)[\dim X]\\ &\simeq \mathcal{H}om_{\mathcal{D}_X}(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}^\bullet, \mathcal{D}_X^\Omega), \mathcal{D}_X^\Omega),\end{aligned}$$

is locally isomorphic to $\mathcal{P}^\bullet$ under the map $\mathcal{P}^\bullet \to \mathcal{H}om_{\mathcal{D}_X}(\mathcal{H}om_{\mathcal{D}_X}(\mathcal{P}^\bullet, \mathcal{D}_X^\Omega), \mathcal{D}_X^\Omega)$ given by evaluation (because $\mathcal{P}^\bullet$ was locally projective and $\mathcal{D}_X^\Omega$ locally isomorphic to $\mathcal{D}_X$). The duality functor will turn out to have particularly nice behavior on holonomic $\mathcal{D}$-modules, as we summarize below.

Theorem 2.39. *If $\mathcal{F}$ is a coherent $\mathcal{D}_X$-module, then*

(i) *$\mathcal{F}$ is holonomic if and only if $\mathbb{D}_X(\mathcal{F})$ lies in grade 0, and*
(ii) *$\mathbb{D}_X$ provides a duality of categories between $\mathrm{DMod}(X)^{hol}$ and $\mathrm{DMod}(X)^{hol,r}$.*

The proof of Theorem 2.39 relies crucially on the homological result attributed to J. E. Roos, which is proved by passing between resolutions of $\mathcal{F}$ and $\mathrm{gr}_F \mathcal{F}$ for good filtrations on $\mathcal{F}$.

Theorem 2.40 ([HTT08, Theorem 2.6.7])**.** *Let $\mathcal{F}$ be a coherent $\mathcal{D}_X$-module. Then, we have*

(i) *$\operatorname{codim} \mathcal{SS}(\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{F}, \mathcal{D}_X^\Omega)) \geq i$, and*
(ii) *$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{F}, \mathcal{D}_X^\Omega) = 0$ for $i < \operatorname{codim} \mathcal{SS}(\mathcal{F})$,*

where we recall that $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{F}, \mathcal{G})$ is the cohomology in grade i of $\mathcal{RH}om_{\mathcal{D}_X}(\mathcal{F}, \mathcal{G})$.

We omit the proof but now show how it implies Theorem 2.39. In fact, Theorem 2.40 has quite powerful consequences for duality in the derived category of all coherent $\mathcal{D}_X$-modules, but we wish to focus on the holonomic case for now.

Proof of Theorem 2.39. This will be simple given the characterization given by Theorem 2.40. To wit, for (i), first suppose $\mathcal{F}$ is holonomic. Then, by Theorem 2.40(ii), we see that $H^i(\mathbb{D}_X(\mathcal{F})) = 0$ for $i < 0$ (note the shift in grade due to to shift in the definition of $\mathbb{D}_X$). Further, by Theorem 2.40(i), we have that

$$\dim \mathcal{SS}(\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{F}, \mathcal{D}_X^\Omega)) - \dim(X) \leq \dim(X) - i$$

for all i, which shows by Bernstein's inequality that

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{F}, \mathcal{D}_X^\Omega) = H^{i-\dim X}(\mathcal{D}(X)) = 0 \text{ for } i > \dim(X),$$

hence $H^i(\mathbb{D}_X(\mathcal{F})) = 0$ for $i > 0$ and $\mathbb{D}_X(\mathcal{F})$ lies only in grade 0.

On the other hand, if $\mathbb{D}_X(\mathcal{F})$ lies only in grade 0, then by duality we see that

$$\mathcal{F} \simeq \mathbb{D}_X(\mathbb{D}_X(\mathcal{F})) \simeq \mathbb{D}_X(\mathbb{D}_X(\mathcal{F})^0),$$

so applying Theorem 2.40 to $\mathbb{D}_X(\mathcal{F})$, we see that

$$2\dim(X) - \dim \mathcal{SS}(\mathcal{E}xt^{\dim(X)}_{\mathcal{D}_X}(\mathbb{D}_X(\mathcal{F}), \mathcal{D}_X^\Omega)) \geq \dim(X)$$

and thus that $\dim(X) \geq \dim \mathcal{SS}(\mathcal{F})$, meaning that $\mathcal{F}$ is holonomic. This finishes the proof of (i). Now (ii) is more or less a restatement of (i) combined with the fact that $\mathbb{D}_X^2 = \mathrm{id}$. ☐

2.5.2. *Pushforwards and pullbacks of holonomic $\mathcal{D}$-modules.* The second key ingredient to the proof of Theorem 2.38 is the following special case. We will omit the other components of the proof to go into more detail about this one.

Proposition 2.41. *Let $\phi : X \to Y$ be an open embedding with Y affine and $X := Y_f$ the locus of non-vanishing of a single equation. Then, the derived pushforward $\phi_\star$ preserves $D(\mathrm{DMod}(X)^{hol})$.*

The proof of Proposition 2.41 will rely upon the following two key lemmas. The first of these is used to prove the second, which is known as the b-function lemma and is of independent interest.

Lemma 2.42. *Let $U \subset X$ be an open set and $\mathcal{F}$ a $\mathcal{D}_X$-module. Then, if $\mathcal{F}|_U$ is a holonomic $\mathcal{D}_U$-module, there exists a holonomic submodule $\mathcal{F}' \subset \mathcal{F}$ such that $\mathcal{F}'|_U = \mathcal{F}|_U$.*

Proof. By Lemma 2.29, we may assume that $\mathcal{F}$ is coherent. Now, write $\mathcal{G} = \mathcal{F}|_U$ and $\mathcal{M} = H^0(\mathbb{D}(\mathcal{F}))$. Applying Theorem 2.40(i), we see that $\operatorname{codim} \mathcal{SS}(\mathcal{M}) \geq \dim(X)$, hence $\mathcal{M}$ is holonomic. Now, set $\mathcal{F}' = \mathbb{D}_X(\mathcal{M})$. We claim that $\mathcal{F}'$ is the desired submodule.

First, it is holonomic by Theorem 2.40. Now, there is a natural map $\mathbb{D}_X(\mathcal{F}) \to H^0(\mathbb{D}_X(\mathcal{F})) = \mathcal{M}$, thus a natural map $\mathcal{F}' = \mathbb{D}_X(\mathcal{M}) \to \mathbb{D}_X^2(\mathcal{F}) \simeq \mathcal{F}$. Now, because $\phi^\Delta \circ \mathbb{D}_X \simeq \mathbb{D}_U \circ \phi^\Delta$ for the exact functor ϕ^Δ, we see that

$$\mathcal{F}'|_U \simeq \phi^\Delta \mathcal{F}' \simeq \mathbb{D}_U(\phi^\Delta H^0(\mathbb{D}_X(\mathcal{F})))) \simeq \mathbb{D}_U(H^0(\phi^\Delta \mathbb{D}_X(\mathcal{F}))) \simeq \mathbb{D}_U(H^0(\mathcal{D}_U(\mathcal{G}))) \simeq \mathcal{G}.$$

Thus, we see that applying ϕ^Δ to the natural map $\mathcal{F}' = \mathbb{D}_X(\mathcal{M}) \to \mathbb{D}_X^2(\mathcal{F}) \simeq \mathcal{F}$ yields the desired isomorphism $\mathcal{F}'|_U \simeq \mathcal{F}|_U$. □

Lemma 2.43. *Let X be a smooth affine variety. For $f \in \mathcal{O}_X$, let $Y := X_f$ be the locus of non-vanishing of f with $i : Y \hookrightarrow X$ the inclusion. Then, for any holonomic $\mathcal{D}_Y$-module $\mathcal{F}$ and any section $u \in i_+(\mathcal{F})$, there exists $d(n) \in \mathcal{D}_X[n]$ and $b(n) \in k[n]$ such that*

$$d(n)(f^{n+1}u) = b(n)f^n u.$$

Proof. For an indeterminate λ, let $X_\lambda := X \underset{k}{\times} k(\lambda)$ be the extension of scalars of X to the field $k(\lambda)$ and define Y_λ analogously. Then, consider the $\mathcal{D}_{Y_\lambda}$-module $\mathcal{F}_\lambda$ which is isomorphic as a $\mathcal{O}_{Y_\lambda}$-module to

$$f^\lambda \cdot k(\lambda) \underset{k}{\otimes} \mathcal{F},$$

with the action of $\mathcal{D}_{Y_\lambda}$ given by

$$\xi \cdot (f^\lambda \cdot m) = f^\lambda \cdot \frac{\lambda \xi(f)}{f} \cdot m + f^\lambda \xi(m).$$

It is clear that $\mathcal{F}_\lambda$ is coherent because $\mathcal{F}$ is; further, a good filtration $F_i\mathcal{F}$ on $\mathcal{F}$ induces a good filtration $F_i\mathcal{F}_\lambda := (F_i\mathcal{F})_\lambda$ which is evidently also good. Therefore, $\mathcal{SS}(\mathcal{F}_\lambda)$ has the same dimension over $k(\lambda)$ as $\mathcal{SS}(\mathcal{F})$ has over k, meaning that $\mathcal{F}_\lambda$ is holonomic.[11]

Now, let $i_\lambda : Y_\lambda \hookrightarrow X_\lambda$ denote the corresponding inclusion and notice that $i_\lambda^\Delta(i_\lambda)_+(\mathcal{F}_\lambda) \simeq \mathcal{F}_\lambda$ is holonomic, hence by Lemma 2.42 we may find some holonomic $\mathcal{D}_{X_\lambda}$-submodule $\mathcal{G}_\lambda \subset (i_\lambda)_+\mathcal{F}_\lambda$ such that $i_\lambda^\Delta(\mathcal{G}_\lambda) \simeq \mathcal{F}_\lambda$. Then, we see that $(i_\lambda)_+\mathcal{F}_\lambda/\mathcal{G}_\lambda$ is supported on $X - Y$, hence any element of $(i_\lambda)_+\mathcal{F}_\lambda/\mathcal{G}_\lambda$ is annihilated by a large enough power of f. In particular, this is true for the image $f^\lambda \cdot u$ of u in $(i_\lambda)_+\mathcal{F}_\lambda$, so there exists some K such that

$$f^\lambda \cdot f^K u \in \mathcal{G}_\lambda.$$

Now, recall that $\mathcal{G}_\lambda$ is holonomic and hence has finite length by Corollary 2.35, meaning that the decreasing chain of submodules

$$\mathcal{D}_{Y_\lambda} \cdot f^\lambda \cdot f^K u \supset \mathcal{D}_{Y_\lambda} \cdot f^\lambda \cdot f^{K+1} u \supset \mathcal{D}_{Y_\lambda} \cdot f^\lambda \cdot f^{K+2} u \cdots$$

of $\mathcal{G}_\lambda$ must stabilize. In particular, this means that we may find some N so that $f^\lambda \cdot f^N \in \mathcal{D}_{Y_\lambda} \cdot f^\lambda \cdot f^{N+1} u$, so there is some $d_\lambda \in \mathcal{D}_{Y_\lambda}$ such that $d_\lambda(f^\lambda \cdot f^{N+1}u) = f^\lambda \cdot f^N u$. Then, writing $d_\lambda = \frac{p(\lambda)}{q(\lambda)}$ for $p(\lambda) \in \mathcal{D}_Y[\lambda]$ and $q(\lambda) \in k[\lambda]$, we see that

$$p(\lambda) f^\lambda \cdot f^{N+1} u = q(\lambda) f^\lambda \cdot f^N u.$$

Then, specializing to $\lambda = n - N$ and taking $d(n) = p(n - N)$ and $b(n) = q(n - N)$, we find that

$$d(n) f^{n+1} u = b(n) f^n u$$

in $i_+(\mathcal{F})$, as desired. □

[11] We note here that nowhere in our discussion of holonomic $\mathcal{D}$-modules did we assume that k was algebraically closed, so discussing the property of being holonomic over the field $k(\lambda)$ is valid.

The polynomial $b(n)$ of minimal degree for which the conclusion of Lemma 2.43 holds is called the b-function of f. We give some concrete examples.

Example 2.44. Take $X = \mathrm{Spec}(k[x])$ and $f = x^2 + 1$. Then, we may compute

$$\frac{1}{4}[(x^2+1)\partial_x^2 - (2n+1)x\partial_x](x^2+1)^{n+1} = n(n+1)(x^2+1)^n,$$

so $b(n) = n(n+1)$ is the b-function (subject to verifying that no linear b-function exists).

Example 2.45. Take $X = \mathrm{Spec}(k[x,y])$ and $f = x^2 + y^2 + 1$. Then, we may compute

$$\frac{1}{2}[(x^2+1)\partial_x^2 + x^2\partial_y^2 - 2(n+1)x\partial_x](x^2+y^2+1)^{n+1} = (n+1)(x^2+y^2+1)^n,$$

so $b(n) = n+1$ is the b-function. The evident asymmetry between x and y here illustrates that there can be multiple differential operators $d(n)$ giving a single b-function.

Let us now sketch how Lemma 2.43 implies Proposition 2.41.

Proof of Proposition 2.41. In this situation, we recall that $\phi_\star = \phi_+$ because ϕ is an affine map. We can reduce to the case where there is a single holonomic $\mathcal{D}_X$-module $\mathcal{F}$, and by taking an increasing holonomic filtration of $\mathcal{F}$, where $\phi_+(\mathcal{F})$ is generated by a single section u.

The first step is now to show that $\phi_+(\mathcal{F})$ is coherent. For this, by Lemma 2.43, we see that the sequence

$$\mathcal{D}_Y \cdot u \subset \mathcal{D}_Y \cdot f^{-1}u \subset \mathcal{D}_Y \cdot f^{-2}u \cdots$$

eventually stabilizes, hence there is some element $f^{-N}u$ which generates $\phi_+(\mathcal{F})$.

Second, to show that $\phi_+(\mathcal{F})$ is holonomic, we note that we are in exactly the situation of the proof of Lemma 2.43. Then, take the construction of $\mathcal{F}_\lambda$ from the proof. Recall that $(\phi_\lambda)_+\mathcal{F}_\lambda$ was generated by u, which by the conclusion of Lemma 2.43 lies in the $\mathcal{D}_{Y_\lambda}$-span of $f^\lambda \cdot f^K u$, which is in its holonomic submodule $\mathcal{G}_\lambda$. So $(\phi_\lambda)_+\mathcal{F}_\lambda$ is itself holonomic.

We would like to pass from this to the fact that $\phi_+\mathcal{F}$ is holonomic. For this, choose a finite number of $d^i_\lambda \in \mathcal{D}_{Y_\lambda}$, the vanishing locus of whose image in $\mathrm{Sym}_{\mathcal{O}_{Y_\lambda}} T_{Y_\lambda}$ is $\mathcal{SS}((\phi_\lambda)_+\mathcal{F}_\lambda)$. Specializing to $\lambda = n$ for small enough $n < 0$, we obtain $\{d^i_n\} \in \mathcal{D}_Y$ who kill $f^n u$ and whose vanishing locus is of dimension at most $\mathcal{SS}((\phi_\lambda)_+\mathcal{F}_\lambda) = \dim(Y)$. The former holds for all n, and the latter for all but finitely many values of n because it holds over $k(\lambda)$ and is expressed by a non-vanishing condition (and an element of $k(\lambda)$ which does not vanish can only specialize to 0 for finitely many values of n).

Recall now that for small enough $n < 0$, $f^n u$ generates $\phi_+\mathcal{F}$. Now, for such an n, take $\mathcal{H}$ to be the submodule generated by $f^n u$. The $\{d^i_n\}$ exhibit a annihilating set for $\mathrm{gr}\,\mathcal{H}$ in $\mathrm{Sym}_{\mathcal{O}_Y} T_Y$ of dimension $\dim(Y)$, hence the singular support of $\mathcal{H}$ is supported on it and has dimension $\dim(Y)$, showing that $\mathcal{H}$ is holonomic. But we chose n so that $\mathcal{H} = \phi_+\mathcal{F}$, giving the desired. ☐

2.5.3. *Duality for holonomic $\mathcal{D}$-modules.* We now give for completeness a description of duality for the derived category of $\mathcal{D}$-modules with holonomic cohomology. We omit the proof, though we will discuss a particular consequence, the classification of all irreducible holonomic $\mathcal{D}_X$-modules on a given variety X. As is somewhat apparent by the definition of $\mathbb{D}$, this theory will be analogous to the theory of Grothendieck duality for coherent sheaves in algebraic geometry. Let $\phi : X \to Y$ be a morphism of smooth varieties. We define functors $\phi_! : D(\mathrm{DMod}(X)^{\mathrm{hol}}) \to D(\mathrm{DMod}(Y)^{\mathrm{hol}})$ and $\phi^\star : D(\mathrm{DMod}(Y)^{\mathrm{hol}}) \to D(\mathrm{DMod}(X)^{\mathrm{hol}})$ by

$$\phi_! = \mathbb{D}_Y \phi_\star \mathbb{D}_X \text{ and } \phi^\star = \mathbb{D}_X \phi^! \mathbb{D}_Y.$$

Combined with our original derived pushforward and pullback, this yields the six functors $\phi_!$, $\phi^!$, $\phi_\star$, $\phi^\star$, $\mathbb{D}_X$, and $\mathbb{D}_Y$, which will be related by the following duality theorem.[12]

Theorem 2.46 ([Ber83, Section 3.9])**.** *We have the following relationships between $\phi_!$, $\phi^!$, $\phi_\star$, $\phi^\star$, $\mathbb{D}_X$, and $\mathbb{D}_Y$:*

(i) *$\phi_!$ is left adjoint to $\phi^!$,*
(ii) *$\phi^\star$ is left adjoint to $\phi_\star$,*
(iii) *there is a natural map $\phi_! \to \phi_\star$ which is an isomorphism if ϕ is proper, and*
(iv) *if ϕ is smooth, then $\phi^! = \phi^\star[2(\dim Y - \dim X)]$.*

[12]Technically speaking, we need also to show that $\mathbb{D}_X$, $\mathbb{D}_Y$ preserve $D(\mathrm{DMod}(X)^{\mathrm{hol}})$. However, as we are omitting the proof of Theorem 2.46, we do not find this necessary.

Let us see a sample application which will be quite relevant to the second half of this essay. Let $i : Z \hookrightarrow X$ be a locally closed affine embedding of a smooth subvariety. Then, for any $\mathcal{O}_Z$-coherent $\mathcal{D}_Z$-module $\mathcal{F}$, we define its *minimal extension* to be

$$i_{!\star}\mathcal{F} := \operatorname{Im}(i_!\mathcal{F} \to i_\star\mathcal{F})$$

as a $\mathcal{D}_X$-module. Notice that $i_{!\star}\mathcal{F}$ is a holonomic $\mathcal{D}_X$-module (and not a complex) because $\mathcal{D}_X$, $\mathcal{D}_Z$, and $i_\star$ all send holonomic $\mathcal{D}$-modules to holonomic $\mathcal{D}$-modules and the property of being holonomic is preserved under taking submodules. When $\mathcal{F}$ is furthermore irreducible, this construction will allow us to obtain all irreducible holonomic $\mathcal{D}_X$-modules by the following theorem, which we again restrict ourselves to only quoting.

Theorem 2.47 ([HTT08, Theorem 3.4.2])**.** *Suppose that $\mathcal{F}$ is an irreducible $\mathcal{O}_Z$-coherent $\mathcal{D}_Z$-module. Then its minimal extension $i_{!\star}\mathcal{F}$ is an irreducible holonomic $\mathcal{D}_X$-module. Moreover, all irreducible holonomic $\mathcal{D}_X$-modules take this form.*

2.6. ***D*-modules on $\mathbb{P}^1$.** In this subsection, we consider the toy example of $\mathcal{D}$-modules on $\mathbb{P}^1$ using the theory that we have developed in this section. We note in particular that $\mathbb{P}^1$ is the flag variety of the semisimple algebraic group SL_2, thus our discussion here will provide a link to the discussion of the Beilinson-Bernstein theorem in the second half of the essay. First, Kashiwara's Theorem gives us the following key characterization of $\mathcal{D}$-modules on projective spaces.

Proposition 2.48. *Let $\mathbb{P}(V)$ be the projective space of a vector space V. Then, the global sections functor provides an equivalence of categories*

$$\Gamma : \operatorname{DMod}(\mathbb{P}(V)) \to \Gamma(\mathbb{P}(V), \mathcal{D}_{\mathbb{P}(V)}) - \mathit{mod}.$$

Before proving Proposition 2.48, we require some general analysis of $\mathcal{D}$-modules on projective space. Define the maps $j : V - \{0\} \hookrightarrow V$ and $\pi : V - \{0\} \to \mathbb{P}(V)$, and let $\{x_i\}$ be a basis for V. The proof will rely crucially on the following lemma, which says that for a $\mathcal{D}_{V-\{0\}}$-module arising as the pullback of a $\mathcal{D}_{\mathbb{P}(V)}$-module, the graded structure on its global sections can be recovered from the $\mathcal{D}_{V-\{0\}}$-action.

Lemma 2.49. *Let $\mathcal{F}$ be a $\mathcal{D}_{\mathbb{P}(V)}$-module and $\mathcal{G} := \pi^\Delta\mathcal{F}$ its pullback to $V - \{0\}$. Then, the i^{th} graded component of $\Gamma(V - \{0\}, \mathcal{G})$ given by the $\mathbb{G}_m$-equivariant structure is the i-eigenspace of the action of the Euler operator $\xi := \sum_i x_i\partial_i$ on $\Gamma(V - \{0\}, \mathcal{G})$.*

Proof. Because $\mathcal{G}$ took the form $\pi^\Delta\mathcal{F}$, the isomorphism

$$\phi : \operatorname{act}^\Delta \mathcal{G} \simeq p_2^\Delta\mathcal{G}$$

giving the $\mathbb{G}_m$-equivariant structure is given as the pullback of the identity isomorphism $\mathcal{F} \to \mathcal{F}$ via the morphisms $\pi \circ \operatorname{act} = \pi \circ p_2$. Therefore, ϕ respects the $\mathcal{D}_{\mathbb{G}_m\times(V-\{0\})}$-actions on $\operatorname{act}^\Delta \mathcal{G}$ and $p_2^\Delta\mathcal{G}$. Recall now that the grading on global sections of $\mathcal{G}$ is given as follows; an element $m \in \Gamma(V - \{0\}, \mathcal{G})$ decomposes as $m = \sum_i m_i$, where

$$\phi(1 \otimes m) = \sum_i t^i \otimes m_i.$$

Let us reinterpret this in terms of the $\mathcal{D}$-module structure. Notice that $\mathcal{D}_{\mathbb{G}_m\times(V-\{0\})} = \mathcal{D}_{V-\{0\}}[t, t^{-1}, \partial_t]$, and that because $T_{\mathbb{G}_m}$ is killed by p_2, the action of $\mathcal{D}_{\mathbb{G}_m} \hookrightarrow \mathcal{D}_{\mathbb{G}_m\times(V-\{0\})}$ on $p_2^*\mathcal{F}$ is given solely by its left action on $\mathcal{O}_{\mathbb{G}_m}$. We may thus compute

$$t\partial_t \cdot \phi(1 \otimes m) = \sum_i it^i \otimes m_i. \tag{6}$$

On the other hand, the map $T_{\mathbb{G}_m\times(V-\{0\})} \to \operatorname{act}^* T_{V-\{0\}}$ sends $t\partial_t$ to $\sum_i x_i\partial_i$. Computing in this way, we see that

$$t\partial_t \cdot \phi(1 \otimes m) = \phi\Big(t\partial_t \cdot (1 \otimes m)\Big) = \phi\Big(1 \otimes \sum_i x_i\partial_i m\Big). \tag{7}$$

Combining our computations (6) and (7) of the action of $t\partial_t$, we see that $\xi := \sum_i x_i\partial_i$ acts on the i^{th} graded component of $\Gamma(V - \{0\}, \mathcal{G})$ by multiplication by i. □

Proof of Proposition 2.48. We first show that $\Gamma(\mathbb{P}(V), -)$ is exact. In view of Lemma 2.49, we may factor Γ as the composition of the functors

$$\operatorname{DMod}(\mathbb{P}(V)) \xrightarrow{\pi^\Delta} \operatorname{DMod}(V - \{0\}) \xrightarrow{\Gamma(V-\{0\},-)} \Gamma(V - \{0\}, \mathcal{O}_{V-\{0\}}) - \operatorname{mod} \xrightarrow{\ker(\operatorname{act}_\xi)} \Gamma(\mathbb{P}(V), \mathcal{D}_{\mathbb{P}(V)}) - \operatorname{mod}.$$

Here, π^Δ is exact because π is flat, so it suffices to show that $\ker(\mathrm{act}_\xi) \circ \Gamma(V - \{0\}, -)$ is exact. For this, consider a short exact sequence

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \tag{8}$$

of $\mathcal{D}_{V-\{0\}}$-modules. Because j is an open embedding, we see that j_+ coincides with the ordinary pushfoward of $\mathcal{O}$-modules, hence we have that $\Gamma(V - \{0\}, -) \simeq \Gamma(V, -) \circ j_+$. Consider now the exact sequence

$$0 \to j_+\mathcal{F}_1 \to j_+\mathcal{F}_2 \to j_+\mathcal{F}_3 \to R^1 j_+\mathcal{F}_1 \to \cdots .$$

Applying the exact functor π^Δ, we see that the sequence $\pi^\Delta R^1 j_+\mathcal{F}_1 = 0$, hence $R^1 j_+\mathcal{F}_1$ is supported at $\{0\} \in V$. Therefore, by Kashiwara's Theorem, we may write $R^1 j_+\mathcal{F}_1 = i_+\mathcal{G}$, where $i : \{0\} \hookrightarrow V$ is the inclusion and $\mathcal{G}$ is a $\mathcal{D}$-module on the single point space $\{0\}$. Then $\mathcal{G}$ is the (possibly infinite) direct sum of copies k_0 of the base field, and we saw in the course of the proof of Kashiwara's Theorem that $i_+(k_0) \simeq [\partial_1, \ldots, \partial_n]k \cdot 1_0$, where each x_i annihilates the formal element 1_0. The action of ξ on monomials $\prod_i \partial_i^{\alpha_i} \cdot 1_0$ in $\Gamma(V, i_+(k_0))$ is thus given by $-n - \sum_i \alpha_i$, a strictly negative integer. In particular, this implies that the kernel of the action of ξ on $\Gamma(V, R^1 j_+\mathcal{F}_1) = \Gamma(V, i_+\mathcal{G})$ is 0. This shows that applying the composition $\ker(\mathrm{act}_\xi) \circ \Gamma(V, -) \circ j_+$ to (8) yields an exact sequence

$$0 \to \Gamma(V - \{0\}, \mathcal{F}_1)^0 \to \Gamma(V - \{0\}, \mathcal{F}_2)^0 \to \Gamma(V - \{0\}, \mathcal{F}_3)^0 \to 0,$$

showing that $\Gamma(\mathbb{P}(V), -)$ is exact.

To show that $\Gamma(\mathbb{P}(V), -)$ gives an equivalence of categories, consider the functor

$$\mathrm{Loc}(M) := \mathcal{D}_{\mathbb{P}(V)} \underset{\Gamma(\mathbb{P}(V), \mathcal{D}_{\mathbb{P}(V)})}{\otimes} M,$$

where we view M and $\Gamma(\mathbb{P}(V), \mathcal{D}_{\mathbb{P}(V)})$ as constant sheaves on $\mathbb{P}(V)$. It is easy to check that Loc is left adjoint to Γ; further, to show that they form an equivalence, it suffices by abstract nonsense to show that that $\Gamma(\mathbb{P}(V), \mathcal{F}) = 0$ implies $\mathcal{F} = 0$. We will give an identical argument to this one in detail in the proof of Corollary 4.12, so we content ourself with this sketch here.

It remains now to check that $\Gamma(\mathbb{P}(V), \mathcal{F}) = 0$ implies $\mathcal{F} = 0$. Take an $\mathcal{F}$ so that $\Gamma(\mathbb{P}(V), \mathcal{F}) = 0$, and consider the graded module $\Gamma(V - \{0\}, \pi^\Delta\mathcal{F})$ which is trivial in grade 0. An analysis similar to the one we performed in the proof of Lemma 2.49 shows that ∂_i maps $\Gamma(V - \{0\}, \pi^\Delta\mathcal{F})^{k+1}$ to $\Gamma(V - \{0\}, \pi^\Delta\mathcal{F})^k$ and x_i maps $\Gamma(V - \{0\}, \pi^\Delta\mathcal{F})^k$ to $\Gamma(V - \{0\}, \pi^\Delta\mathcal{F})^{k+1}$. Therefore, the action of the Euler operator ξ and the fact that $\Gamma(V - \{0\}, \pi^\Delta\mathcal{F})^0 = 0$ show that $\Gamma(V - \{0\}, \pi^\Delta\mathcal{F})^k = 0$ for $k > 0$. Similarly, if all x_i kill $\Gamma(V - \{0\}, \pi^\Delta\mathcal{F})^k$ for some k, then $\Gamma(V - \{0\}, \pi^\Delta\mathcal{F})^k$ is supported at $\{0\}$, hence 0. Applying this to the fact that $\Gamma(V - \{0\}, \pi^\Delta\mathcal{F})^0 = 0$ shows that $\Gamma(V - \{0\}, \pi^\Delta\mathcal{F})^k = 0$ for $k < 0$, hence $\Gamma(V - \{0\}, \pi^\Delta\mathcal{F}) = 0$. This implies that $\mathcal{F} = 0$ (for instance because for any coherent submodule $\mathcal{H} \subset \mathcal{F}$, the twists $\mathcal{H}(n)$ are eventually globally generated). $\square$

Remark. Varieties X such that the global sections functor defines an equivalence of categories between $\mathcal{D}_X$-modules and $\Gamma(X, \mathcal{D}_X)$-modules are called *$\mathcal{D}$-affine.* In Proposition 2.48, we have shown that the projective spaces $\mathbb{P}^n$ are $\mathcal{D}$-affine. In these terms, the following half of this essay will be devoted to showing (via a different technique involving Lie algebra representations) that the flag variety G/B of a semisimple algebraic group is also $\mathcal{D}$-affine.

Using Proposition 2.48, we may now calculate explicitly in the category of $\mathcal{D}$-modules on $\mathbb{P}^1$. Pick two coordinates z and $w = z^{-1}$ corresponding to a cover of $\mathbb{P}^1$ by two copies U_z and U_w of $\mathbb{A}^1$. The sheaf of differential operators $\mathcal{D}_{\mathbb{P}^1}$ is then defined by

$$\begin{aligned} \Gamma(U_z, \mathcal{D}_{\mathbb{P}^1}) &= k[z, \partial_z]/\langle[\partial_z, z] = 1, \\ \Gamma(U_w, \mathcal{D}_{\mathbb{P}^1}) &= k[w, \partial_w]/\langle[\partial_w, w] = 1\rangle, \text{ and} \\ \Gamma(U_z \cap U_w, \mathcal{D}_{\mathbb{P}^1}) &= k[z, z^{-1}, \partial_z]/\langle[\partial_z, z] = 1, [\partial_z, z^{-1}] = -z^{-2}\rangle, \end{aligned}$$

where the restriction maps are the natural inclusion $z \mapsto z$, $\partial_z \mapsto \partial_z$ on $\Gamma(U_z, \mathcal{D}_{\mathbb{P}^1})$ and the map

$$w \mapsto z^{-1}, \qquad \partial_w \mapsto -z^2\partial_z$$

on $\Gamma(U_w, \mathcal{D}_{\mathbb{P}^1})$. Now, writing all monomials in $k[z, z^{-1}, \partial_z]$ in the form $z^i\partial_z^j$, we find that

$$\Gamma(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}) = \mathrm{span}\Big(1,\ z^i\partial_z^j \text{ for } i \leq j+1,\ j > 0\Big).$$

One may check that $\Gamma(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1})$ is generated by ∂_z, $z\partial_z$, and $z^2\partial_z$. Suggestively, if we normalize these as

$$e = -\partial_z, \qquad h = -2z\partial_z, \qquad f = z^2\partial_z, \tag{9}$$

then we may check that $[e, f] = h$, $[h, e] = 2e$, and $[h, f] = -2f$. Therefore, we see that (9) defines a map of associative algebras $U(\mathfrak{sl}_2) \to \Gamma(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1})$. It is also easy (if a bit tedious) to check that $c = \frac{1}{2}h^2 + h + 2fe$, the Casimir element of $U(\mathfrak{sl}_2)$, is sent to zero by this map. For $U(\mathfrak{sl}_2)$, it is known that c generates the entire center, hence we have just manually constructed a map

$$U(\mathfrak{sl}_2)/Z(\mathfrak{sl}_2) \cdot U(\mathfrak{sl}_2) \to \Gamma(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1}).$$

Evidently, this map associates to each $\mathcal{D}_{\mathbb{P}^1}$-module a $U(\mathfrak{sl}_2)$-module where $Z(\mathfrak{sl}_2)$ acts trivially. As we shall see in the sequel, this is an instance of the Beilinson-Bernstein correspondence. Now, let us identify the $U(\mathfrak{sl}_2)$-representations that some specific $\mathcal{D}_{\mathbb{P}^1}$-modules correspond to.

Example 2.50. If $\phi : X \to \mathbb{P}^1$ is a closed embedding, then

$$\phi_{!\star}(\mathcal{F}) = \mathrm{Im}(\phi_!(\mathcal{F}) \to \phi_\star(\mathcal{F})) \simeq \phi_\star(\mathcal{F}) = \phi_+(\mathcal{F})$$

because ϕ is automatically proper, hence we may apply Theorem 2.46 to see that the map $\phi_! \to \phi_\star$ is an isomorphism. Therefore, Theorem 2.47 implies that $\phi_\star(\mathcal{F})$ should be an irreducible holonomic $\mathcal{D}_{\mathbb{P}^1}$-module when $\mathcal{F}$ is coherent and irreducible. Consider now a closed embedding $\phi : \{x\} \hookrightarrow \mathbb{P}^1$ of a point x which we will assume lies in the coordinate chart U_z. Any $\mathcal{D}$-module on $\{x\}$ is simply a vector space, so the only interesting $\mathcal{D}$-module we produce in this way is the pushfoward $\phi_+(k_x)$ of the trivial $\mathcal{D}_{\{x\}}$-module k_x. Explicitly, it takes the form

$$\phi_+(k_x) := \phi_.(\mathcal{D}_{\mathbb{P}^1 \leftarrow \{x\}} \underset{\mathcal{O}_{\{x\}}}{\otimes} k_x)$$

and thus

$$\Gamma(\mathbb{P}^1, \phi_+(k_x)) \simeq \Gamma(\{x\}, \mathcal{D}_{\mathbb{P}^1 \leftarrow \{x\}}) \simeq k_x \underset{\mathcal{O}_{\mathbb{P}^1,x}}{\otimes} \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(\Omega_{\mathbb{P}^1}, \mathcal{D}_{\mathbb{P}^1})_x \simeq (\mathcal{D}_{\mathbb{P}^1})_x \underset{\mathcal{O}_{\mathbb{P}^1,x}}{\otimes} k_x \simeq [\partial_z] k_x,$$

where the action of $\Gamma(\mathbb{P}^1, \mathcal{D}_{\mathbb{P}^1})$ is by left multiplication. Computing the actions explicitly, we see that

$$\begin{aligned}
-\partial_z \cdot \partial_z^i &= -\partial_z^{i+1} \\
-2z\partial_z \cdot \partial_z^i &= -2x\partial_z^{i+1} + 2(i+1)\partial_z^i \\
z^2\partial_z \cdot \partial_z^i &= x^2\partial_z^{i+1} - 2(i+1)x\partial_z^i + i(i+1)\partial_z^{i-1},
\end{aligned}$$

which when $x = 0$ corresponds to the dual Verma module $M_{-2}^\vee$ for $\mathfrak{sl}_2$ with lowest weight 2. When $x \neq z$, we observe that the resulting $\mathfrak{sl}_2$-representation is *not* a lowest weight representation for the given choice of f, h, e, which we note implicitly defined a choice of Cartan and Borel subalgebra for $\mathfrak{sl}_2$. Indeed, if we instead consider the mapping

$$e' = -\partial_z, \qquad h' = -2(z - x)\partial_z, \qquad f' = (z - w)^2\partial_z,$$

then we see that

$$\begin{aligned}
-\partial_z \cdot \partial_z^i &= -\partial_z^{i+1} \\
-2(z - x)\partial_z \cdot \partial_z^i &= 2(i+1)\partial_z^i \\
(z - x)^2\partial_z \cdot \partial_z^i &= i(i+1)\partial_z^{i-1},
\end{aligned}$$

meaning that the result is a lowest weight representation of $\mathfrak{sl}_2$ for this different choice of Cartan and Borel subalgebra. We conclude this discussion by noting that the point $x = \infty$ should correspond to the missing case of the Verma module M_{-2} for $\mathfrak{sl}_2$ with highest weight -2.

3. The geometric setting

In this section, we establish some geometric preliminaries for the rest of this essay. First, we describe the flag variety of a semisimple algebraic group G and provide a construction of some equivariant vector bundles on it. The primary sources for this are [Spr98] and [Jan03]. Second, we define the Lie algebra $\mathfrak{g}$ and describe the induced action of $\mathfrak{g}$ on varieties with a G-action. Finally, we discuss the Chevalley and Harish-Chandra isomorphisms for $\mathfrak{g}$. The sources for this portion are [Gai05], [Hum08], and [Dix77].

3.1. **Preliminaries on semisimple algebraic groups.** Let G be a connected, simply connected, semisimple algebraic group over an algebraically closed field k of characteristic 0. Pick a Borel subgroup B of G and a maximal torus $T \subset B$. Let U be a maximal unipotent subgroup of B, and recall that $T \simeq B/U$.

Let $\Phi(G,T) = (X^*, R, X_*, R^\vee)$ be the root datum associated to G. That is, X^* and X_* are abstract abelian groups equipped with isomorphisms $X^* \simeq X^*(T)$ and $X_* \simeq X_*(T)$ to the lattices of characters and co-characters of T. In particular, for $\lambda \in X^*$, we denote by e^λ the corresponding character $e^\lambda : T \to \mathbb{G}_m$. Let R and $R^\vee$ be the roots and co-roots of G, viewed as subsets of X^* and X_*. Denote by $\langle -, - \rangle : X^* \otimes X_* \to \mathbb{Z}$ the perfect pairing induced from the usual pairing between $X^*(T)$ and $X_*(T)$. Our choice of B defines a choice of positive roots $R^+ \subset R$. Let the corresponding sets of simple roots and co-roots be $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\}$, respectively, and let $\{\omega_1, \ldots, \omega_n\} \subset X$ be the fundamental weights, which lie in X^* because G was chosen to be simply connected. Let $\rho = \omega_1 + \cdots + \omega_n$ be half the sum of the positive weights.

Let $W = N(T)/T$ be the Weyl group of G, and let s_i be the simple reflection corresponding to α_i. For $w \in W$, let $\ell(w)$ denote its length, which is the length of the shortest reduced decomposition for w. Denote by w_0 the longest element in W.

3.2. **Flag and Schubert varieties.** We are now ready to introduce our basic geometric setting, the flag variety $X = G/B$. Let us first recall the following facts about Borel subgroups in G.

Proposition 3.1 ([Spr98, Theorem 6.2.7]). *Any two Borel subgroups in G are conjugate.*

Proposition 3.2 ([Spr98, Theorem 6.4.9]). *For any Borel subgroup B in G, we have $N_G(B) = B$.*

For any Borel subgroup B of G, let $X = G/B$ be the corresponding flag variety of G, interpreted as a quotient under the action of B by right translation. The following corollary allows us to describe X without fixing a choice of B.

Corollary 3.3 ([DG70, XXII.5.8.3]). *For each B, $X = G/B$ is a projective variety representing the functor*

$$\mathcal{B}(S) = \{\textit{Borel subgroups of } G \underset{k}{\times} S\}.^{13}$$

Proof. On k-points, the correspondence is simple; it maps the point gB of X to the Borel subgroup gBg^{-1}. That this is an isomorphism then follows from Proposition 3.2. We refer to [DG70] for the proof in the general case; however, we will only use this correspondence on k-points in this essay. ☐

We now summarize some well-known properties of X in the following two propositions.

Proposition 3.4. *The flag variety X satisfies the following properties:*

(i) *the projection $\pi : G \to X$ gives G the structure of a principal B-bundle over X;*
(ii) *there is an equivalence of categories $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(G)^B$, where B-equivariance is taken with respect to the action of B on G by right translation.*

Proof. For (i), see [Spr98, Lemma 8.5.2]. For (ii), see [FGI$^+$05, Theorem 4.46], though that result is much more general than is needed here. ☐

Proposition 3.5. *The fixed points of the action of $g \in G$ by left translation on X are in correspondence with Borel subgroups containing g under the identification of Corollary 3.3.*

Proof. Under the identification of Corollary 3.3, notice that $g \in G$ fixes $g'B$ if and only if g normalizes $g'B(g')^{-1}$, which by Proposition 3.2, occurs if and only if g lies in $g'B(g')^{-1}$. ☐

3.3. **Equivariant vector bundles on the flag variety.** We are now ready to give a fundamental construction which relates the representation theory of G to the geometry of X. For any B-module V, define the *associated sheaf* $\mathcal{L}(V)$ on X of sections of $G \times^B V$ given by

$$\Gamma(U, \mathcal{L}(V)) = \{f \in \mathcal{O}_{\pi^{-1}(U)} \underset{k}{\otimes} V \mid f(g \cdot b) = b^{-1} \cdot f(g)\}.$$

The following gives some basic compatibility properties of this construction.

Proposition 3.6. *We have the following:*

[13] For general S, we must take the definition of Borel subgroup given in [DG70, XXII.5.2.3]. That is, a Borel subgroup of a group scheme G over S is a smooth subgroup B of finite type with connected fibers over S such that each geometric fiber $B \underset{S}{\times} \bar{k}_s$ is a Borel subgroup of $G \underset{S}{\times} \bar{k}_s$ as a linear algebraic group over $\bar{k}_s$. We include this only for completeness, as we will only use the case where $S = \mathrm{Spec}(k)$.

(i) *$V \mapsto \mathcal{L}(V)$ defines an exact functor $B - mod \to \mathrm{QCoh}(X)^G$;*
(ii) *for each V, $\mathcal{L}(V)$ is a vector bundle of rank $\dim V$;*
(iii) *for representations V, W of B, we have $\mathcal{L}(V \otimes W) \simeq \mathcal{L}(V) \otimes \mathcal{L}(W)$.*

Proof. For (i), see [Jan03, Proposition I.5.9] and the remarks in [Jan03, Section II.4.2]. For (ii) and (iii), see the remarks in [Jan03, Section II.4.1]. □

This construction provides a geometric way to realize induced representations from B to G. In particular, because the $\mathcal{L}(V)$ are G-equivariant sheaves, their global sections $\Gamma(X, \mathcal{L}(V))$ acquire a natural G-module structure; an alternate notation for these G-modules is $\mathrm{ind}_B^G V$.

When V was originally a G-module instead of only a B-module, we may characterize $\mathcal{L}(V)$ more explicitly. Indeed, consider the vector bundle $\mathcal{V} := \mathcal{O}_X \otimes_k V$ with a G-equivariant structure inherited from the action of G on both $\mathcal{O}_X$ and V; by this, we mean that the isomorphism

$$\mathrm{act}^* \mathcal{O}_X \underset{k}{\otimes} V \simeq \mathrm{act}^* \mathcal{V} \to p_2^* \mathcal{V} \simeq p_2^* \mathcal{O}_X \underset{k}{\otimes} V$$

is given by

$$\mathrm{act}^* \mathcal{O}_X \underset{k}{\otimes} V \to p_2^* \mathcal{O}_X \underset{k}{\otimes} V \underset{k}{\otimes} \mathcal{O}_G \to p_2^* \mathcal{O}_X \underset{k}{\otimes} V,$$

where the first map is given by the standard isomorphism $\mathrm{act}^* \mathcal{O}_X \to p_2^* \mathcal{O}_X$ and the action map $V \to V \underset{k}{\otimes} \mathcal{O}_G$ and the second map is multiplication in $\mathcal{O}_G$. The following proposition shows that this provides an alternate construction for $\mathcal{L}(V)$.

Proposition 3.7. *Let V be a G-module. There is an isomorphism of G-equivariant vector bundles*

$$\mathcal{V} \simeq \mathcal{L}(V).$$

Proof. By definition, we have that

$$\Gamma(X, \mathcal{L}(V)) = \{f \in \mathcal{O}_G \otimes_k V \mid f(g \cdot b) = b^{-1} \cdot f(g)\}.$$

Consider the map $\phi^\flat : V \to \Gamma(X, \mathcal{L}(V))$ defined by

$$\phi^\flat(v) = \Big(g \mapsto g^{-1} \cdot v\Big),$$

and let $\phi : \mathcal{V} \to \mathcal{L}(V)$ be the corresponding map of $\mathcal{O}_X$-modules. It is easy to check that this map is a G-equivariant isomorphism. □

We now consider an important special case. For any $\lambda \in X^*$, we obtain a representation $k_{-\lambda}$ of B via $B \to B/U \simeq T \xrightarrow{e^{-\lambda}} \mathbb{G}_m$, giving rise to a family of line bundles $\mathcal{L}^\lambda = \mathcal{L}(k_{-\lambda})$ on X. Notice that $\mathcal{L}^0 = \mathcal{O}_X$, and, by Proposition 3.6(iii), $\mathcal{L}^\lambda \otimes \mathcal{L}^\mu = \mathcal{L}^{\lambda+\mu}$. In fact, it is known that all line bundles on X take this form (see [FI73, Proposition 3.1]). The line bundles $\mathcal{L}^\lambda$ satisfy some well-known properties, which we summarize below.

Proposition 3.8. *We have the following:*
(i) *the canonical bundle ω_X of X is given by $\mathcal{L}^{-2\rho}$;*
(ii) *the G-module $\Gamma(X, \mathcal{L}^\lambda)$ is non-zero if and only if λ is dominant;*
(iii) *the line bundle $\mathcal{L}^\lambda$ is globally generated if λ is dominant and ample if λ is regular.*

Proof. For (i), see [Jan03, Section II.4.2]; for (ii), see [Spr98, Theorem 8.5.8]; for (iii), see [Jan03, Propositions II.4.4 and II.4.5]. □

Finally, we have the classical Borel-Weil theorem, which shows that we may recover the irreducible representations of G from the global sections of $\mathcal{L}^\lambda$.[14]

Theorem 3.9 (Borel-Weil)**.** *For any dominant weight λ, the G-module $\Gamma(X, \mathcal{L}^\lambda)^*$ is the (unique) irreducible representation of G with highest weight λ.*

[14]Much more is known about the cohomologies of the $\mathcal{L}^\lambda$. Namely, the Borel-Weil-Bott theorem shows that each $\mathcal{L}^\lambda$ has a unique non-zero cohomology group which is isomorphic to a certain irreducible representation of G. However, this is not strictly necessary for our main discussion, so we will not discuss it further.

3.4. **Preliminaries on Lie algebras.** Following [DG80], we define the Lie algebra of G as follows. Write $k[\varepsilon] = k[\varepsilon]/\varepsilon^2$ for the ring of dual numbers and for a scheme X, write $X_\varepsilon = X \underset{k}{\times} \mathrm{Spec}(k[\varepsilon])$. For an algebraic group G, consider the functor $\mathrm{Lie}(G)$ whose S points are given by

$$1 \to \mathrm{Lie}(G)(S) \to G(S_\varepsilon) \to G(S) \to 1.$$

Here for a scheme S over k, by $G(S)$ and $G(S_\varepsilon)$ we mean $\mathrm{Hom}_{\mathrm{Sch}_{/k}}(S, G)$ and $\mathrm{Hom}_{\mathrm{Sch}_{/k}}(S_\varepsilon, G)$. Then, we take the *Lie algebra of* G to be $\mathfrak{g} = \mathrm{Lie}(G)(k)$. Let us see what this means concretely. Notice that $G(\mathrm{Spec}(k[\varepsilon])) = \mathrm{Hom}_k(\mathcal{O}_G, k[\varepsilon])$ and that $G(k) = \mathrm{Hom}_k(\mathcal{O}_G, k)$. Now, because elements of $\mathfrak{g}$ are identified with elements of $G(\mathrm{Spec}(k[\varepsilon]))$ mapping to the inclusion of the identity in $G(k)$, we obtain an isomorphism $\mathfrak{g} \simeq \mathrm{Der}_k(\mathcal{O}_G, k)$.

Suppose now that G is connected, simply-connected, and semisimple. Let $\mathfrak{h} \subset \mathfrak{g}$ and $\mathfrak{b} \subset \mathfrak{g}$ denote the Cartan and Borel subalgebras corresponding to T and B, respectively. Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ so that $\mathfrak{h} \simeq \mathfrak{b}/\mathfrak{n}$. Let $\mathfrak{b}^-$ be the opposite Borel subalgebra and set $\mathfrak{n}^- = [\mathfrak{b}^-, \mathfrak{b}^-]$ so that $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. We can make these constructions for all points $x \in X$ under the correspondence between points of X and Borel subgroups of G; for any such x, we will denote by $\mathfrak{b}_x, \mathfrak{b}_x^-, \mathfrak{n}_x, \mathfrak{n}_x^-$ the subalgebras corresponding to the Borel subgroup of x in the sense of Corollary 3.3. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$; the preceding direct sum decomposition yields the following well-known decomposition for $U(\mathfrak{g})$.

Theorem 3.10 (Poincaré-Birkhoff-Witt)**.** *As a $(U(\mathfrak{n}^-), U(\mathfrak{n}))$-bimodule, $U(\mathfrak{g})$ admits the decomposition*

$$U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}).$$

Differentiating characters $T \to \mathbb{G}_m$ and co-characters $\mathbb{G}_m \to T$ gives rise to canonical embeddings $X^* \hookrightarrow \mathfrak{h}^*$ and $X_* \hookrightarrow \mathfrak{h}$ such that the perfect pairing $\langle -, - \rangle$ is simply the pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$. Under this correspondence, note that the roots R defined earlier correspond to the weights of $\mathfrak{h}$ on $\mathfrak{g}$, the positive roots R^+ correspond to the weights of $\mathfrak{h}$ on $\mathfrak{n}$, and the negative roots correspond to the weights of $\mathfrak{h}$ on $\mathfrak{n}^-$. From now on, we will identify elements of $(X^*, R, X_*, R^\vee)$ with their images under this embedding.

We say that $\lambda \in \mathfrak{h}^*$ is an *integral weight* if it lies in X^*; in particular, because G is simply connected, we see that λ is integral if and only if $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}$ for all i. Note that the integral weights λ are exactly those which lift to characters $e^\lambda \in X^*(T)$ of the algebraic group. We say that a weight $\lambda \in \mathfrak{h}^*$ is called *dominant* if $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all i and *regular* if $\langle \lambda, \alpha_i^\vee \rangle > 0$ for all i. These occur if and only if λ is a non-negative (resp. positive) linear combination of fundamental weights.

3.5. **Lie algebra actions on G-equivariant sheaves.** Let X be a scheme equipped with a G-action $\mathrm{act} : G \times X \to X$. Let us view the action map $G \times X \to X$ as a map of functors $G \to \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ is the functor

$$\mathrm{Aut}(X)(R) = \mathrm{Aut}_{\mathrm{Sch}_{/k}}(X \underset{k}{\times} R, X \underset{k}{\times} R).$$

This is a map of group objects, so we may consider the corresponding induced map on Lie algebras

$$\mathrm{Lie}(G) \to \mathrm{Lie}(\mathrm{Aut}(X)).$$

Taking this map on k-points, we obtain a map $\mathrm{Lie}(G)(k) \to \mathrm{Lie}(\mathrm{Aut}(X))(k)$, where we have[15]

$$\begin{aligned} \mathrm{Lie}(\mathrm{Aut}(X))(k) &= \{\phi \in \mathrm{Aut}(X_\varepsilon) \mid \phi|_X = \mathrm{id}\} \\ &= \{\phi \in \mathrm{Hom}(\mathcal{O}_X[\varepsilon], \mathcal{O}_X[\varepsilon]) \mid \phi|_{\mathcal{O}_X} = \mathrm{id}\} \\ &= \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X). \end{aligned}$$

Thus, we have constructed a map

$$\rho : \mathfrak{g} \to \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X) \tag{10}$$

which is the result of differentiating the G-action on $\mathcal{O}_X$. Unraveling the construction above, we see that $\rho(\xi)$ is given by the composition

$$\mathcal{O}_X \xrightarrow{\mathrm{act}^*} \mathcal{O}_G \otimes \mathcal{O}_X \xrightarrow{\xi} \mathcal{O}_X.$$

The following proposition shows that we may perform a similar construction for any G-equivariant quasicoherent sheaf on X.

[15]Recall that $\mathrm{Lie}(\mathrm{Aut}(X))(R)$ satisfies the exact sequence $1 \to \mathrm{Lie}(\mathrm{Aut}(X))(R) \to \mathrm{Aut}(X)(R_\varepsilon) \to \mathrm{Aut}(X)(R) \to 1$.

Proposition 3.11. *Let $\mathcal{F}$ be a G-equivariant quasi-coherent sheaf. Then, there is a natural map*

$$\rho_{\mathcal{F}} : \mathfrak{g} \to \mathrm{End}_k(\mathcal{F})$$

such that for $\xi \in \mathfrak{g}$ and f and s local sections of $\mathcal{O}_X$ and $\mathcal{F}$, respectively, we have that

$$\rho_{\mathcal{F}}(\xi)(f \cdot s) = f \cdot \rho_{\mathcal{F}}(\xi)(s) + \rho(\xi)(f) \cdot s, \tag{11}$$

where ρ is the map of (10).

Proof. Let $\phi : \mathrm{act}^* \mathcal{F} \to p_2^* \mathcal{F}$ be the isomorphism giving the G-equivariant structure on $\mathcal{F}$. We construct $\rho_{\mathcal{F}}$ explicitly. For any affine open $U \subset X$ and $\xi \in \mathfrak{g}$, let $\rho_{\mathcal{F}}(\xi)$ be given by the composition

$$\rho_{\mathcal{F}}(\xi) : \Gamma(U, \mathcal{F}) \xrightarrow{\mathrm{act}^*} \Gamma(G \times U, \mathrm{act}^* \mathcal{F}) \xrightarrow{\phi} \Gamma(G \times U, p_2^* \mathcal{F}) \simeq \mathcal{O}_G \otimes \Gamma(U, \mathcal{F}) \xrightarrow{\xi} \Gamma(U, \mathcal{F}),$$

where ξ acts on $\mathcal{O}_G$ via the identification $\mathfrak{g} \simeq \mathrm{Der}_k(\mathcal{O}_G, k)$ of Subsection 3.4 in the final map.

Let us now check that relation (11) holds. Let $e : \{e\} \hookrightarrow G$ be the inclusion of the identity into G. Tracing the image of $f \cdot s$ through the composition above, we see that $\rho_{\mathcal{F}}(\xi)$ takes it to

$$\begin{aligned} \rho_{\mathcal{F}}(\xi)(f \cdot s) &= \xi(\phi(\mathrm{act}^*(f \cdot s))) \\ &= \xi(\mathrm{act}^*(f) \cdot \phi(\mathrm{act}^*(s))) \\ &= e^*(\mathrm{act}^*(f)) \cdot \xi(\phi(\mathrm{act}^*(s))) + \xi(\mathrm{act}^*(f)) \cdot e^*(\phi(\mathrm{act}^*(s))) \\ &= f \cdot \rho_{\mathcal{F}}(\xi)(s) + \rho(\xi)(f) \cdot s. \end{aligned}$$ ☐

Remark. For $\mathcal{F} = \mathcal{O}_X$, the map $\rho_{\mathcal{O}_X}$ defined in Proposition 3.11 is simply the original action map ρ of (10), and the condition (11) simply states that $\rho(\xi)$ is a derivation $\mathcal{O}_X \to \mathcal{O}_X$. Where there is no confusion, we will abuse notation to write ρ for $\rho_{\mathcal{F}}$.

Applying Proposition 3.11 for $\mathcal{F} = \mathcal{L}$, we obtain a map

$$\rho : \mathfrak{g} \to \mathrm{End}_k(\mathcal{L})$$

satisfying (11). This map extends to a map $U(\mathfrak{g}) \to \mathrm{End}_k(\mathcal{L})$ with image controlled by the following.

Corollary 3.12. *The action of Proposition 3.11 defines a map*

$$\phi : U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_{X,\mathcal{L}}).$$

Proof. It suffices to show that the image of ρ lies in $\Gamma(X, F_1\mathcal{D}_{X,\mathcal{L}})$. For sections $f_1, f_0 \in \Gamma(X, \mathcal{O}_X)$, we must check that

$$[f_1, [f_0, \rho(\xi)]] = 0.$$

Indeed, for any local section $s \in \mathcal{L}$, we have by repeated application of (11) that

$$[f_1, [f_0, \rho(\xi)]](s) = f_1 f_0 \rho(\xi)(s) - f_1 \rho(\xi)(f_0 s) - f_0 \rho(\xi)(f_1 s) + \rho(\xi)(f_1 f_0 s) = 0.$$ ☐

3.6. **Chevalley and Harish-Chandra isomorphisms.** In this section, we construct and characterize two important maps associated to a semisimple Lie algebra $\mathfrak{g}$, the Chevalley and Harish-Chandra isomorphisms.

3.6.1. *Chevalley isomorphism.* Consider the space $\mathrm{Sym}(\mathfrak{g}^*)^G$ of polynomial functions on $\mathfrak{g}$ invariant under the adjoint action of G on $\mathfrak{g}$. Recall that the Killing form $(\xi, \mu) \mapsto \mathrm{Tr}(\mathrm{ad}_\xi \, \mathrm{ad}_\mu)$ provides an identification $\mathfrak{g} \simeq \mathfrak{g}^*$ which intertwines the two adjoint actions of G on $\mathfrak{g}$ and $\mathfrak{g}^*$. Consider now the induced identification $\mathrm{Sym}(\mathfrak{g}^*) \simeq \mathrm{Sym}(\mathfrak{g})$; because G was taken to be connected, the invariants of the G-action on $\mathrm{Sym}(\mathfrak{g})$ are the same as those of the $\mathfrak{g}$-action.

Given a function $f \in \mathrm{Sym}(\mathfrak{g}^*)^G$ and a choice of Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, we may ask when the restriction of f to $\mathfrak{b}$ factors through $\mathfrak{b} \to \mathfrak{b}/\mathfrak{n} \simeq \mathfrak{h}$. This restriction corresponds to taking the image of f under $\mathrm{Sym}(\mathfrak{g}^*)^G \to \mathrm{Sym}(\mathfrak{b}^*)^H$, where the result in $\mathrm{Sym}(\mathfrak{b}^*)$ will still be H-invariant because H acts semisimply on $\mathfrak{g}^*$. Let us see what this means under the identification $\mathfrak{g} \simeq \mathfrak{g}^*$. Because $\mathrm{Tr}(\mathrm{ad}_\xi \, \mathrm{ad}_\mu) = 0$ for $\xi \in \mathfrak{n}$ and $\mu \in \mathfrak{b}$, we see that $\mathfrak{b}^* \simeq \mathfrak{g}/\mathfrak{n} \simeq \mathfrak{b}^-$, hence this restriction map translates to the map

$$\mathrm{Sym}(\mathfrak{g})^G \to \mathrm{Sym}(\mathfrak{b}^-)^H,$$

given by the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$. It is now clear that the $\mathfrak{h}$-invariants of $\mathrm{Sym}(\mathfrak{b}^-)$ lie in $\mathrm{Sym}(\mathfrak{h})$ and hence that the image of f in $\mathrm{Sym}(\mathfrak{b}^-)^H$ lies in $\mathrm{Sym}(\mathfrak{h}) \simeq \mathrm{Sym}(\mathfrak{h}^*)$. Analyzing more carefully, the fact that $N(H)$ normalizes $\mathfrak{h}$, we now see that f lands in the $N(H)$-invariants of $\mathrm{Sym}(\mathfrak{h}^*)$ and thus that this construction defines a map

$$\widetilde{\phi} : \mathrm{Sym}(\mathfrak{g}^*)^G \to \mathrm{Sym}(\mathfrak{h}^*)^W.$$

The map $\widetilde{\phi}$ is known as the *Chevalley homomorphism* and by the following theorem is an isomorphism. We denote the inverse map $\phi : \mathrm{Sym}(\mathfrak{h}^*)^W \to \mathrm{Sym}(\mathfrak{g}^*)^G$.

Theorem 3.13 ([Gai05, Theorem 2.3])**.** *The map $\widetilde{\phi}$ is an isomorphism* $\mathrm{Sym}(\mathfrak{h}^*)^W \simeq \mathrm{Sym}(\mathfrak{g}^*)^G$.

We omit the proof of Theorem 3.13 in favor of an heuristic description of the inverse map ϕ. For a function $f \in \mathrm{Sym}(\mathfrak{h}^*)^W$, set $g = \phi(f)$. By the definition of the Chevalley homomorphism, for any $\xi \in \mathfrak{g}$, we may compute the value of $g(\xi)$ by conjugating ξ into $\mathfrak{b}$ and considering its image under the maps $\mathfrak{b} \to \mathfrak{b}/\mathfrak{n} \simeq \mathfrak{h} \xrightarrow{f} k$. Because f was W-invariant, we see that choosing any Borel $\mathfrak{b}'$ containing both $\mathfrak{h}$ and ξ and applying the same procedure would give the same result. For *regular semisimple* ξ, these will comprise all Borel subalgebras containing ξ, a fact which will come into play later.

3.6.2. *Harish-Chandra isomorphism.* In this subsection, we describe the Harish-Chandra isomorphism, which characterizes the possible actions of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$. By Schur's lemma, the action of $Z(\mathfrak{g})$ on any finite-dimensional irreducible $U(\mathfrak{g})$-module factors through k, meaning that $Z(\mathfrak{g})$ acts via a map of k-algebras $\chi : Z(\mathfrak{g}) \to k$, which we call a *central character.* For any central character χ, define

$$U(\mathfrak{g})_\chi = U(\mathfrak{g})/U(\mathfrak{g}) \cdot \ker \chi$$

to be the quotient of $U(\mathfrak{g})$ by the (two-sided) ideal generated by $\ker \chi$. Note that $U(\mathfrak{g})_\chi$-modules are simply $U(\mathfrak{g})$-modules where $Z(\mathfrak{g})$ acts by χ.

We now classify the central characters χ by relating them to characters $\lambda \in \mathfrak{h}^*$. By Theorem 3.10, given a choice of Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b}$, $U(\mathfrak{g})$ splits as a direct sum

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- \cdot U(\mathfrak{g}) + U(\mathfrak{g}) \cdot \mathfrak{n}), \tag{12}$$

where we note that any pure tensor in $U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n})$ which does not lie in $U(\mathfrak{h})$ lies in at least one of $\mathfrak{n}^- \cdot U(\mathfrak{g})$ or $U(\mathfrak{g}) \cdot \mathfrak{n}$. Let the map

$$\psi_{\mathfrak{b}} : U(\mathfrak{g}) \to U(\mathfrak{h})$$

to be the projection onto $U(\mathfrak{h})$ under this direct sum. We call the restriction of $\psi_{\mathfrak{b}}$ to $Z(\mathfrak{g})$ the *Harish-Chandra homomorphism* relative to $\mathfrak{b}$.[16] The following lemma justifies the name.

Lemma 3.14. *The map $\psi_{\mathfrak{b}}$ is an algebra homomorphism $Z(\mathfrak{g}) \to U(\mathfrak{h})$.*

Proof. This results from a careful analysis of $Z(\mathfrak{g})$ using Theorem 3.10. We claim that $Z(\mathfrak{g}) \cap (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}) \subset U(\mathfrak{g})\mathfrak{n} \cap \mathfrak{n}^- U(\mathfrak{g})$. This would imply that ψ is a map of algebras, as for $i = 1, 2$, if we had $z_i = h_i + g_i$ with $h_i \in U(\mathfrak{h})$ and $g_i \in U(\mathfrak{g})\mathfrak{n} \cap \mathfrak{n}^- U(\mathfrak{g})$, then the decomposition

$$z_1 z_2 = h_1 h_2 + (h_1 g_2 + g_1 h_2 + g_1 g_2)$$

would satisfy $h_1 h_2 \in U(\mathfrak{h})$ and $h_1 g_2 + g_1 h_2 + g_1 g_2 \in U(\mathfrak{g})\mathfrak{n} \cap \mathfrak{n}^- U(\mathfrak{g})$.

It remains to prove the claim. It is evidently symmetric in $\mathfrak{n}$ and $\mathfrak{n}^-$, so it suffices to show that $Z(\mathfrak{g}) \cap (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}) \subset U(\mathfrak{g})\mathfrak{n}$. For this, write any element $z \in Z(\mathfrak{g})$ in the PBW ordering and consider the adjoint action of $\mathfrak{h}$ on z. Then, take any monomial

$$z' = e_{-\alpha_{i_1}} \cdots e_{-\alpha_{i_k}} g f_{\alpha_{j_1}} \cdots f_{\alpha_{j_{k'}}}$$

of z with $g \in U(\mathfrak{h})$ and the α_i simple roots. Then, we see that the adjoint action of $\mathfrak{h}$ on z' is via $\alpha_{j_1} + \cdots + \alpha_{j_{k'}} - \alpha_{i_1} - \cdots - \alpha_{i_k}$, which must vanish, meaning that either $k = k' = 0$ or $k, k' > 0$, giving the claim. □

Viewing $\mathrm{Sym}(\mathfrak{h})$ as the space of polynomial functions on $\mathfrak{h}^*$, we may interpret $\psi_{\mathfrak{b}}$ as a map

$$\mathfrak{h}^* \to \mathrm{MaxSpec}(Z(\mathfrak{g})),$$

where the central characters χ are in bijection with $\mathrm{MaxSpec}(Z(\mathfrak{g}))$. For $\lambda \in \mathfrak{h}^*$, denote by $\chi_\lambda^{\mathfrak{b}}$ the central character corresponding to λ relative to $\mathfrak{b}$. We may immediately understand the following important concrete example of the action of $Z(\mathfrak{g})$ through a central character.

Corollary 3.15. *For any weight $\lambda \in \mathfrak{h}^*$, $Z(\mathfrak{g})$ acts via $\chi_\lambda^{\mathfrak{b}}$ on the $U(\mathfrak{g})$-module*

$$M_\lambda^{\mathfrak{b}} := U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} k_\lambda,$$

which is known as the Verma module of highest weight λ.

Proof. This follows immediately from the fact that $Z(\mathfrak{g}) \cap (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}) \subset U(\mathfrak{g})\mathfrak{n}$, as $\mathfrak{n}$ acts trivially in k_λ, meaning that the action of $Z(\mathfrak{g})$ can be through only the part preserved by $\psi_{\mathfrak{b}}$. □

[16] While the definition of $\psi_{\mathfrak{b}}$ depends on the choice of both $\mathfrak{h}$ and $\mathfrak{b}$, we write $\psi_{\mathfrak{b}}$ instead of $\psi_{\mathfrak{h},\mathfrak{b}}$ because the latter is somewhat cumbersome.

Thus far, we have considered what we call the Harish-Chandra homomorphism *relative to* $\mathfrak{b}$, a map $\psi_{\mathfrak{b}} : Z(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{h})$ which may (and does) depend on the embedding $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ we chose when applying the PBW decomposition. This dependency is not strictly necessary, as both $Z(\mathfrak{g})$ and $\mathfrak{h}$ can be defined abstractly without reference to a specific choice of Cartan or Borel subalgebra. More concretely, what we have done so far does not allow us to compute the action of $Z(\mathfrak{g})$ on Verma modules

$$U(\mathfrak{g}) \underset{U(\mathfrak{b}_x)}{\otimes} k_\lambda$$

relative to arbitrary Borel subalgebras. As we have hinted in our nomenclature thus far, it is possible to modify the construction of $\psi_{\mathfrak{b}}$ to obtain a map $\psi : Z(\mathfrak{g}) \to U(\mathfrak{h})$ that is independent of the choice of Cartan subalgebra. Indeed, define the map ψ, which we will call the *Harish-Chandra homomorphism*, to be the composition

$$Z(\mathfrak{g}) \overset{\psi_{\mathfrak{b}}}{\to} U(\mathfrak{h}) \overset{\xi \mapsto \xi - \rho_{\mathfrak{b}}(\xi)}{\to} U(\mathfrak{h}),$$

where the map $U(\mathfrak{h}) \to U(\mathfrak{h})$ is the one induced by mapping $\mathfrak{h} \to U(\mathfrak{h})$ via $\xi \mapsto \xi - \rho_x(\xi)$. Here, we write $\rho_{\mathfrak{b}}$ for the longest positive root relative to $\mathfrak{b}$ to emphasize that the choice of $\rho_{\mathfrak{b}} \in \mathfrak{g}$ depends on the choice of $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. We see that ψ then admits the following nice characterization, which is known as the *Harish-Chandra isomorphism*.

Theorem 3.16 ([Dix77, Theorem 7.4.5])**.** *The map ψ is an isomorphism $Z(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{h})^W$ which is independent of the choice of $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$.*

We omit the proof of Theorem 3.16 and instead discuss some consequences. Notice that ψ gives a different map

$$\mathfrak{h}^* \to \mathrm{MaxSpec}(Z(\mathfrak{g})). \tag{13}$$

For $\lambda \in \mathfrak{h}^*$, denote now by χ_λ the central character corresponding to λ. We then have the following characterization of the space of central characters.

Corollary 3.17. *We have the following:*

(i) *every central character χ lies in the image of the map (13), and*
(ii) *$\chi_\lambda = \chi_\mu$ if and only if there is some $w \in W$ such that $\lambda = w(\mu)$.*

Proof. This follows formally from the Harish-Chandra isomorphism and the following standard fact about quotients of affine schemes by actions of finite groups (see [Har92, Lecture 10]). Let $\mathrm{Spec}(A)$ be an affine variety over k equipped with the action of a finite group G. Then, the map $\mathrm{MaxSpec}(A) \to \mathrm{MaxSpec}(A^G)$ is the quotient map for the G-action on $\mathrm{MaxSpec}(A)$.

Applying this for $A = \mathrm{Sym}(\mathfrak{h})$ and $G = W$ acting on $\mathrm{Sym}(\mathfrak{h})$, (i) follows because the quotient map is surjective, and (ii) follows because the fibers of the quotient map are exactly W-orbits. □

This provides a significantly more flexible perspective from which to compute the action of $Z(\mathfrak{g})$ on Verma modules corresponding to arbitrary Borel subalgebras. In particular, we have the following two computations which will be crucial later on.

Corollary 3.18. *For any weight $\lambda \in \mathfrak{h}^*$ and any Borel subalgebra $\mathfrak{b}_x$ with corresponding longest positive root $\rho_x := \rho_{\mathfrak{b}_x}$, $Z(\mathfrak{g})$ acts on the Verma module*

$$M_\lambda^{\mathfrak{b}_x} := U(\mathfrak{g}) \underset{U(\mathfrak{b}_x)}{\otimes} k_\lambda$$

of highest weight λ relative to $\mathfrak{b}_x$ via $\chi_{\lambda+\rho_x}$.

Proof. This is a direct translation of Corollary 3.15 into the language of Theorem 3.16. □

Remark. By Corollary 3.18, we see that the action of $Z(\mathfrak{g})$ on the Verma module $M_{\lambda-\rho_x}^{\mathfrak{b}_x}$ is given by χ_λ, which is independent of the choice of $\mathfrak{b}_x$.

Corollary 3.19. *For any weight $\lambda \in \mathfrak{b}^*$ and any Borel subalgebra $\mathfrak{b}_x$ with corresponding longest positive root ρ_x, $Z(\mathfrak{g})$ acts on the right $U(\mathfrak{g})$-module*

$${}_\lambda M^{\mathfrak{b}_x} := k_\lambda \underset{U(\mathfrak{b}_x)}{\otimes} U(\mathfrak{g})$$

via the central character $\chi_{\lambda-\rho_x}$.

Proof. Recall that there is a standard anti-involution $\tau : U(\mathfrak{g}) \to U(\mathfrak{g})$ induced by the map $\xi \mapsto -\xi$ on $\mathfrak{g}$. This map gives rise to an isomorphism $U(\mathfrak{g}) \simeq U(\mathfrak{g})^{\mathrm{op}}$ that fixes $Z(\mathfrak{g})$ pointwise and interchanges $\mathfrak{b}_x$ and $\mathfrak{b}_x^-$ for each x (see [Dix77, Proposition 2.2.17]). Viewing ${}_\lambda M^{\mathfrak{b}_x}$ as a left $U(\mathfrak{g})$-module under this correspondence, we see that it identifies with

$$M_\lambda^{\mathfrak{b}_x^-} = U(\mathfrak{g}) \underset{U(\mathfrak{b}_x^-)}{\otimes} k_\lambda,$$

where $\mathfrak{b}_x^-$ is the Borel subalgebra opposite to $\mathfrak{b}_x$. By Corollary 3.18, $Z(\mathfrak{g})$ acts via $\chi_{\lambda+\rho_x^-} = \chi_{\lambda-\rho_x}$ on $M_\lambda^{\mathfrak{b}_x}$, giving the desired conclusion. □

For context, we translate the content of Corollary 3.17 into our original situation of a fixed chosen Borel subalgebra $\mathfrak{b}$. Define the *dotted action* of W on $\mathfrak{h}^*$ by

$$w \cdot \lambda = w(\lambda + \rho_{\mathfrak{b}}) - \rho_{\mathfrak{b}}$$

for any $\lambda \in \mathfrak{h}^*$. Then, we see that $\chi_\lambda^{\mathfrak{b}} = \chi_\mu^{\mathfrak{b}}$ if and only if λ and μ are in the same W-orbit under the dotted action.

4. Beilinson-Bernstein localization

In this section, we may now state and prove the main result of this essay, the Beilinson-Bernstein localization theorem. This result relates the category of modules over a sheaf $\mathcal{D}_X^\lambda$ of twisted differential operators on the flag variety X of a semisimple algebraic group G to the category of representations of its Lie algebra $\mathfrak{g}$ acting with a given central character. The proof divides into two main steps. First, we identify the global sections of $\mathcal{D}_X^\lambda$ with a quotient of the universal enveloping algebra of $\mathfrak{g}$. The proof passes to the associated graded rings and crucially involves Kostant's theorem on the ring of regular functions of the cone of nilpotent elements in $\mathfrak{g}$. We give a proof of this using the *Springer* resolution of the nilpotent cone.

Second, we show that the category of $\mathcal{D}_X^\lambda$-modules is equivalent to the category of modules over the global sections $\Gamma(\mathcal{D}_X^\lambda)$. The main idea of the proof is to tensor a $\mathcal{D}_X^\lambda$-module with known vector bundles associated to G-representations to create a result on which the global sections functor has nice properties. A splitting trick of [BB81] then allows us to pass back to the original module.

We must discuss a fine point of notational convention before we proceed. Throughout this section, we will assume that we have a fixed choice of a distinguished Borel subgroup $B \subset G$ and the corresponding Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, which will correspond to the set of positive roots R^+. The positivity, dominance, and regularity of weights will be taken with respect to this choice.

We draw from a number of sources for this section. For preliminary constructions involving twisted differential operators and Lie algebroids, we refer to the original papers [BB81], [BB93], and [Kas89]. Our treatment of Kostant's theorem and the geometry of the nilpotent cone is based primarily on that of [Gai05], though we appeal to [BL96], [Kos59], and [HTT08] to fill in some details of the proofs. Finally, for the proof that the global sections functor gives an equivalence of categories, we follow the original proof of [BB81], though we consulted [HTT08] and [Kas89] to understand some of the more subtle points.

4.1. **Twisted differential operators.**

Before we state the Beilinson-Bernstein correspondence, we must first introduce a generalization of the sheaf of differential operators.

Definition 4.1. For a smooth algebraic variety X, a sheaf of *twisted differential operators* on X is a sheaf $\mathcal{D}$ of $\mathcal{O}_X$-algebras on X equipped with a filtration $\{F_i\mathcal{D}\}_{i\geq 0}$ such that

(i) the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{D}$ gives an isomorphism $\mathcal{O}_X \simeq F_0\mathcal{D}$,
(ii) the natural map $\mathrm{Sym}_{\mathcal{O}_X} F_1\mathcal{D}/F_0\mathcal{D} \to \mathrm{gr}\,\mathcal{D}$ is an isomorphism of $\mathcal{O}_X$-algebras, and
(iii) the map $F_1\mathcal{D}/F_0\mathcal{D} \to T_X$ defined by

$$\xi \mapsto \Big(f \mapsto \xi f - f\xi\Big)$$

is an isomorphism.

Remark. We interpret condition (iii) in Definition 4.1 as follows. For sections $\xi \in F_1\mathcal{D}$ and $f \in \mathcal{O}_X \simeq F_0\mathcal{D}$, the commutativity of $\mathrm{gr}\,\mathcal{D}$ ensures that $f \mapsto \xi f - f\xi \in F_0\mathcal{D} \simeq \mathcal{O}_X$.

Definition 4.2. For a sheaf of twisted differential operators $\mathcal{D}$, a $\mathcal{D}$-*module* on X is a quasicoherent sheaf equipped with a $\mathcal{D}$-action compatible with the $\mathcal{O}_X$-action via the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{D}$.

By Propositions 2.4 and 2.5, we see that $\mathcal{D}_X$ is a sheaf of twisted differential operators on X. This is an instance of the following more general construction. For $\mathcal{L}$ a line bundle on X, define a *$\mathcal{L}$-twisted differential operator on X* of order at most n to be a k-linear map $d : \mathcal{L} \to \mathcal{L}$ such that for sections $f_0, f_1, \dots, f_n \in \mathcal{O}_X$, we have

$$[f_n, [f_{n-1}, [\cdots, [f_0, d]]]] = 0$$

as a k-linear map $\mathcal{L} \to \mathcal{L}$, where a section $f \in \mathcal{O}_X$ gives rise to a k-linear map $\mathcal{L} \to \mathcal{L}$ via the $\mathcal{O}_X$-action on $\mathcal{L}$. These operators form a sheaf of rings $\mathcal{D}_{X,\mathcal{L}}$ on X, which we call the *sheaf of $\mathcal{L}$-twisted differential operators* on X.[17] Notice that a $\mathcal{O}_X$-twisted differential operator on X is simply what we previously termed a differential operator on X, and hence that $\mathcal{D}_{X,\mathcal{O}_X} = \mathcal{D}_X$. In analogy with $\mathcal{D}_X$, we endow $\mathcal{D}_{X,\mathcal{L}}$ with the order filtration $F_n\mathcal{D}_{X,\mathcal{L}}$. By the following proposition, this construction results in a sheaf of twisted differential operators.

Proposition 4.3. *For any line bundle $\mathcal{L}$ on X, $\mathcal{D}_{X,\mathcal{L}}$ is a sheaf of twisted differential operators on X.*

Proof. We check the conditions in Definition 4.1, though in a different order. For (i), an element of $F_0\mathcal{D}_{X,\mathcal{L}}$ is simply an $\mathcal{O}_X$-linear map $\mathcal{L} \to \mathcal{L}$, so $F_0\mathcal{D}_{X,\mathcal{L}} \simeq \mathcal{O}_X$ because $\mathcal{L}$ was a line bundle. For (ii) and (iii), it is enough to check this locally, in which case $\mathcal{L}|_U \simeq \mathcal{O}_U$ and the statements follow from the corresponding ones for $\mathcal{D}_U$ (Propositions 2.4 and 2.5). □

By construction, $\mathcal{L}$ is a $\mathcal{D}_{X,\mathcal{L}}$-module, and for $\mathcal{L} = \mathcal{O}_X$, the $\mathcal{D}_{X,\mathcal{L}}$-module structure induced on $\mathcal{O}_X$ is simply the natural $\mathcal{D}_X$-module structure of Example 2.11.

4.2. **Lie algebroids and enveloping algebras on the flag variety.** Let us now restrict to the case where $X = G/B$ is the flag variety of a connected, simply connected, semisimple algebraic group G. We now describe a way to understand the notion of a sheaf of $\mathfrak{g}$-modules on X. The key advantage of this notion over simply considering the $\mathfrak{g}$-module alone will be the ability to consider all Borel subalgebras and subgroups of $\mathfrak{g}$ and G at once.

Define the *Lie algebroid* $\widetilde{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$ to be a sheaf of Lie algebras isomorphic to $\mathcal{O}_X \otimes_k \mathfrak{g}$ as a $\mathcal{O}_X$-module and with Lie bracket $[-,-] : \widetilde{\mathfrak{g}} \otimes_k \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ extending the bracket on $\mathfrak{g}$ such that for $f \in \mathcal{O}_X$ and $\xi_1, \xi_2 \in \mathfrak{g}$, we have

$$[\xi_1, f \cdot \xi_2] = f[\xi_1, \xi_2] + \rho(\xi_1)(f)\xi_2.$$

Then, define the *universal enveloping algebra* $\mathcal{U}(\widetilde{\mathfrak{g}})$ to be the enveloping algebra of $\widetilde{\mathfrak{g}}$. That is, it is a sheaf of $\mathcal{O}_X$-algebras isomorphic to $\mathcal{O}_X \otimes_k U(\mathfrak{g})$ as a quasi-coherent $\mathcal{O}_X$-module with multiplicative structure given by extending the structure on $U(\mathfrak{g})$ subject to the twisting relation

$$[\xi, f] = \rho(\xi)(f)$$

for $\xi \in \mathfrak{g}$ and $f \in \mathcal{O}_X$. Recall here that $\rho : \mathfrak{g} \to T_X$ is the map (10) induced by the G-action on X; it naturally extends to a map of $\mathcal{O}_X$-algebras

$$\widetilde{\rho} : \mathcal{U}(\widetilde{\mathfrak{g}}) \to \mathcal{D}_X. \tag{14}$$

For any G-module V, let $\mathcal{V} := \mathcal{O}_X \otimes_k V$ be the G-equivariant sheaf associated to V as a B-module. Then, $\mathcal{V}$ acquires the structure of a $\mathcal{U}(\widetilde{\mathfrak{g}})$-module by differentiating the action of G; explicitly, $\xi \in \mathfrak{g}$, viewed as a local section of $\mathcal{U}(\widetilde{\mathfrak{g}})$, acts by

$$\xi \cdot (f \otimes v) = \rho(\xi)(f) \otimes v + f \otimes \xi \cdot v$$

on a local section $f \otimes v$ of $\mathcal{V}$. When V is only a $U(\mathfrak{g})$-module, this action still exhibits $\mathcal{V}$ as a $\mathcal{U}(\widetilde{\mathfrak{g}})$-module, though it is no longer a G-equivariant sheaf. Conversely, for any $\mathcal{U}(\widetilde{\mathfrak{g}})$-module $\mathcal{M}$, its local sections $\Gamma(U, \mathcal{M})$ acquire the structure of $U(\mathfrak{g})$-modules via the inclusion $U(\mathfrak{g}) \hookrightarrow \Gamma(U, \mathcal{O}_X) \otimes_k U(\mathfrak{g})$. For two $\mathcal{U}(\widetilde{\mathfrak{g}})$-modules $\mathcal{M}_1$ and $\mathcal{M}_2$, we may equip $\mathcal{M}_1 \otimes \mathcal{M}_2$ with the structure of a $\mathcal{U}(\widetilde{\mathfrak{g}})$-module by letting a local section $\xi \in \mathcal{U}(\widetilde{\mathfrak{g}})$ act via

$$\xi \cdot (m_1 \otimes m_2) = \xi \cdot m_1 \otimes m_2 + m_1 \otimes \xi \cdot m_2$$

on a local section $m_1 \otimes m_2 \in \mathcal{M}_1 \otimes \mathcal{M}_2$. It is easy to check that this defines a valid action of $\mathcal{U}(\widetilde{\mathfrak{g}})$ and that the operation $\mathcal{M}_1 \otimes -$ is functorial.

Define the center $\mathcal{Z}(\widetilde{\mathfrak{g}})$ of $\mathcal{U}(\widetilde{\mathfrak{g}})$ by applying the corresponding construction for $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ and view $\mathcal{Z}(\widetilde{\mathfrak{g}})$ as a subsheaf of $\mathcal{U}(\widetilde{\mathfrak{g}})$. Further, restricting the map (14) to $\widetilde{\mathfrak{g}} \subset \mathcal{U}(\widetilde{\mathfrak{g}})$, we define

$$\widetilde{\mathfrak{b}} := \ker(\widetilde{\rho} : \widetilde{\mathfrak{g}} \to T_X)$$

[17]If the line bundle $\mathcal{L}$ is replaced by a vector bundle $\mathcal{M}$, we may still define a sheaf of $\mathcal{M}$-twisted differential operators. However, it will not be a sheaf of twisted differential operators in the sense of Definition 4.1.

and $\widetilde{\mathfrak{n}} := [\widetilde{\mathfrak{b}}, \widetilde{\mathfrak{b}}]$. Observe that $\widetilde{\mathfrak{b}}$ and $\widetilde{\mathfrak{n}}$ are subsheaves of $\widetilde{\mathfrak{g}}$. We may describe $\widetilde{\mathfrak{b}}$ explicitly as

$$\widetilde{\mathfrak{b}} = \{f \in \widetilde{\mathfrak{g}} \mid f(x) \in \mathfrak{b}_x \text{ for all } x \in X\},$$

where here $\mathfrak{b}_x \subset \mathfrak{g}$ is the Borel subalgebra corresponding to $x \in X$. As a result, we see that

$$\widetilde{\mathfrak{n}} = \{f \in \widetilde{\mathfrak{g}} \mid f(x) \in \mathfrak{n}_x \text{ for all } x \in X\},$$

and that $\widetilde{\mathfrak{b}}/\widetilde{\mathfrak{n}} \simeq \widetilde{\mathfrak{h}} := \mathcal{O}_X \underset{k}{\otimes} \mathfrak{h}$. For a map $\lambda : \widetilde{\mathfrak{b}} \to \mathcal{O}_X$ and a $\mathcal{U}(\widetilde{\mathfrak{g}})$-module $\mathcal{M}$, we say that $\mathcal{U}(\widetilde{\mathfrak{b}}) \subset \mathcal{U}(\widetilde{\mathfrak{g}})$ *acts via* λ on $\mathcal{M}$ if the action of $\mathcal{U}(\widetilde{\mathfrak{b}})$ factors through λ. If $\lambda \in \mathfrak{h}^*$ is a weight, we denote the map $\widetilde{\mathfrak{b}} \to \widetilde{\mathfrak{b}}/\widetilde{\mathfrak{n}} \simeq \widetilde{\mathfrak{h}} \xrightarrow{\lambda} \mathcal{O}_X$ it induces also by λ.[18]

4.3. **A family of twisted differential operators.** Recall from Subsection 3.3 that for each character $e^{-\lambda} \in X^*(T)$ we have a G-equivariant line bundle $\mathcal{L}^\lambda$ given by the corresponding one-dimensional representation of B. Specializing the previous construction, we note that $\mathcal{D}_{X,\mathcal{L}^\lambda}$ is a sheaf of twisted differential operators on X. This construction for $\mathcal{D}_{X,\mathcal{L}^\lambda}$ requires λ to be a integral weight (so that it could lift to a character of T). We now extend it to a family $\mathcal{D}_X^\lambda$ of sheaves of twisted differential operators parametrized by any weight $\lambda \in \mathfrak{h}^*$.[19]

Let $\lambda \in \mathfrak{h}^*$ be an arbitrary weight. We construct a sheaf $\mathcal{D}_X^\lambda$ as follows. Consider the map $\rho : \widetilde{\mathfrak{b}} \to \mathcal{O}_X$ given on each fiber by taking the action of ρ_x on $\mathfrak{b}_x$. Define the *right* ideal $\mathcal{I}^\lambda(\widetilde{\mathfrak{g}}) \subset \mathcal{U}(\widetilde{\mathfrak{g}})$ generated by the local sections

$$\xi - (\rho - \lambda)(\xi) \in \mathcal{U}(\widetilde{\mathfrak{g}})$$

for $\xi \in \widetilde{\mathfrak{b}}$. The following lemma shows that it is possible to quotient by $\mathcal{I}^\lambda(\widetilde{\mathfrak{g}})$.

Lemma 4.4. *For $\lambda \in \mathfrak{h}^*$ an arbitrary weight, $\mathcal{I}^\lambda(\widetilde{\mathfrak{g}})$ is a two-sided ideal.*

Proof. It suffices for us to show that $[\widetilde{\mathfrak{b}}, \widetilde{\mathfrak{g}}] \subset \widetilde{\mathfrak{b}}$. But this follows because the map $\widetilde{\rho}$ of (14) was a map of $\mathcal{O}_X$-algebras, hence for any $\xi \in \widetilde{\mathfrak{b}}$ and $\mu \in \widetilde{\mathfrak{g}}$, we have

$$\widetilde{\rho}([\xi, \mu]) = \widetilde{\rho}(\xi)\widetilde{\rho}(\mu) - \widetilde{\rho}(\mu)\widetilde{\rho}(\xi) = 0,$$

hence $[\xi, \mu] \in \widetilde{\mathfrak{b}}$. □

Remark. Lemma 4.4 illustrates the essential role that sheaves play in the theory. In particular, a naive definition of an analogous left ideal $I^\lambda(\mathfrak{g}) \subset U(\mathfrak{g})$ generated by $\xi - (\rho - \lambda)(\xi)$ for $\xi \in \mathfrak{b}$ does not result in a two-sided ideal. The problem is that $\mathfrak{b} \subset \mathfrak{g}$ is *not* globally the kernel of the map $\mathfrak{g} \to \Gamma(X, T_X)$. By considering instead the sheaf $\widetilde{\mathfrak{b}}$, we are able to detect more local behavior.

Now, define the sheaf of $\mathcal{O}_X$-algebras $\mathcal{D}_X^\lambda$ to be the quotient

$$\mathcal{D}_X^\lambda := \mathcal{U}(\widetilde{\mathfrak{g}})/\mathcal{I}^\lambda(\widetilde{\mathfrak{g}}),$$

and define the maps

$$\Phi_\lambda : \mathcal{U}(\widetilde{\mathfrak{g}}) \to \mathcal{D}_X^\lambda \tag{15}$$

and

$$\phi_\lambda : U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X^\lambda) \tag{16}$$

to be the projection map and the map given by composing the action of Φ_λ on global sections with the natural inclusion $U(\mathfrak{g}) \hookrightarrow \Gamma(X, \mathcal{O}_X) \otimes_k U(\mathfrak{g})$, respectively.

Proposition 4.5. *For any weight $\lambda \in \mathfrak{h}^*$, $\mathcal{D}_X^\lambda$ is a sheaf of twisted differential operators.*

Proof. We check each property in turn. Condition (i) is obvious. For condition (iii), note that when restricted to $F_1\mathcal{U}(\widetilde{\mathfrak{g}})/F_0\mathcal{U}(\widetilde{\mathfrak{g}}) \simeq \widetilde{\mathfrak{g}}$, the image of the inclusion $\mathcal{I}^\lambda(\widetilde{\mathfrak{b}}) \to \mathcal{U}(\widetilde{\mathfrak{g}})$ is simply $\widetilde{\mathfrak{b}}$, hence the isomorphism follows from the exact sequence $0 \to \widetilde{\mathfrak{b}} \to \widetilde{\mathfrak{g}} \to T_X \to 0$. Finally, for condition (ii), note that the image of $\mathcal{I}^\lambda(\widetilde{\mathfrak{g}})$ in $F_i\mathcal{U}(\widetilde{\mathfrak{g}})/F_{i-1}\mathcal{U}(\widetilde{\mathfrak{g}}) \simeq \mathrm{Sym}^i_{\mathcal{O}_X} \widetilde{\mathfrak{g}}$ is given by $\widetilde{\mathfrak{b}} \cdot \mathrm{Sym}^i_{\mathcal{O}_X} \widetilde{\mathfrak{g}}$, hence we see that

$$\mathrm{gr}^i\, \mathcal{U}(\widetilde{\mathfrak{g}})/\mathcal{I}^\lambda(\widetilde{\mathfrak{g}}) \simeq \mathrm{Sym}^i_{\mathcal{O}_X} T_X,$$

as needed, where we are using the fact that $\widetilde{\mathfrak{g}}/\widetilde{\mathfrak{b}} \simeq T_X$. □

Corollary 4.6. *For any weight $\lambda \in \mathfrak{h}^*$, $\mathcal{U}(\widetilde{\mathfrak{b}})$ acts via $\rho - \lambda$ on $\mathcal{D}_X^\lambda$.*

---

[18] We refer the reader to [BB93] for more information about constructions of this sort and their generalizations.

[19] There is a more general theory of twisted differential operators, for which we refer the reader to [BB93]. However, we will in this essay only be concerned with sheaves of twisted differential operators of the form $\mathcal{D}_X^\lambda$.

Proof. This follows because the action of $\mathcal{U}(\widetilde{\mathfrak{b}})$ is by applying Φ_λ and the multiplying on the left. $\square$

Now, let $\lambda \in \mathfrak{h}^*$ be an integral weight. Because $\mathcal{L}^\lambda$ is G-equivariant, by Corollary 3.12 we have a map $\phi_\lambda : U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_{X,\mathcal{L}^{\lambda-\rho}})$. Together with the inclusion $\mathcal{O}_X \to \mathcal{D}_{X,\mathcal{L}^{\lambda-\rho}}$, this defines a map of $\mathcal{O}_X$-algebras $\Psi_\lambda : \mathcal{U}(\widetilde{\mathfrak{g}}) \to \mathcal{D}_{X,\mathcal{L}^{\lambda-\rho}}$. We would like to show that Ψ_λ realizes $\mathcal{D}_{X,\mathcal{L}^{\lambda-\rho}}$ as exactly the quotient $\mathcal{U}(\widetilde{\mathfrak{g}})/\mathcal{I}^\lambda(\widetilde{\mathfrak{g}})$ of $\mathcal{U}(\widetilde{\mathfrak{g}})$, which would mean that our new family $\mathcal{D}_X^\lambda$ extends the family $\mathcal{D}_{X,\mathcal{L}^{\lambda-\rho}}$.

Lemma 4.7. *Let $\lambda \in \mathfrak{h}^*$ be an integral weight. Then, the map $\Psi_\lambda : \mathcal{U}(\widetilde{\mathfrak{g}}) \to \mathcal{D}_{X,\mathcal{L}^{\lambda-\rho}}$ defines an isomorphism*

$$\mathcal{D}_X^\lambda \to \mathcal{D}_{X,\mathcal{L}^{\lambda-\rho}},$$

where here we take ρ corresponding to the choice of B used to construct $\mathcal{L}^{\lambda-\rho}$.

Proof. First, we claim that the map $\mathcal{U}(\widetilde{\mathfrak{g}}) \to \mathcal{D}_{X,\mathcal{L}^{\lambda-\rho}}$ kills $\mathcal{I}^\lambda(\widetilde{\mathfrak{g}})$. For this, it suffices to note that for each $x \in X$, the corresponding Borel subalgebra $\mathfrak{b}_x$ acts on the fiber $\mathcal{L}_x^\lambda$ by $e^{\rho_x - \lambda}$ (this is ρ_x instead of ρ because we are on $\mathfrak{b}_x$ instead of $\mathfrak{b}$). The resulting map is an isomorphism by the following commutative diagram on the level of the associated graded

$$\begin{tikzcd} \operatorname{Sym}_{\mathcal{O}_X} T_X & \operatorname{gr}\mathcal{D}_{X,\mathcal{L}^\lambda} \arrow[l] \\ \operatorname{gr}\mathcal{D}_X^\lambda, \arrow[u] \arrow[ur] & \end{tikzcd}$$

where the top and left maps are isomorphisms by Propositions 4.3 and 4.5. $\square$

4.4. **The localization functor and the localization theorem.** Given a $\mathcal{D}_X^\lambda$-module $\mathcal{M}$ on X, the map (16) endows its global sections $\Gamma(X, \mathcal{M})$ with the structure of a $U(\mathfrak{g})$-module. Conversely, given a $U(\mathfrak{g})$-module M, the map ϕ_λ of (16) allows us to construct the $\mathcal{D}_X^\lambda$-module

$$\operatorname{Loc}_\lambda(M) := \mathcal{D}_X^\lambda \underset{U(\mathfrak{g})}{\otimes} M,$$

where $U(\mathfrak{g})$ is viewed as a (locally) constant sheaf of algebras on X, M is viewed as the corresponding (locally) constant sheaf of modules over $U(\mathfrak{g})$, and the map $U(\mathfrak{g}) \to \mathcal{D}_X^\lambda$ is induced by ϕ_λ.[20] We call $\operatorname{Loc}_\lambda(M)$ the *localization* of M and $\operatorname{Loc}_\lambda$ a *localization functor*. The following shows that it is left adjoint to the global sections functor $\Gamma := \Gamma(X, -)$.

Proposition 4.8. *The functors* $\operatorname{Loc}_\lambda : U(\mathfrak{g}) - mod \rightleftarrows \mathcal{D}_X^\lambda - mod : \Gamma$ *are adjoint.*

Proof. This follows from the following chain of natural isomorphisms of bifunctors

$$\operatorname{Hom}_{\mathcal{D}_X^\lambda}(\operatorname{Loc}_\lambda(-), -) = \operatorname{Hom}_{\mathcal{D}_X^\lambda}(\mathcal{D}_X^\lambda \underset{U(\mathfrak{g})}{\otimes} -, -) \simeq \operatorname{Hom}_{U(\mathfrak{g})}(-, \operatorname{Hom}_{\mathcal{D}_X^\lambda}(\mathcal{D}_X^\lambda, -)) \simeq \operatorname{Hom}_{U(\mathfrak{g})}(-, \Gamma(X, -)).$$

$\square$

To further characterize the relationship between $\operatorname{Loc}_\lambda$ and Γ, we must analyze ϕ_λ more carefully. By the following theorem, whose proof we defer to Subsection 4.5, ϕ_λ allows us to identify $\Gamma(X, \mathcal{D}_X^\lambda)$ with $U(\mathfrak{g})_\chi$ for $\chi = \chi_{-\lambda}$.

Theorem 4.9. *The map $\phi_\lambda : U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X^\lambda)$ defines an isomorphism*

$$U(\mathfrak{g})_{\chi_{-\lambda}} \simeq \Gamma(X, \mathcal{D}_X^\lambda).$$

An immediate consequence of Theorem 4.9 is the following.

Corollary 4.10. *If M is a $U(\mathfrak{g})_\chi$-module for some $\chi \neq \chi_{-\lambda}$, then* $\operatorname{Loc}_\lambda(M) = 0$.

Proof. Take some $z \in Z(\mathfrak{g})$ with $\chi_\lambda(z) \neq \chi(z)$. For any local section $\xi \otimes m$ of $\operatorname{Loc}_\lambda(M)$, we have

$$\xi \otimes m = \frac{1}{\chi_{-\lambda}(z) - \chi(z)} \cdot \xi \cdot (z - \chi(z)) \otimes m = \frac{1}{\chi_{-\lambda}(z) - \chi(z)} \xi \otimes (z - \chi(z)) \cdot m = 0,$$

where the first equality follows from Theorem 4.9. We conclude that $\operatorname{Loc}_\lambda(M) = 0$. $\square$

[20]Equivalently, the map factors as $U(\mathfrak{g}) \to \mathcal{U}(\widetilde{\mathfrak{g}}) \to \mathcal{D}_X^\lambda$.

Thus, we see that Loc_λ kills the subcategories $U(\mathfrak{g})_\chi - \mathrm{mod}$ of $U(\mathfrak{g}) - \mathrm{mod}$ unless $\chi = \chi_{-\lambda}$. On the other hand, by Theorem 4.9, Γ is a functor $\mathcal{D}_X^\lambda - \mathrm{mod} \to U(\mathfrak{g})_{\chi_{-\lambda}} - \mathrm{mod}$.[21] The following theorem and its corollary, which is the main result of this essay, show that when λ is regular, these functors give an equivalence of categories. We defer the proof of Theorem 4.11 to Subsection 4.7.

Theorem 4.11. *Let $\lambda \in \mathfrak{h}^*$ be an arbitrary weight. Then, we have the following:*
(i) *if λ is dominant, then $\Gamma : \mathcal{D}_X^\lambda - mod \to U(\mathfrak{g})_{\chi_{-\lambda}} - mod$ is exact, and*
(ii) *if λ is regular, for $\Gamma : \mathcal{D}_X^\lambda - mod \to U(\mathfrak{g})_{\chi_{-\lambda}} - mod$, if $\Gamma(X, \mathcal{F}) = 0$, then $\mathcal{F} = 0$.*[22]

Corollary 4.12 (Beilinson-Bernstein localization)**.** *If λ is regular, then*
$$\mathrm{Loc}_\lambda : U(\mathfrak{g})_{\chi_{-\lambda}} - mod \rightleftharpoons \mathcal{D}_X^\lambda - mod : \Gamma$$
is an equivalence of categories.

Proof. This follows formally from Theorem 4.11. Indeed, by Proposition 4.8, Loc_λ and Γ are adjoint, hence it suffices to check that the unit and counit maps are isomorphisms.

First, consider the map $\mathrm{Loc}_\lambda \circ \Gamma \to \mathrm{id}$. It becomes an isomorphism after application of Γ by the adjunction. However, Γ is exact and conservative by Theorem 4.11, which implies that the original map is an isomorphism. Indeed, for any $\mathcal{M} \in U(\mathfrak{g})_{\chi_\lambda} - \mathrm{mod}$, taking the exact sequence
$$0 \to K \to \mathrm{Loc}_\lambda(\Gamma(X, \mathcal{M})) \to \mathcal{M} \to C \to 0$$
gives an exact sequence
$$0 \to \Gamma(X, K) \to \Gamma(X, \mathrm{Loc}_\lambda(\Gamma(X, \mathcal{M}))) \to \Gamma(X, \mathcal{M}) \to \Gamma(X, C) \to 0$$
with $\Gamma(X, \mathrm{Loc}_\lambda(\Gamma(X, \mathcal{M}))) \simeq \Gamma(X, \mathcal{M})$, which shows that $\Gamma(X, K) = \Gamma(X, C) = 0$, hence $C = K = 0$. Next, consider the map $\mathrm{id} \to \Gamma \circ \mathrm{Loc}_\lambda$. For any $M \in U(\mathfrak{g})_{\chi_{-\lambda}} - \mathrm{mod}$, take a free resolution
$$U(\mathfrak{g})^I_{\chi_{-\lambda}} \to U(\mathfrak{g})^J_{\chi_{-\lambda}} \to M \to 0$$
for M for some index sets I and J. Now, by Theorem 4.11 $\Gamma \circ \mathrm{Loc}_\lambda$ is right exact as the composition of an exact functor and a left adjoint, hence we obtain the commutative diagram of right exact sequences
$$\begin{tikzcd}
U(\mathfrak{g})^I_{\chi_{-\lambda}} \arrow[r] \arrow[d, "\sim"'] & U(\mathfrak{g})^J_{\chi_{-\lambda}} \arrow[r] \arrow[d, "\sim"'] & M \arrow[r] \arrow[d] & 0 \\
U(\mathfrak{g})^I_{\chi_{-\lambda}} \arrow[r] & U(\mathfrak{g})^J_{\chi_{-\lambda}} \arrow[r] & \Gamma(X, \mathrm{Loc}_\lambda(M)) \arrow[r] & 0
\end{tikzcd}$$
where the first two columns are isomorphisms. By the five lemma, $M \to \Gamma(X, \mathrm{Loc}_\lambda(M))$ is an isomorphism, completing the proof. ☐

At first glance, Corollary 4.12 applies only to weights λ lying in the principal Weyl chamber $\{\mu \in \mathfrak{h}^* \mid \langle \mu, \alpha_i^\vee \rangle > 0\}$. However, because W acts simply transitively on the Weyl chambers, for any weight λ which avoids the root hyperplanes $\{\lambda \in \mathfrak{h}^* \mid \langle \mu, \alpha_i^\vee \rangle = 0\}$, we may find a unique $w \in W$ such that $\mu = w(\lambda)$ is regular. Therefore, Corollaries 3.17 and 4.12 give equivalences of categories
$$U(\mathfrak{g})_{\chi_{-\lambda}} - \mathrm{mod} = U(\mathfrak{g})_{\chi_{-\mu}} - \mathrm{mod} \simeq \mathcal{D}_X^\mu - \mathrm{mod}.$$

Remark. We summarize here the relevant notational conventions underlying the statement of Corollary 4.12. Recall that we chose the positive roots R^+ to correspond to the Borel subgroup B, and the G-equivariant line bundle $\mathcal{L}^\lambda$ to correspond to the character $e^{-\lambda}$ rather than e^λ (we make this choice so that dominant and regular λ correspond to globally generated and ample $\mathcal{L}^\lambda$ in Proposition 3.8). In particular, the $\mathfrak{b}$-action on local sections of $\mathcal{L}^\lambda$ is via the weight $-\lambda$ rather than λ. We emphasize, however, that under our definition of
$$\mathcal{D}_X^\lambda = \mathcal{U}(\mathfrak{g})/\langle \xi - (\rho - \lambda)(\xi) \rangle \mathcal{U}(\mathfrak{g}),$$

---

[21]This fact places Loc_λ in analogy with the following situation for schemes which explains why it is known as the localization functor. For X a scheme, there is a left adjoint $\mathrm{Loc}_X : \Gamma(X, \mathcal{O}_X) - \mathrm{mod} \to \mathrm{QCoh}(X)$ to the functor of global sections given by $\mathrm{Loc}_X(M) := \mathcal{O}_X \underset{\Gamma(X, \mathcal{O}_X)}{\otimes} M$. For quasicoherent sheaves, Loc_X is an equivalence if and only if X is affine; however, for $\mathcal{D}$-modules, the key consequence of the Beilinson-Bernstein correspondence is that the localization functor is an equivalence more frequently.

[22]Recall that a functor F is called *conservative* if, for $F(f)$ an isomorphism, f is an isomorphism. When F is an exact functor between abelian categories, the condition of (ii) implies that F is conservative.

the sheaf $\mathcal{D}_X^\lambda$ corresponds to $\mathcal{D}_{X,\mathcal{L}^{\lambda-\rho}}$. In particular, we have that $\mathcal{D}_X^\rho \simeq \mathcal{D}_X$. As we shall see in the proof of Proposition 4.13, this shift is necessary to ensure that $\mathbb{Z}(\mathfrak{g})$ acts by the same central character on all fibers of $\mathcal{D}_X^\lambda$. Finally, we note the action of $Z(\mathfrak{g})$ on the global sections of $\mathcal{D}_X^\lambda$-modules is via $\chi_{-\lambda}$ (rather than χ_λ).

4.5. **The map $U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X^\lambda)$.** In this subsection, we prove Theorem 4.9 subject to an algebraic statement related to the geometry of the cone of nilpotent elements in $\mathfrak{g}$, which we defer to the next subsection. We begin by checking that the statement of the theorem makes sense.

Proposition 4.13. *The map $\phi_\lambda : U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X^\lambda)$ factors through $U(\mathfrak{g})_{\chi_{-\lambda}}$.*

Proof. We recall that the map ϕ_λ was obtained by considering

$$U(\mathfrak{g}) \hookrightarrow \Gamma(X, \mathcal{U}(\widetilde{\mathfrak{g}})) \to \Gamma(X, \mathcal{D}_X^\lambda).$$

Letting $\mathcal{J}_{-\lambda}(\widetilde{\mathfrak{g}}) \subset \mathcal{U}(\widetilde{\mathfrak{g}})$ be the ideal generated by $z - \chi_{-\lambda}(z)$ for $z \in \mathcal{Z}(\widetilde{\mathfrak{g}})$, it suffices for us to show that the composition

$$\mathcal{J}_{-\lambda}(\widetilde{\mathfrak{g}}) \hookrightarrow \mathcal{U}(\widetilde{\mathfrak{g}}) \to \mathcal{D}_X^\lambda$$

is zero. In fact, it suffices to check this on fibers. For each $x \in X$, by right exactness of $k_x \otimes -$, we see that

$$k_x \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X^\lambda = k_x \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X^\lambda \simeq U(\mathfrak{g})/(k_x \underset{\mathcal{O}_X}{\otimes} \mathcal{I}^\lambda(\widetilde{\mathfrak{g}})),$$

where we observe that $k_x \underset{\mathcal{O}_X}{\otimes} \mathcal{I}^\lambda(\widetilde{\mathfrak{g}})$ is the *right ideal* of $U(\mathfrak{g})$ generated by $\xi - (\rho_x - \lambda)(\xi)$ for $\xi \in \mathfrak{b}_x$.[23] In other words, we have shown that

$$k_x \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X^\lambda \simeq k_{\rho_x - \lambda} \underset{U(\mathfrak{b}_x)}{\otimes} U(\mathfrak{g}).$$

By Corollary 3.19, the action of $Z(\mathfrak{g})$ on this *right* $U(\mathfrak{g})$-representation is by $\chi_{\rho_x - \lambda - \rho_x} = \chi_{-\lambda}$, hence $\mathcal{J}_{-\lambda}(\widetilde{\mathfrak{g}})_x$ dies on each fiber, as needed. □

Remark. The proof of Proposition 4.13 allows us to identify

$$\Gamma(X, k_x \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X^\lambda) \simeq k_{\rho_x - \lambda} \underset{U(\mathfrak{b}_x)}{\otimes} U(\mathfrak{g})$$

as a right Verma module which under the isomorphism between $U(\mathfrak{g})$ and $U(\mathfrak{g})^{\mathrm{op}}$ will transform to a dual Verma module.

By Proposition 4.13, ϕ_λ induces a map $U(\mathfrak{g})_{\chi_{-\lambda}} \to \Gamma(X, \mathcal{D}_X^\lambda)$, which we will denote by ψ_λ. We would like to show that ψ_λ is an isomorphism. For this, we will require the following algebraic consequence, Proposition 4.14, of a theorem of Kostant, Theorem 4.19. Kostant's theorem will require some additional geometric background to state, so for now we describe only Proposition 4.14 and how it may be used to prove Theorem 4.9. In the next subsection, we will state and prove Kostant's theorem and then use it to derive Proposition 4.14.

Recall now that the G-action on X gives rise to a map $\rho : \mathfrak{g} \to \Gamma(X, T_X)$ of (10) given by differentiating the action. This map induces a map $\mathrm{Sym}(\mathfrak{g}) \to \Gamma(X, \mathrm{Sym}_{\mathcal{O}_X}(T_X))$, which we may characterize as follows.

Proposition 4.14. *Let $\mathrm{Sym}(\mathfrak{g})^G_+$ denote the elements of positive grade in $\mathrm{Sym}(\mathfrak{g})^G$. The map*

$$\mathrm{Sym}(\mathfrak{g})/\mathrm{Sym}(\mathfrak{g}) \cdot \mathrm{Sym}(\mathfrak{g})^G_+ \to \Gamma(X, \mathrm{Sym}_{\mathcal{O}_X} T_X)$$

is an isomorphism.

Modulo the following lemma, we are now ready to prove Theorem 4.9.

Lemma 4.15. *The inclusion $Z(\mathfrak{g}) \to U(\mathfrak{g})$ gives rise to an isomorphism*

$$\mathrm{gr}\, Z(\mathfrak{g}) \simeq \mathrm{Sym}(\mathfrak{g})^G.$$

Proof. By definition, we have for each i the short exact sequence

$$0 \to F_{i-1}U(\mathfrak{g}) \to F_iU(\mathfrak{g}) \to \mathrm{Sym}^i(\mathfrak{g}) \to 0$$

induced by the order filtration on $U(\mathfrak{g})$. Viewing this as a sequence of $\mathfrak{g}$-representations, it splits by complete reducibility. We may therefore apply the functor of G-invariants (which coincides with the functor of $\mathfrak{g}$-invariants) to obtain an exact sequence

$$0 \to F_{i-1}Z(\mathfrak{g}) \to F_iZ(\mathfrak{g}) \to \mathrm{Sym}^i(\mathfrak{g})^G \to 0$$

which induces the desired isomorphism $\mathrm{gr}\, Z(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{g})^G$. □

[23] We emphasize here that $\mathcal{I}^\lambda(\widetilde{\mathfrak{g}})_x$ is *not* a left ideal of $U(\mathfrak{g})$.

Proof of Theorem 4.9. Equip $U(\mathfrak{g})_{\chi_{-\lambda}}$ with a filtration inherited from the filtration of $U(\mathfrak{g})$ by order. By definition ϕ_λ respects the filtration, so it suffices for us to show that the induced map

$$\operatorname{gr}\phi_\lambda : \operatorname{gr} U(\mathfrak{g})_{\chi_{-\lambda}} \to \operatorname{gr}\Gamma(X, \mathcal{D}_X^\lambda)$$

is an isomorphism. Our approach for this will be to construct a chain of maps

$$\operatorname{Sym}(\mathfrak{g})/\operatorname{Sym}(\mathfrak{g}) \cdot \operatorname{Sym}(\mathfrak{g})_+^G \twoheadrightarrow \operatorname{gr} U(\mathfrak{g})_{\chi_{-\lambda}} \to \operatorname{gr}\Gamma(X, \mathcal{D}_X^\lambda) \hookrightarrow \Gamma(X, \operatorname{gr}\mathcal{D}_X^\lambda) \simeq \Gamma(X, \operatorname{Sym}_{\mathcal{O}_X} T_X),$$

with the first and last map surjective and injective, respectively, whose composition is the map of Proposition 4.14. Because it is an isomorphism, each of the maps will be an isomorphism, giving the claim.

We now construct the two desired maps. For the first map, we have a natural surjective map $\operatorname{Sym}(\mathfrak{g}) \to \operatorname{gr} U(\mathfrak{g})_{\chi_{-\lambda}}$; it suffices to check that it kills $\operatorname{Sym}(\mathfrak{g})_+^G$. But recall from Lemma 4.15 that $\operatorname{gr} Z(\mathfrak{g}) \simeq \operatorname{Sym}(\mathfrak{g})^G$, hence for any element $f \in \operatorname{Sym}(\mathfrak{g})_+^G$, we see that the image of f in $\operatorname{Sym}(\mathfrak{g})$ lies in $(\operatorname{gr} Z(\mathfrak{g}))_+ \subset \operatorname{gr} U(\mathfrak{g})$. But $\chi_{-\lambda}$ sends $Z(\mathfrak{g}) \to k \simeq F_0 U(\mathfrak{g})$, hence this image dies in $\operatorname{gr} U(\mathfrak{g})_{\chi_{-\lambda}}$. For the second map, we have for all i a short exact sequence of sheaves

$$0 \to F_{i-1}\mathcal{D}_X^\lambda \to F_i\mathcal{D}_X^\lambda \to \operatorname{gr}_i \mathcal{D}_X^\lambda \to 0,$$

hence applying the left-exact functor $\Gamma(X, -)$ yields an injection

$$\operatorname{gr}^i \Gamma(X, \mathcal{D}_X^\lambda) = F_i\Gamma(X, \mathcal{D}_X^\lambda)/F_{i-1}\Gamma(X, \mathcal{D}_X^\lambda) \hookrightarrow \Gamma(X, \operatorname{gr}_i \mathcal{D}_X^\lambda),$$

where we are using the fact that $F_i\Gamma(X, \mathcal{D}_X^\lambda) = \Gamma(X, F_i\mathcal{D}_X^\lambda)$. It is clear that the resulting map $\operatorname{gr}\Gamma(X, \mathcal{D}_X^\lambda) \hookrightarrow \Gamma(X, \operatorname{gr}\mathcal{D}_X^\lambda)$ respects the ring structure, hence this gives the desired map. □

4.6. **The geometry of the nilpotent cone and Kostant's Theorem.** It now remains for us to prove Proposition 4.14. We will reinterpret the statement in terms of the geometry of the *nilpotent cone* $\mathcal{N}$ of nilpotent elements in $\mathfrak{g}$. Therefore, let us begin by considering the geometric situation more carefully. We note that $\mathcal{N}$ is in fact a variety, as the condition that ad_x is a nilpotent linear map $\mathfrak{g} \to \mathfrak{g}$ is a polynomial one. Now, recall that in Subsection 4.2, we defined the vector bundles $\widetilde{\mathfrak{g}}$, $\widetilde{\mathfrak{b}}$, and $\widetilde{\mathfrak{n}}$ on X whose fibers over $x \in X$ were $\mathfrak{g}$, $\mathfrak{b}_x$, and $\mathfrak{n}_x$, respectively. Let $\mathfrak{h}//W := \operatorname{Spec}(\operatorname{Sym}(\mathfrak{h}^*)^W)$ be the quotient of $\mathfrak{h}$ by the W-action, and consider the diagram

$$\begin{tikzcd} \operatorname{Sym}_X \widetilde{\mathfrak{b}} \arrow[r] \arrow[d] & \mathfrak{h} \arrow[d] \\ \mathfrak{g} \arrow[r] & \mathfrak{h}//W \end{tikzcd} \tag{17}$$

where the maps $\operatorname{Sym}_X \widetilde{\mathfrak{b}} \to \mathfrak{g}$ and $\operatorname{Sym}_X \widetilde{\mathfrak{b}} \to \mathfrak{h}$ are given by the inclusion $\widetilde{\mathfrak{b}} \to \mathfrak{g}$ and the projection $\widetilde{\mathfrak{b}}/\widetilde{\mathfrak{n}} \to \mathfrak{h}$ and the map $\mathfrak{g} \to \mathfrak{h}//W$ is given by the Chevalley isomorphism. We now give some initial characterizations of (17).

Lemma 4.16. *The diagram (17) commutes, the variety $\mathfrak{h}//W$ is smooth, and the maps $\mathfrak{h} \to \mathfrak{h}//W$ and $\mathfrak{g} \to \mathfrak{h}//W$ are flat.*

Proof. For commutativity, we check that the corresponding maps of structure sheaves agree. We would like to show that

$$\begin{tikzcd} \operatorname{Sym}_{\mathcal{O}_X} \widetilde{\mathfrak{b}}^* & \operatorname{Sym}(\mathfrak{h}^*) \arrow[l] \\ \operatorname{Sym}(\mathfrak{g}^*) \arrow[u] & \operatorname{Sym}(\mathfrak{h}^*)^W \arrow[l] \arrow[u] \end{tikzcd}$$

commutes, for which it suffices to note that for $f \in \operatorname{Sym}(\mathfrak{h}^*)^W$, the image of f tracing up and to the left is a function on $\operatorname{Sym}_X \widetilde{\mathfrak{b}}$ which factors through the map

$$\operatorname{Sym}_X \widetilde{\mathfrak{b}} \to \operatorname{Sym}_X \widetilde{\mathfrak{h}} \xrightarrow{f} \mathbb{A}^1.$$

On the other hand, the image of f tracing left and up is a function on $\operatorname{Sym}_X \widetilde{\mathfrak{b}}$ which assigns to a pair $(\mathfrak{b}_x, \xi)$ with $\xi \in \mathfrak{b}_x$ the value of the corresponding function $\phi(f)$ at ξ, where ϕ is the map of the Chevalley isomorphism. But by Theorem 3.13, such a map factors through the projection $\mathfrak{b}_x \to \mathfrak{b}_x/\mathfrak{n}_x \to \mathfrak{h} \xrightarrow{f} \mathbb{A}^1$, hence the two resulting maps on $\operatorname{Sym}_X \widetilde{\mathfrak{b}}$ agree, as needed.

The fact that $\mathfrak{h}//W$ is smooth follows from the fact that it is affine as the quotient of a finite reflection group acting on a vector space (see [Hum90, Theorem 3.5]). Now, $\mathfrak{h} \to \mathfrak{h}//W$ is a map with fibers of dimension 0 with $\mathfrak{h}$ Cohen-Macaulay and $\mathfrak{h}//W$ regular, hence it is flat.

Finally, to show that the map $\mathfrak{g} \to \mathfrak{h}//W$ induced by the Chevalley isomorphism is flat, we must show that $\mathrm{Sym}(\mathfrak{g}^*)$ is flat over $\mathrm{Sym}(\mathfrak{h}^*)^W \simeq \mathrm{Sym}(\mathfrak{g}^*)^G$. We give a sketch here of the very elegant argument presented in [BL96, Theorem 1.2]. Fix a choice of $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$, which gives rise to a canonical decomposition $\mathfrak{g}^* \simeq \mathfrak{h}^* \oplus (\mathfrak{g}/\mathfrak{h})^*$. Place an alternate filtration $F_i' \mathrm{Sym}(\mathfrak{g}^*)$ on $\mathrm{Sym}(\mathfrak{g}^*)$ so that

$$F_i' \mathrm{Sym}(\mathfrak{g}^*) = \mathrm{Sym}((\mathfrak{g}/\mathfrak{h})^*) \cdot F_i \mathrm{Sym}(\mathfrak{g}^*),$$

where F_i is the standard filtration. We then check that under this filtration $\mathrm{gr}\,\mathrm{Sym}(\mathfrak{g}^*) \simeq \mathrm{Sym}(\mathfrak{h}^*) \otimes \mathrm{Sym}((\mathfrak{g}/\mathfrak{h}))^*$, where the grading on the second term is on $\mathrm{Sym}(\mathfrak{h}^*)$ only. Let us now consider $\mathrm{gr}\,\mathrm{Sym}(\mathfrak{g}^*)$ as a $\mathrm{gr}\,\mathrm{Sym}(\mathfrak{g}^*)^G$-algebra with this modified filtration. Now, the image of $\mathrm{gr}\,\mathrm{Sym}(\mathfrak{g}^*)^G$ in $\mathrm{gr}\,\mathrm{Sym}(\mathfrak{g}^*)$ is the same as its image after restriction to $\mathrm{Sym}(\mathfrak{h}^*)$. But by the Chevalley isomorphism, this is simply $\mathrm{gr}\,\mathrm{Sym}(\mathfrak{h}^*)^W$. We see therefore that the map $\mathrm{gr}\,\mathrm{Sym}(\mathfrak{g}^*)^G \to \mathrm{gr}\,\mathrm{Sym}(\mathfrak{g}^*)$ factors through

$$\mathrm{gr}\,\mathrm{Sym}(\mathfrak{g}^*)^G \simeq \mathrm{Sym}(\mathfrak{h}^*)^W \to \mathrm{Sym}(\mathfrak{h}^*) \hookrightarrow \mathrm{Sym}(\mathfrak{h}^*) \otimes \mathrm{Sym}((\mathfrak{g}/\mathfrak{h}))^* \simeq \mathrm{gr}\,\mathrm{Sym}(\mathfrak{g}^*),$$

where we showed that $\mathrm{Sym}(\mathfrak{h}^*)$ was flat over $\mathrm{Sym}(\mathfrak{h}^*)^W$. The result then follows from the following two facts. First, a module over a graded algebra is flat if and only if it is free ([Eis95, Exercise 6.11]), which implies that $\mathrm{Sym}(\mathfrak{h}^*)$ is free over $\mathrm{Sym}(\mathfrak{h}^*)^W$, hence $\mathrm{gr}\,\mathrm{Sym}(\mathfrak{g}^*)$ is free over $\mathrm{gr}\,\mathrm{Sym}(\mathfrak{g}^*)^G$. Second, if $\mathrm{gr}\,M$ is a free $\mathrm{gr}\,A$-module, then M is a free A-module ([BL96, Lemma 4.2]); this follows from noting that an isomorphism $(\mathrm{gr}\,A)^n \to \mathrm{gr}\,M$ comes from a map $A^n \to M$ with a special filtration on A^n. This implies that $\mathrm{Sym}(\mathfrak{g}^*)$ is free over $\mathrm{Sym}(\mathfrak{g}^*)^G$, as needed. □

Now, we say that $\xi \in \mathfrak{g}$ is *regular* if its centralizer $\mathfrak{z}_{\mathfrak{g}}(\xi) = \{\mu \in \mathfrak{g} \mid \mathrm{ad}_\xi(\mu) = 0\}$ has minimal dimension. The locus of regular elements is determined by the non-vanishing of a determinantal ideal, hence forms a Zariski-dense open set in $\mathfrak{g}$, which we denote by $\mathfrak{g}_{\mathrm{reg}}$. Recall that this minimal dimension is called the *rank* of $\mathfrak{g}$ and for semisimple Lie algebras the rank is given by $\dim \mathfrak{h}$ (see for instance [Ser01, Corollary III.5.2]). Importantly, the following proposition shows that the Chevalley map ϕ is smooth on $\mathfrak{g}_{\mathrm{reg}}$.

Proposition 4.17. *The restriction of the Chevalley map ϕ to $\mathfrak{g}_{reg}$ is smooth.*

Proof. We refer the reader to [HTT08, Theorem 10.3.7] for the proof, which proceeds by considering the differential form $\det(\Omega)$ on $\mathfrak{g}$ defined by $\Omega_a(x, y) = (a, [x, y])$, where $(-, -)$ is the Killing form. This form turns out to be non-vanishing exactly on $\mathfrak{g}_{\mathrm{reg}}$, and its Hodge star may be computed to be the Jacobian of the Chevalley map, giving the result.[24] □

This allows us to give a characterization of (17). Let us write $\mathfrak{g}' := \mathfrak{g} \underset{\mathfrak{h}//W}{\times} \mathfrak{h}$.

Lemma 4.18. *Let $(\mathrm{Sym}_X \widetilde{\mathfrak{b}})_{reg}$ be the preimage of $\mathfrak{g}_{reg}$ in $\mathrm{Sym}_X \widetilde{\mathfrak{b}}$. Then, the following diagram is Cartesian.*

$$\begin{tikzcd}
(\mathrm{Sym}_X \widetilde{\mathfrak{b}})_{reg} \arrow[r] \arrow[d] & \mathfrak{h} \arrow[d] \\
\mathfrak{g}_{reg} \arrow[r] & \mathfrak{h}//W
\end{tikzcd}$$

Proof. Denote by $\mathfrak{g}'_{\mathrm{reg}}$ the preimage of $\mathfrak{g}_{\mathrm{reg}}$ in $\mathfrak{g}'$. The diagram induces a map

$$p_{\mathrm{reg}} : (\mathrm{Sym}_X \widetilde{\mathfrak{b}})_{\mathrm{reg}} \to \mathfrak{g}'_{\mathrm{reg}} := \mathfrak{g}_{\mathrm{reg}} \underset{\mathfrak{h}//W}{\times} \mathfrak{h}$$

which is proper because $(\mathrm{Sym}_X \widetilde{\mathfrak{b}})_{\mathrm{reg}} \to \mathfrak{g}_{\mathrm{reg}}$ is proper and $\mathfrak{g}'_{\mathrm{reg}} \to \mathfrak{g}_{\mathrm{reg}}$ is separated.

Let us see now that it is also birational. Let $\mathfrak{g}_{\mathrm{ss}}$ denote the set of semi-simple elements in $\mathfrak{g}$; recall that it is an open Zariski-dense set in $\mathfrak{g}$, and let $\mathfrak{g}_{\mathrm{reg,ss}} := \mathfrak{g}_{\mathrm{reg}} \cap \mathfrak{g}_{\mathrm{ss}}$ be the set of elements which are both regular and semi-simple. Let $(\mathrm{Sym}_X \widetilde{\mathfrak{b}})_{\mathrm{reg,ss}}$ denote the preimage of $\mathfrak{g}_{\mathrm{reg,ss}}$ in $(\mathrm{Sym}_X \widetilde{\mathfrak{b}})_{\mathrm{reg}}$, and let $\mathfrak{h}_{\mathrm{reg}}$ and $\mathfrak{h}_{\mathrm{reg}}//W$ denote the regular elements in $\mathfrak{h}$ and $\mathfrak{h}//W$, respectively. Here, here we notice that $\mathfrak{h}_{\mathrm{reg}}$ is

[24]This is a somewhat non-trivial result, so we are in some sense cheating by citing it here. However, it does not introduce any logical circularity, so we choose to omit the fairly computational proof to streamline our description of the geometric situation.

simply the complement of the root hyperplanes in $\mathfrak{h}$, the W-action restricts to $\mathfrak{h}_{\mathrm{reg}}$ and it makes sense to consider $\mathfrak{h}_{\mathrm{reg}}//W$. Then, for birationality, it suffices for us to show that

$$\begin{tikzcd} (\mathrm{Sym}_X \widetilde{\mathfrak{b}})_{\mathrm{reg,ss}} \arrow[r] \arrow[d] & \mathfrak{h}_{\mathrm{reg}} \arrow[r] \arrow[d] & \mathfrak{h} \arrow[d] \\ \mathfrak{g}_{\mathrm{reg,ss}} \arrow[r] & \mathfrak{h}_{\mathrm{reg}}//W \arrow[r] & \mathfrak{h}//W \end{tikzcd} \tag{18}$$

is Cartesian. For this, we need only check that the left square is Cartesian. We claim that each $\xi \in \mathfrak{g}_{\mathrm{reg,ss}}$ is contained in exactly $|W|$ Borel subalgebras of $\mathfrak{g}$; indeed, because $\xi \in \mathfrak{g}_{\mathrm{reg,ss}}$, its centralizer $\mathfrak{z}_{\mathfrak{g}}(\xi)$ is the unique Cartan subalgebra containing it. Therefore, the set of Borel subalgebras containing ξ is exactly those which contain this Cartan subalgebra, of which there are $|W|$ by definition. To specify one of these Borel subalgebras is to specify exactly one of the $|W|$ possible images of ξ in the abstract Cartan subalgebra, hence we see that (18) is Cartesian and thus p_{reg} is birational.

Now, notice that the map $\mathfrak{g}'_{\mathrm{reg}} \to \mathfrak{h}$ is smooth as the pullback of the smooth map $\mathfrak{g}_{\mathrm{reg}} \to \mathfrak{h}//W$ (by Lemma 4.17) along the flat map $\mathfrak{h} \to \mathfrak{h}//W$. Therefore, we see that $\mathfrak{g}'_{\mathrm{reg}}$ is regular. Thus, by Zariski's main theorem (see [Gro61, III.4.4.9]), it suffices for us to show that p_{reg} has finite fibers, for which it suffices to check that p_{reg} is injective on tangent spaces. Indeed, note that the tangent space to $\mathrm{Sym}_X \widetilde{\mathfrak{b}}$ at a point $(\mathfrak{b}, \xi)$ with $\xi \in \mathfrak{b}$ is given by $\mathfrak{g}/\mathfrak{b} \oplus \mathfrak{b}$ and the map to the tangent space to $\mathfrak{g} \underset{\mathfrak{h}//W}{\times} \mathfrak{h}$ is given by the map

$$(\mu_1, \mu_2) \mapsto (\mu_2, \pi([\mu_1, \xi]))$$

with π the projection $\mathfrak{b} \to \mathfrak{h}$. Note here that this map is well defined because π vanishes on $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}$. Thus, we must check that for $\mu_1 \in \mathfrak{g}$ and $\xi \in \mathfrak{b}_{\mathrm{reg}}$, if $[\mu_1, \xi] \in \mathfrak{n}$, then $\mu_1 \in \mathfrak{b}$. Indeed, if $\mu_1 \notin \mathfrak{b}$, then by the classification of Proposition 4.20, the lowest weight of $[\mu_1, \xi]$ will be non-positive, meaning that $[\mu_1, \xi] \notin \mathfrak{n}$, as needed. □

Because $\mathfrak{g}_{\mathrm{reg}}$ is open in $\mathfrak{g}$, Lemma 4.18 implies that the induced map

$$p : \mathrm{Sym}_X \widetilde{\mathfrak{b}} \to \mathfrak{g} \underset{\mathfrak{h}//W}{\times} \mathfrak{h}$$

is birational. We now consider the following two diagrams.

$$\begin{tikzcd} \mathrm{Sym}_X \widetilde{\mathfrak{n}} \arrow[r] \arrow[d] \arrow[dr, phantom, "(a)"] & \{0\} \arrow[d] \\ \mathrm{Sym}_X \widetilde{\mathfrak{b}} \arrow[r] & \mathfrak{h} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{N} \arrow[r] \arrow[d] \arrow[dr, phantom, "(b)"] & \{0\} \arrow[d] \\ \mathfrak{g} \arrow[r] & \mathfrak{h}//W \end{tikzcd}$$

We claim that they commute and are Cartesian; for (a), this is clear, and for (b), this follows from the fact that nilpotent elements in $\mathfrak{g}$ are exactly those which are killed by all functions in the image of the Chevalley homomorphism without a constant term. We may now fit all three diagrams together to summarize the geometric situation.

$$\begin{tikzcd} \mathrm{Sym}_X \widetilde{\mathfrak{n}} \arrow[r, dashed, "\widetilde{p}"] \arrow[d, "h"'] \arrow[dr, phantom, "(c)"] & \mathcal{N} \arrow[r, "\pi"] \arrow[d, "j"'] \arrow[dr, phantom, "(d)"] & \{0\} \arrow[d, "i"'] \\ \mathrm{Sym}_X \widetilde{\mathfrak{b}} \arrow[r, dashed, "p"] \arrow[dr] & \mathfrak{g} \underset{\mathfrak{h}//W}{\times} \mathfrak{h} \arrow[r, "\phi'"] \arrow[d] \arrow[dr, phantom, "(e)"] & \mathfrak{h} \arrow[d] \\ & \mathfrak{g} \arrow[r, "\phi"] & \mathfrak{h}//W \end{tikzcd} \tag{19}$$

Here, the maps of (cd) are given by those of (a), and the maps of (de) are given by those of (b). The diagram commutes by construction; we now claim that each square is Cartesian. Indeed, for (d), this follows because (de) and (e) are both Cartesian, and for (c), this follows because (cd) and (d) are both Cartesian. We may now state Kostant's theorem.

Theorem 4.19 (Kostant)**.** *The map $\widetilde{p} : \mathrm{Sym}_X \widetilde{\mathfrak{n}} \to \mathcal{N}$ of (19) is a resolution of singularities, known as the Springer resolution. Further, the map $\widetilde{p}$ satisfies $\widetilde{p}_* \mathcal{O}_{\mathrm{Sym}_X \widetilde{\mathfrak{n}}} \simeq \mathcal{O}_{\mathcal{N}}$.*[25]

Before giving the proof, let us first see how it implies Proposition 4.14.

Proof of Proposition 4.14. By Theorem 4.19, the map $\widetilde{p} : \mathrm{Sym}_X \widetilde{\mathfrak{n}} \to \mathcal{N}$ provides an isomorphism

$$\widetilde{p}_* \mathcal{O}_{\mathrm{Sym}_X \widetilde{\mathfrak{n}}} \simeq \mathcal{O}_{\mathcal{N}}. \tag{20}$$

Recall from the definition of $\widetilde{\mathfrak{b}}$ that we had an exact sequence

$$0 \to \widetilde{\mathfrak{b}} \to \widetilde{\mathfrak{g}} \to T_X \to 0.$$

Note that this sequence is a priori only left exact, but is right exact by Lemma 4.7. Therefore, we see that $T_X \simeq \widetilde{\mathfrak{g}}/\widetilde{\mathfrak{b}} \simeq \widetilde{\mathfrak{n}}^-$. Therefore, applying the identification given by the Killing form we have $T_X^* \simeq (\widetilde{\mathfrak{n}}^-)^* \simeq \widetilde{\mathfrak{n}}$ and hence

$$\mathrm{Sym}_X T_X^* \simeq \mathrm{Sym}_X \widetilde{\mathfrak{n}}. \tag{21}$$

Further, the construction of the Chevalley isomorphism given in Theorem 3.13 provides an explicit description for $\mathcal{N}$ as the elements in $\mathfrak{g}$ which vanish under every element of $\mathrm{Sym}(\mathfrak{g}^*)^G$ without a constant term; under the identification $\mathfrak{g} \simeq \mathfrak{g}^*$, this shows exactly that

$$\mathcal{N} \simeq \mathrm{Spec}(\mathrm{Sym}(\mathfrak{g})/\mathrm{Sym}(\mathfrak{g}) \cdot \mathrm{Sym}(\mathfrak{g})^G_+). \tag{22}$$

Combining the identifications (21) and (22), the isomorphism of (20) induces an isomorphism

$$\mathrm{Sym}(\mathfrak{g})/\mathrm{Sym}(\mathfrak{g}) \cdot \mathrm{Sym}(\mathfrak{g})^G_+ \simeq \Gamma(\mathcal{N}, \mathcal{O}_{\mathcal{N}}) \to \Gamma(\mathrm{Sym}_X T_X^*, \mathcal{O}_{\mathrm{Sym}_X T_X^*}) \simeq \Gamma(X, \mathrm{Sym}_{\mathcal{O}_X} T_X).$$

It remains now to check that this isomorphism is the map given in the statement of Proposition 4.14. Concretely, we must show that the diagram

$$\begin{tikzcd} \mathrm{Sym}_X \widetilde{\mathfrak{n}} \simeq \mathrm{Sym}_X(\widetilde{\mathfrak{n}}^-)^* \arrow[r, "\sim"] \arrow[d] & \mathrm{Sym}_X T_X^* \arrow[d] \\ \mathcal{N} \arrow[r, hook] & \mathfrak{g} \simeq \mathfrak{g}^* \end{tikzcd}$$

commutes, where the map $\mathrm{Sym}_X T_X^* \to \mathfrak{g}^*$ is induced by differentiating the action of G on X. Indeed, tracing through the identifications, this amounts to the fact that the double adjunction of a map $\widetilde{\mathfrak{g}} \to T_X$ gives the map itself. ☐

It remains now for us to prove Theorem 4.19. Our strategy will be to lift the resolution of singularities $\mathrm{Sym}_X \widetilde{\mathfrak{b}} \to \mathfrak{g}'$ given by Lemma 4.18 to a resolution $\mathrm{Sym}_X \widetilde{\mathfrak{n}} \to \mathcal{N}$. The remaining result will then follow from a more detailed analysis of the regular elements in $\mathfrak{g}$. In particular, we will first classify all regular elements that lie within a specified Borel subalgebra $\mathfrak{b}$.

Proposition 4.20. *An element $\xi \in \mathfrak{b}$ is regular if and only if there exists a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$, a subset $I \subset \{1, \ldots, n\}$ of the simple roots, and a decomposition*

$$\xi = \xi_{ss} + \sum_{\alpha \in R^+} \xi_\alpha$$

with $\xi_{ss} \in \mathfrak{h}$ and $\xi_\alpha \in \mathfrak{g}_\alpha$ such that the following properties hold:

- *$\alpha_i(\xi_{ss}) = 0$ for all i in I,*
- *$\alpha_i(\xi_{ss}) \neq 0$ for $i \notin I$,*
- *$\xi_\alpha = 0$ if α_i appears in the decomposition of α for $i \in I$, and*
- *$\xi_{\alpha_i} \neq 0$ for $i \notin I$.*

Proof. Take $\xi \in \mathfrak{g}$ and let $\xi = \xi_{\mathrm{ss}} + \xi_n$ with ξ_{ss} semi-simple, ξ_n nilpotent, and $[\xi_{\mathrm{ss}}, \xi_n] = 0$ be its Jordan decomposition. Write $\mathfrak{z} = \mathfrak{z}_{\mathfrak{g}}(\xi_{\mathrm{ss}})$ and note that $\mathfrak{z}$ is the direct sum of a semisimple Lie algebra and an abelian one with total rank $\mathrm{rank}\, \mathfrak{g}$. Further, because $\mathfrak{z}_{\mathfrak{z}}(\xi_n) \subset \mathfrak{z}_{\mathfrak{g}}(\xi)$ we see that ξ is regular if and only if ξ_n is regular inside the semisimple part of $\mathfrak{z}$.

[25] In fact, it is known that $\widetilde{p}$ is a *rational resolution of singularities*, meaning that $R\widetilde{p}_* \mathcal{O}_{\mathrm{Sym}_X \widetilde{\mathfrak{n}}} \simeq \mathcal{O}_{\mathcal{N}}$ in the derived category. For $k = \mathbb{C}$, Grauert-Riemenschneider vanishing (see [GR70, Satz 2.3]) shows that it suffices to check that the canonical bundle of $\mathrm{Sym}_X \widetilde{\mathfrak{n}}$ is trivial.

Now, suppose $\xi \in \mathfrak{b}$ was regular. Pick a Borel subalgebra $\mathfrak{b}' \subset \mathfrak{b}$ of $\mathfrak{z}$ containing ξ and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{z}$ inside $\mathfrak{b}'$. Then $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$. Then, this determines a decomposition

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$$

along with a choice $I \subset \{1, \ldots, n\}$ of a subset of the simple roots for which we may write ξ in the form

$$\xi = \xi_{\mathrm{ss}} + \sum_{\alpha \in R^+} \xi_\alpha$$

with $\xi_\alpha \in \mathfrak{g}_\alpha$ such that $\alpha_i(\xi_{\mathrm{ss}}) = 0$ for $i \in I$, $\alpha_i(\xi_{\mathrm{ss}}) \neq 0$ for $i \notin I$, and $\xi_\alpha = 0$ if α_i for $i \in I$ appears in the decomposition of α as the sum of simple roots.

To complete the classification of regular elements, then it suffices to understand when nilpotent elements of the form $\sum_{\alpha \in R^+} \xi_\alpha$ are regular. We claim that this occurs if and only if $\xi_{\alpha_i} \neq 0$ for all $i \notin I$; we omit the proof of this and instead refer the reader to the original in [Kos59, Theorem 5.3], which proceeds by combinatorial considerations on principal $\mathfrak{sl}_2$'s in $\mathfrak{g}$. Putting everything together, we have obtained exactly the desired conditions on our decomposition. □

We may now use Proposition 4.20 to analyze the codimension of $\mathfrak{g} - \mathfrak{g}_{\mathrm{reg}}$ in $\mathfrak{g}$.

Lemma 4.21. *The codimension of* $\mathfrak{g} - \mathfrak{g}_{reg}$ *in* $\mathfrak{g}$ *is at least* 2.

Proof. It suffices for us to show that for every Borel subalgebra $\mathfrak{b}$, the codimension of $\mathfrak{b} - \mathfrak{b} \cap (\mathfrak{g} - \mathfrak{g}_{\mathrm{reg}})$ is at least 2.[26] Given a choice of $\mathfrak{b}$, we have a canonical projection

$$\mathfrak{b} \to \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$$

independent of the choice of $\mathfrak{h}$. Translating the result of Proposition 4.20, we see that the image of $\mathfrak{b} - \mathfrak{b}_{\mathrm{reg}}$ under this projection is contained in the finite union of the hyperplanes $\{\xi_{\alpha_i} \neq 0\}$ and further that the fiber of $\mathfrak{b} - \mathfrak{b}_{\mathrm{reg}}$ above each point in $\bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha$ has codimension at least 1, allowing us to conclude that $\mathfrak{b} - \mathfrak{b}_{\mathrm{reg}}$ itself has codimension at least 2 in $\mathfrak{b}$, as needed. □

Finally, before beginning the proof, we need one more auxiliary result, a standard lemma from commutative algebra.

Lemma 4.22. *Consider a Cartesian diagram*

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[d] \\ X \arrow[r, hook] & Y \arrow[r] & Z \end{tikzcd}$$

of affine schemes with $Z' \to Z$ *flat and* $X \hookrightarrow Y$ *and* $X' \hookrightarrow Y'$ *closed embeddings. Then, if* X *has codimension at least* 2 *in* Y, X' *has codimension at least* 2 *in* Y'.

Proof. Recall by construction that

$$\mathcal{O}_{X'} = \mathcal{O}_X \underset{\mathcal{O}_Z}{\otimes} \mathcal{O}_{Z'}.$$

Take a prime $\mathfrak{p} \otimes \mathfrak{q}$ of $\mathcal{O}_{X'}$, where $\mathfrak{p}$ and $\mathfrak{q}$ are primes of $\mathcal{O}_X$ and $\mathcal{O}_{Z'}$ which agree upon restriction to $\mathcal{O}_Z$. Because X has codimension at least 2 in Y, there exists a non-trivial chain of primes

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \mathfrak{p}$$

in $\mathcal{O}_X$ with corresponding preimages $\mathfrak{p}'_0 \subset \mathfrak{p}'_1 \subset \mathfrak{p}'$ in $\mathcal{O}_Z$. Now, $\mathcal{O}_{Z'}$ is flat over $\mathcal{O}_Z$, so we may apply the going down theorem to obtain a corresponding chain of primes

$$\mathfrak{q}_0 \subset \mathfrak{q}_1 \subset \mathfrak{q}$$

in $\mathcal{O}_{Z'}$ such that $\mathfrak{p}_i$ and $\mathfrak{q}_i$ both restrict to $\mathfrak{p}'_i$. This gives a chain of primes $\mathfrak{p}_i \otimes \mathfrak{q}_i$ contained in $\mathfrak{p} \otimes \mathfrak{q}$ which shows that $\mathrm{height}(\mathfrak{p} \otimes \mathfrak{q}) \geq 2$, as needed. □

[26] In fact, it is true that $\mathrm{codim}(\mathfrak{g} - \mathfrak{g}_{\mathrm{reg}}) \geq 3$.

Proof of Theorem 4.19. Note that square (c) is Cartesian and $p : \mathrm{Sym}_X \widetilde{\mathfrak{b}} \to \mathfrak{g}'$ is birational by Lemma 4.18, so to show that $\widetilde{p}$ is birational, it suffices merely to check that the intersection of $\mathcal{N}$ and $\mathfrak{g}'_{\mathrm{reg}}$ in $\mathfrak{g}'$ is non-empty. Indeed, the preimage of any regular nilpotent element of $\mathfrak{g}$ lies in this intersection, so $\widetilde{p}$ is birational.

Let us now show that $\widetilde{p}_* \mathcal{O}_{\mathrm{Sym}_X \widetilde{\mathfrak{n}}} \simeq \mathcal{O}_{\mathcal{N}}$. For this, we first consider the map $p : \mathrm{Sym}_X \widetilde{\mathfrak{b}} \to \mathfrak{g}'$. Denote the composition

$$q : \mathrm{Sym}_X \widetilde{\mathfrak{b}} \to X \times \mathfrak{g} \to \mathfrak{g}$$

of the closed embedding $\mathrm{Sym}_X \widetilde{\mathfrak{b}} \to X \times \mathfrak{g}$ and the projection $X \times \mathfrak{g} \to \mathfrak{g}$. Note that q is projective because X is a projective variety; therefore, p is projective because the projection $\mathfrak{g}' \to \mathfrak{g}$.

We now claim that $\mathfrak{g}'$ is normal. By Lemma 4.16, $\mathfrak{g}'$ is the pullback of a smooth variety along a flat morphism between smooth varieties, hence it is a local complete intersection and therefore Cohen-Macaulay. Now, recall $p : \mathrm{Sym}_X \widetilde{\mathfrak{b}} \to \mathfrak{g}'$ is birational and restricts to an isomorphism to $\mathfrak{g}'_{\mathrm{reg}} \subset \mathfrak{g}'$, which is regular because it is isomorphic to $(\mathrm{Sym}_X \widetilde{\mathfrak{b}})_{\mathrm{reg}}$, which is smooth. Thus, to show that $\mathfrak{g}'$ is normal, it suffices to check that $\mathfrak{g}' - \mathfrak{g}'_{\mathrm{reg}}$ has codimension at least 2. By Lemma 4.21, we know that $\mathfrak{g} - \mathfrak{g}_{\mathrm{reg}}$ has codimension at least 2; because $\mathfrak{h} \to \mathfrak{h}//W$ is flat by Lemma 4.16, by Lemma 4.22 we see that $\mathfrak{g}' - \mathfrak{g}'_{\mathrm{reg}}$ has codimension 2 in $\mathfrak{g}'$. This implies that $\mathfrak{g}'$ is Cohen-Macaulay and regular in codimension 1, hence normal. Therefore, by Zariski's Main Theorem applied to the proper map p (see [Gro61, III.4.3.12]), we see that $p_* \mathcal{O}_{\mathrm{Sym}_X \widetilde{\mathfrak{b}}} \simeq \mathcal{O}_{\mathfrak{g}'}$.

We now claim that $\mathrm{Sym}_X \widetilde{\mathfrak{b}} \to \mathfrak{h}$ is flat. Indeed, note that $\mathrm{Sym}_X \widetilde{\mathfrak{b}}$ is smooth and equidimensional as a vector bundle over the flag variety, hence this is a map of Cohen-Macaulay schemes, so it suffices by [Gro65, IV.6.1.5] to check that each fiber is Cohen-Macaulay and has dimension $\dim \mathfrak{g} - \dim \mathfrak{h}$. But notice that the fiber over a point $\xi \in \mathfrak{h}$ is a vector bundle over X whose fiber over $x \in X$ is $\{\mu \in \mathfrak{b}_x \mid \pi_x(\mu) = \xi\}$ for $\pi_x : \mathfrak{b}_x/\mathfrak{n}_x \to \mathfrak{h}$ the canonical projection. Thus, we see that each fiber is smooth of the correct dimension, so the map is flat as needed. Therefore, we may apply flat base change on (cd) and (d) to see that

$$\pi_* \widetilde{p}_* \mathcal{O}_{\mathrm{Sym}_X \widetilde{\mathfrak{n}}} \simeq i^* \phi'_* p_* \mathcal{O}_{\mathrm{Sym}_X \widetilde{\mathfrak{b}}} \simeq i^* \phi'_* p_* \mathcal{O}_{\mathrm{Sym}_X \widetilde{\mathfrak{b}}} \simeq \pi_* j^* \mathcal{O}_{\mathfrak{g}'} \simeq \pi_* \mathcal{O}_{\mathcal{N}}.$$

But recall that $\mathcal{N}$ was affine, so this implies that $\widetilde{p}_* \mathcal{O}_{\mathrm{Sym}_X \widetilde{\mathfrak{n}}} \simeq \mathcal{O}_{\mathcal{N}}$.[27] □

4.7. **The global sections functor on the flag variety.** In this subsection, we analyze the functor of global sections on the flag variety to prove Theorem 4.11. Let $\mathcal{M}$ be a $\mathcal{D}_X^\lambda$-module. For $\mu \in X$ an appropriately chosen dominant integral weight, we will instead consider $\mathcal{M} \otimes \mathcal{L}^\mu$. For this, let $V_{-\mu} = \Gamma(X, \mathcal{L}^\mu)$ and $V^\mu = \Gamma(X, \mathcal{L}^\mu)^*$; by Theorem 3.9, $V_{-\mu}$ and V^μ are the (finite dimensional) irreducible representations of G of lowest weight $-\mu$ and highest weight μ, respectively. Denote by $\mathcal{V}_{-\mu} := \mathcal{O}_X \otimes_k V_{-\mu}$ and $\mathcal{V}^\mu := \mathcal{O}_X \otimes_k V^\mu$ the corresponding sheaves.

Lemma 4.23. *Let $\{\mu_1, \ldots, \mu_k\}$ be a labeling of the set of weights of V^μ so that $\mu_i < \mu_j$ implies that $i > j$. Then, there are B-stable filtrations*

$$0 = V_0 \subset V_1 \subset \cdots \subset V_k = V^\mu$$

and

$$0 = W_0 \subset W_1 \subset \cdots \subset W_k = V_{-\mu},$$

such that V_i/V_{i-1} is the one dimensional B-module corresponding to μ_i and W_i/W_{i-1} is the one dimensional B-module corresponding to $-\mu_{k-i+1}$.

Proof. Such a filtration of $U(\mathfrak{b})$-modules is immediate from the classification of finite-dimensional representations of semisimple Lie algebras. Because G was connected and simply connected, it lifts to a filtration of B-modules. □

Lemma 4.24. *Let $\{\mu_1, \ldots, \mu_k\}$ be a labeling of the multiset of weights of V^μ so that $\mu_i < \mu_j$ implies that $i > j$. Then, there are filtrations*

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_k = \mathcal{V}^\mu$$

and

$$0 = \mathcal{W}_0 \subset \mathcal{W}_1 \subset \cdots \subset \mathcal{W}_k = \mathcal{V}_{-\mu}$$

of G-equivariant sheaves such that $\mathcal{V}_i/\mathcal{V}_{i-1} \simeq \mathcal{L}^{-\mu_i}$ and $\mathcal{W}_i/\mathcal{W}_{i-1} \simeq \mathcal{L}^{\mu_{k-i+1}}$.

[27] Instead of passing through $\mathfrak{g}'$, we could have instead shown that $\mathcal{N}$ is normal by further analyzing the geometry of G-orbits in $\mathcal{N}$. Then, a direct application of Zariski's main theorem to the birational map $\mathrm{Sym}_X \widetilde{\mathfrak{n}} \to \mathcal{N}$ would give this part of the result.

Proof. Construct these filtrations by applying the functor $\mathcal{L}(-)$ to the filtrations of Lemma 4.23. The result satisfies $\mathcal{V}_k := \mathcal{L}(V^\mu) = \mathcal{V}^\mu$ and $\mathcal{W}_k := \mathcal{L}(V_{-\mu}) = \mathcal{V}_{-\mu}$ by Proposition 3.7 and $\mathcal{V}_i/\mathcal{V}_{i-1} \simeq \mathcal{L}^{-\mu_i}$ and $\mathcal{W}_i/\mathcal{W}_{i-1} \simeq \mathcal{L}^{\mu_{k-i+1}}$ by Proposition 3.6(i). □

Because μ is dominant, $\mathcal{L}^\mu$ is globally generated, giving a natural surjection $\mathcal{V}_{-\mu} \twoheadrightarrow \mathcal{L}^\mu$. Dualizing and tensoring by the line bundle $\mathcal{L}^\mu$, we obtain also an injection $\mathcal{O}_X \hookrightarrow \mathcal{L}^\mu \otimes \mathcal{V}^\mu$. Because the $\mathcal{U}(\widetilde{\mathfrak{g}})$-module structure on $\mathcal{L}^\mu$ is induced by the map $\mathfrak{g} \to \Gamma(X, \mathcal{D}_X^\mu)$, we see that these are maps of $\mathcal{U}(\widetilde{\mathfrak{g}})$-modules. By functoriality, we may tensor them with $\mathcal{M}$ to obtain maps of $\mathcal{U}(\widetilde{\mathfrak{g}})$-modules

$$\mathcal{M} \hookrightarrow \mathcal{M} \otimes \mathcal{L}^\mu \otimes \mathcal{V}^\mu \tag{23}$$

and

$$\mathcal{M} \otimes \mathcal{V}_{-\mu} \twoheadrightarrow \mathcal{M} \otimes \mathcal{L}^\mu. \tag{24}$$

Here (23) is evidently injective and (24) is surjective by right exactness of $\mathcal{M} \otimes -$. We now have the following key lemma.

Lemma 4.25. *Let μ be an integral weight and λ an arbitrary weight. For any $\mathcal{D}_X^\lambda$-module $\mathcal{M}$, we have the following:*

(i) *if λ is dominant, the map of (23) is a split injection of $\mathcal{U}(\widetilde{\mathfrak{g}})$-modules, and*
(ii) *if λ is regular, the map of (24) is a split surjection of $\mathcal{U}(\widetilde{\mathfrak{g}})$-modules.*

Proof. Pick a labeling $\{\mu_1, \ldots, \mu_k\}$ of the weights of V^μ; in particular, we note that $\mu_1 = \mu$ and $\mu_i < \mu$ for $i > 1$. Consider now the corresponding filtrations given by Lemma 4.23. Viewing $\mathcal{V}_i$ and $\mathcal{W}_i$ as $\mathcal{U}(\widetilde{\mathfrak{g}})$-modules via the G-equivariant structure, we obtain filtrations

$$0 = \mathcal{M} \otimes \mathcal{L}^\mu \otimes \mathcal{V}_0 \subset \mathcal{M} \otimes \mathcal{L}^\mu \otimes \mathcal{V}_1 \subset \cdots \subset \mathcal{M} \otimes \mathcal{L}^\mu \otimes \mathcal{V}_k = \mathcal{M} \otimes \mathcal{L}^\mu \otimes \mathcal{V}^\mu$$

and

$$0 = \mathcal{M} \otimes \mathcal{W}_0 \subset \mathcal{M} \otimes \mathcal{W}_1 \subset \cdots \subset \mathcal{M} \otimes \mathcal{W}_k = \mathcal{M} \otimes \mathcal{V}_{-\mu}$$

of $\mathcal{U}(\widetilde{\mathfrak{g}})$-modules, where for each i we have short exact sequences

$$0 \to \mathcal{M} \otimes \mathcal{L}^\mu \otimes \mathcal{V}_{i-1} \to \mathcal{M} \otimes \mathcal{L}^\mu \otimes \mathcal{V}_i \to \mathcal{M} \otimes \mathcal{L}^\mu \otimes \mathcal{L}^{-\mu_i} \to 0 \tag{25}$$

and

$$0 \to \mathcal{M} \otimes \mathcal{W}_{i-1} \to \mathcal{M} \otimes \mathcal{W}_i \to \mathcal{M} \otimes \mathcal{L}^{\mu_{k-i+1}} \to 0. \tag{26}$$

We claim now that for any local section $z \in \mathcal{Z}(\widetilde{\mathfrak{g}})$, the section

$$a_i(z) := \prod_{i=1}^{i} (z - \chi_{-\lambda-\mu+\mu_i}(z))$$

acts by zero on $\mathcal{M} \otimes \mathcal{L}^\mu \otimes \mathcal{V}_i$. For this, recall that by Proposition 4.13, $\mathbb{Z}(\widetilde{\mathfrak{g}})$ acts on $\mathcal{M}$ and $\mathcal{L}^\mu$ via $\chi_{-\lambda}$ and $\chi_{-\mu-\rho}$, respectively. Further, notice that if $\mathbb{Z}(\widetilde{\mathfrak{g}})$ acts on $\mathcal{M}_1$ and $\mathcal{M}_2$ via χ_{μ_1} and χ_{μ_2}, then by the definition of the $\mathcal{U}(\mathfrak{g})$-action on $\mathcal{M}_1 \otimes \mathcal{M}_2$, it acts on $\mathcal{M}_1 \otimes \mathcal{M}_2$ via $\chi_{\mathfrak{m}_1+\mu_2+\rho}$.[28]

The claim will now follow by induction on i. For the base case $i = 1$, $\mathbb{Z}(\widetilde{\mathfrak{g}})$ acts on $\mathcal{M}$, $\mathcal{L}^\mu$, and $\mathcal{V}_1$ via $\chi_{-\lambda}$, $\chi_{-\mu-\rho}$, and $\chi_{-\mu_1-\rho}$, respectively, hence it acts on $\mathcal{M} \otimes \mathcal{L}^\mu \otimes \mathcal{V}_1$ via $\chi_{-\lambda-\mu+\mu_1} = \chi_{-\lambda}$. For the inductive step, consider the short exact sequence (25); by the inductive hypothesis, $a_{i-1}(z)$ kills the first term, while $\mathcal{Z}(\widetilde{\mathfrak{g}})$-acts by $\chi_{\lambda-\mu+\mu_i}$ on the last term, hence $a_i(z)$ kills the middle term, completing the induction.

Taking $i = k$, we see that $\prod_{i=1}^k (z - \chi_{-\lambda-\mu+\mu_i}(z))$ kills $\mathcal{M} \otimes \mathcal{L}^\mu \otimes \mathcal{V}^\mu$. By a similar argument, we see that $\prod_{i=1}^k (z - \chi_{-\lambda-\mu_{k-i+1}}(z))$ kills $\mathcal{M} \otimes \mathcal{V}_{-\mu}$. Together, these show that the action of $\mathcal{Z}(\widetilde{\mathfrak{g}})$ is locally finite on $\mathcal{M} \otimes \mathcal{L}^\mu \otimes \mathcal{V}^\mu$ and $\mathcal{M} \otimes \mathcal{V}_{-\mu}$, as the image of any section under the $\mathcal{Z}(\widetilde{\mathfrak{g}})$-action is spanned by the images of finite powers of the generators for $\mathcal{Z}(\widetilde{\mathfrak{g}})$.

Thus, $\mathcal{M} \otimes \mathcal{L}^\mu \otimes \mathcal{V}^\mu$ and $\mathcal{M} \otimes \mathcal{V}_{-\mu}$ split as $\mathcal{U}(\widetilde{\mathfrak{g}})$-modules into the direct sum of generalized weight spaces for $\mathcal{Z}(\widetilde{\mathfrak{g}})$ with weights parametrized by the weights of the $\mathcal{U}(\widetilde{\mathfrak{b}})$-stable filtrations. To complete the proof of the lemma, for (i) it suffices to show that the inclusion (23) exhibits $\mathcal{M}$ as the kernel of the action by $(z - \chi_{-\lambda}(z))^N$ for N large under the inclusion

$$0 \to \mathcal{M} \simeq \mathcal{M} \otimes \mathcal{L}^\mu \otimes \mathcal{V}_1 \hookrightarrow \mathcal{M} \otimes \mathcal{L}^\mu \otimes \mathcal{V}^\mu$$

and for (ii) that (24) exhibits $\mathcal{M} \otimes \mathcal{L}^\mu$ as the quotient by the kernel of the action of $\prod_{i=2}^k (z - \chi_{-\lambda-\mu_i}(z))^N$ for N large under the projection

$$\mathcal{M} \otimes \mathcal{V}_{-\mu} \twoheadrightarrow \mathcal{M} \otimes \mathcal{W}_k/\mathcal{W}_{k-1} \simeq \mathcal{M} \otimes \mathcal{L}^\mu \to 0.$$

[28] Here the addition of ρ is to compensate for the shift we introduced in the definition of the Harish-Chandra isomorphism.

For (i), we need to check that for $\mu_i < \mu$, we have $\chi_{-\lambda} \neq \chi_{-\lambda-\mu+\mu_i}$. By Corollary 3.17, it suffices to check that $-\lambda$ and $-\lambda - \mu + \mu_i$ are not in the same W-orbit. Indeed, if

$$w(-\lambda) = -\lambda - \mu + \mu_i,$$

we would have

$$w(\lambda) - \lambda = \mu - \mu_i > 0,$$

which is impossible because λ is dominant.

For (ii), we need to check that for $\mu_i < \mu$, we have $\chi_{-\lambda-\mu_i} \neq \chi_{-\lambda-\mu}$. Again by Corollary 3.17, it suffices to check that $-\lambda - \mu_i$ and $-\lambda - \mu$ are not in the same W-orbit; indeed, if

$$w(-\lambda - \mu_i) = -\lambda - \mu,$$

we would have

$$w(\lambda) - \lambda = \mu - w(\mu_i) \geq 0$$

where the last inequality follows because $-\mu_i$ is a weight of the lowest weight module $V_{-\mu}$ with lowest weight $-\mu$. This implies that $w(\lambda) = \lambda$ and hence that $w = 1$ because λ is regular. But this means that $\mu = \mu_i$, a contradiction.[29] □

Proof of Theorem 4.11. Let $\mathcal{M}$ be a $\mathcal{D}_X^\lambda$-module. For (i), we wish to show that $H^k(X, \mathcal{M}) = 0$ for all $k > 0$. Recall that because cohomology commutes with direct limits, we have that

$$H^k(X, \mathcal{M}) = \varinjlim H^k(X, \mathcal{F}),$$

where we take the direct image over the directed set of coherent submodules $\mathcal{F}$ of $\mathcal{M}$; it therefore suffices for us to show that $H^k(X, \mathcal{F}) \to H^k(X, \mathcal{M})$ is the zero map for all such $\mathcal{F}$. Fix now a coherent submodule $\mathcal{F}$ of $\mathcal{M}$. Because $\mathcal{L}^\mu$ is ample for μ regular by Proposition 3.8(iii), we may choose a regular integral weight μ so that

$$H^k(X, \mathcal{F} \otimes \mathcal{L}^\mu \otimes_k \Gamma(X, \mathcal{L}^\mu)^*) \simeq H^k(X, \mathcal{F} \otimes \mathcal{L}^\mu) \otimes_k \Gamma(X, \mathcal{L}^\mu)^* = 0$$

by Serre vanishing. Now, consider the following commutative diagram on the level of sheaves of abelian groups.

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ \mathcal{F} \otimes \mathcal{L}^\mu \otimes_k \Gamma(X, \mathcal{L}^\mu)^* \arrow[r] & \mathcal{M} \otimes \mathcal{L}^\mu \otimes_k \Gamma(X, \mathcal{L}^\mu)^* \end{tikzcd}$$

Therefore, the resulting map $H^k(X, \mathcal{F}) \to H^k(\mathcal{M} \otimes \mathcal{L}^\mu \otimes_k \Gamma(X, \mathcal{L}^\mu)^*)$ factors through $H^k(\mathcal{F} \otimes \mathcal{L}^\mu \otimes_k \Gamma(X, \mathcal{L}^\mu)^*) = 0$, so it is the zero map. On the other hand, by Lemma 4.25(i), we see that $H^k(X, \mathcal{M}) \to H^k(\mathcal{M} \otimes \mathcal{L}^\mu \otimes_k \Gamma(X, \mathcal{L}^\mu)^*)$ is injective, meaning that $H^k(X, \mathcal{F}) \to H^k(X, \mathcal{M})$ is zero for all coherent submodules $\mathcal{F}$ of $\mathcal{M}$, hence $H^k(X, \mathcal{M})$ for $k > 0$, as desired.

For (ii), suppose that $\Gamma(X, \mathcal{M}) = 0$. We again pass to the coherent submodules $\mathcal{F}$ of $\mathcal{M}$. For any such $\mathcal{F}$, we have an injection $\Gamma(X, \mathcal{F}) \hookrightarrow \Gamma(X, \mathcal{M})$, hence $\Gamma(X, \mathcal{F}) = 0$. Now, pick μ a regular integral weight so that $\mathcal{F} \otimes \mathcal{L}^\mu$ is globally generated (again this is possible by Proposition 3.8(iii)). Then, by Lemma 4.25(ii), we have a split surjection

$$\mathcal{F} \otimes_k \Gamma(X, \mathcal{L}^\mu) \to \mathcal{F} \otimes \mathcal{L}^\mu,$$

which induces a surjection $\Gamma(X, \mathcal{F}) \otimes_k \Gamma(X, \mathcal{L}^\mu) \twoheadrightarrow \Gamma(X, \mathcal{F} \otimes \mathcal{L}^\mu)$ on global sections. Thus, we find that $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^\mu) = 0$, meaning that $\mathcal{F} \otimes \mathcal{L}^\mu = 0$ by global generation, hence $\mathcal{F} = 0$. Therefore, every coherent submodule $\mathcal{F}$ of $\mathcal{M}$ is zero, so $\mathcal{M} = 0$, as desired. □

[29] It was crucial to the proof of (ii) that we could use regularity to conclude that $w = 1$, which is not necessary for (i).

References

[BB81] Alexander Beĭlinson and Joseph Bernstein, *Localisation de g-modules*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 1, 15–18. MR 610137 (82k:14015)

[BB83] ———, *A generalization of Casselman's submodule theorem*, Representation theory of reductive groups (Park City, Utah, 1982), Progr. Math., vol. 40, Birkhäuser Boston, Boston, MA, 1983, pp. 35–52. MR 733805 (85e:22024)

[BB93] ———, *A proof of Jantzen conjectures*, I. M. Gel′fand Seminar, Adv. Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, pp. 1–50. MR 1237825 (95a:22022)

[Beĭ83] Alexander Beĭlinson, *Localization of representations of reductive lie algebras*, Proceedings of the ICM Warszaw 1983, 1983, pp. 699–709.

[Ber83] Joseph Bernstein, *Algebraic Theory of D-Modules*, `http://www.math.uchicago.edu/~mitya/langlands/Bernstein/Bernstein-dmod.ps`, 1983.

[BGK+87] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, and F. Ehlers, *Algebraic D-modules*, Perspectives in Mathematics, vol. 2, Academic Press Inc., Boston, MA, 1987. MR 882000 (89g:32014)

[BK81] J.-L. Brylinski and M. Kashiwara, *Kazhdan-Lusztig conjecture and holonomic systems*, Invent. Math. **64** (1981), no. 3, 387–410. MR 632980 (83e:22020)

[BL96] Joseph Bernstein and Valery Lunts, *A simple proof of Kostant's theorem that $U(\mathfrak{g})$ is free over its center*, Amer. J. Math. **118** (1996), no. 5, 979–987. MR 1408495 (97h:17012)

[DG70] M. Demazure and A. Grothendieck, *Schémas en groupes. I: Structure des schémas en groupes réductifs*, Lecture Notes in Math., vol. 151, Springer-Verlag, New York, 1970, Séminaire de Géométrie Algébrique du Bois Marie 1962/64. MR 0274458

[DG80] Michel Demazure and Peter Gabriel, *Introduction to algebraic geometry and algebraic groups*, North-Holland Mathematics Studies, vol. 39, North-Holland Publishing Co., Amsterdam, 1980, Translated from the French by J. Bell. MR 563524 (82e:14001)

[Dix77] Jacques Dixmier, *Enveloping algebras*, North-Holland Publishing Co., Amsterdam, 1977, North-Holland Mathematical Library, Vol. 14, Translated from the French. MR 0498740 (58 #16803b)

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995, With a view toward algebraic geometry. MR 1322960 (97a:13001)

[FGI+05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli, *Fundamental algebraic geometry*, Mathematical Surveys and Monographs, vol. 123, American Mathematical Society, Providence, RI, 2005, Grothendieck's FGA explained. MR 2222646 (2007f:14001)

[FI73] R. Fossum and B. Iversen, *On Picard groups of algebraic fibre spaces*, J. Pure Appl. Algebra **3** (1973), 269–289.

[Gai05] Dennis Gaitsgory, *Notes on geometric representation theory*, `http://math.harvard.edu/~gaitsgde/267y/cat0.pdf`, Fall 2005.

[GR70] Hans Grauert and Oswald Riemenschneider, *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent. Math. **11** (1970), 263–292. MR 0302938 (46 #2081)

[Gro61] A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I*, Inst. Hautes Études Sci. Publ. Math. (1961), no. 11, 167. MR 0163910 (29 #1209)

[Gro65] ———, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 24, 231. MR 0199181 (33 #7330)

[Gro66] ———, *On the de Rham cohomology of algebraic varieties*, Inst. Hautes Études Sci. Publ. Math. (1966), no. 29, 95–103. MR 0199194 (33 #7343)

[Har92] Joe Harris, *Algebraic geometry: a first course*, Graduate texts in mathematics, vol. 133, Springer-Verlag, 1992.

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston Inc., Boston, MA, 2008, Translated from the 1995 Japanese edition by Takeuchi. MR 2357361 (2008k:32022)

[Hum90] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR 1066460 (92h:20002)

[Hum08] ———, *Representations of semisimple Lie algebras in the BGG category O*, Graduate studies in mathematics, vol. 94, American Mathematical Society, 2008.

[Jan03] Jens Carsten Jantzen, *Representations of algebraic groups*, second ed., Mathematical Surveys and Monographs, vol. 107, American Mathematical Society, Providence, RI, 2003. MR 2015057 (2004h:20061)

[Kas89] Masaki Kashiwara, *Representation theory and D-modules on flag varieties*, Astérisque **173-174** (1989), 55–109.

[Kos59] Bertram Kostant, *The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group*, Amer. J. Math. **81** (1959), 973–1032. MR 0114875 (22 #5693)

[Ser01] Jean-Pierre Serre, *Complex semisimple Lie algebras*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2001, Translated from the French by G. A. Jones, Reprint of the 1987 edition. MR 1808366 (2001h:17001)

[Spr98] T. A. Springer, *Linear algebraic groups*, second ed., Progress in Mathematics, vol. 9, Birkhäuser Boston Inc., Boston, MA, 1998. MR 1642713 (99h:20075)