# FROBENIUS SPLITTING FOR SCHUBERT VARIETIES

YI SUN

ABSTRACT. This thesis presents an expository account of the use of Frobenius splitting techniques in the study of Schubert varieties. After developing the basic theory of Frobenius splitting, we show that the Schubert and Bott-Samelson varieties are split and use this to derive geometric consequences in arbitrary characteristic. The main result highlighted is that Schubert varieties are normal, Cohen-Macaulay, and have rational resolutions. In addition, we give a proof of the Demazure character formula.

## CONTENTS

## 1. INTRODUCTION

1.1. **Motivation.** Let $G$ be a semi-simple algebraic group over an algebraically closed field $k$ with fixed Borel subgroup $B$ containing a maximal torus $T$. When $k$ has characteristic 0, the Borel-Weil-Bott Theorem allows one to recover the representation theory of $G$ from the geometry of the flag variety $G/B$. More precisely, to each integral weight $\lambda$ is associated a certain line bundle $\mathcal{L}(\lambda)$ on $G/B$, on which all but one cohomology group vanishes. This non-vanishing cohomology group is then the irreducible representation of $G$ with a specified highest weight, for which the Weyl character formula provides an explicit character decomposition.

The flag variety is classically equipped with a decomposition into Schubert varieties coming from the Bruhat decomposition. Given the strength of the results on the flag variety, one might hope to find refinements of them on the smaller Schubert varieties. Indeed, the geometry of the Schubert varieties is quite nice; they are normal, Cohen-Macaulay varieties admitting a rational resolution of singularities. As a result, we have many results of a similar flavor to those present for the flag variety – the pullbacks of the line bundles $\mathcal{L}(\lambda)$ have vanishing higher cohomology, and we may recover a character formula for their unique non-vanishing cohomology group, though it is now only $B$-module rather than a $G$-module.

This is quite a beautiful picture; however, it comes with a caveat. If we allow $k$ to have characteristic $p$, many of these results become much weaker, as many complex analytic tools may no longer be used. In particular, for an arbitrary integral weight $\lambda$, it may not be the case that $H^i(G/B, L(\lambda))$ is non-zero for a unique value of $i$. If we restrict $\lambda$ to be a dominant weight, these vanishing theorems can be salvaged, but even then the unique non-zero cohomology group may no longer be an irreducible representation of $G$. [1]

The situation for Schubert varieties is similarly muddled. Our advertised results will remain true over arbitrary characteristic, but many of the geometric techniques used to obtain them originally are valid only in characteristic 0. However, in [MR], Mehta and Ramanathan were able to bypass many of the difficulties by introducing the new notion of a *Frobenius split* scheme, which ironically is valid *only* in positive characteristic. This property, define as the splitting of the absolute Frobenius homomorphism on a scheme, captures a type of geometric information about a scheme; for instance, Frobenius split schemes are reduced and all higher cohomologies of ample line bundles on them vanish. This idea is just enough to obtain in positive characteristic many of the nice results known in characteristic 0; in fact, it will even imply the original characteristic 0 results by semi-continuity!

The purpose of this thesis is to give an expository account of the use of these Frobenius splitting techniques in the context of Schubert varieties. In practice, this will be require a significant amount of additional input. It is difficult to prove directly that the Schubert varieties are Frobenius split, as they are highly singular. Thus, we consider instead an explicit desingularization, the Bott-Samelson variety, perform almost all of our Frobenius splitting arguments there, and pass the results over to the Schubert variety. In our presentation, we attempt to isolate the Frobenius splitting techniques as much as possible in order to pinpoint the geometric input that it provides.

Our main results will be as follows. First, Schubert varieties are normal, Cohen-Macaulay varieties with rational resolution of singularities. Second, the higher cohomologies of line bundles $\mathcal{L}_w(\lambda)$ associated to dominant weights vanish. Third and last, we have a character formula for the global sections of these line bundles. We emphasize that these results are valid in arbitrary characteristic.

1.2. **Conventions and notations.** Throughout this thesis, all fields will be algebraically closed. We will be using characteristic $p$ methods to obtain results valid in arbitrary characteristic, so we will take particular care to emphasize which characteristic we are using. At the beginning of each section, the characteristic of our field will be specified.

All schemes we deal with will be of finite type over a field $k$. Some morphisms we use will be in the category of schemes, not the category of schemes over $k$; when this occurs, we will mention it explicitly. For a scheme $X$, we will write $\mathrm{QCoh}(X)$ for the category of quasi-coherent sheaves and $\Omega^m_X$ for the sheaf of $m^{\text{th}}$ order Kähler differentials. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves on some scheme $X$, we will write $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ for the sheaf Hom between them. For a Cartier divisor $D$ on $X$, we will write $\mathcal{O}_X(D)$ for the corresponding line bundle; when clear from context, we will sometimes omit the subscript and write only $\mathcal{O}(D)$.

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[1]However, the irreducible representation of $G$ with the "correct" highest weight does embed into the cohomology.

1.3. **Organization and references.** The remainder of this thesis will be organized as follows. In Section 2, we construct the geometric backdrop for our investigations. In particular, we provide a desingularization $Z_{\mathfrak{w}} \to X_w$ of the Schubert variety by the Bott-Samelson variety and study the effects of this desingularization on the line bundles. In Section 3, we introduce the idea of Frobenius splitting and develop basic properties and consequences. The major results are the splitting criterion Proposition 3.16 and the cohomology vanishing and embeddings of Propositions 3.17 and 3.18. In Section 4, we apply the results of Section 3 to show that the Bott-Samelson and Schubert varieties are Frobenius split. We then obtain the key cohomology vanishing of Proposition 4.1 and pass it to characteristic 0. In Section 5, we use the vanishing Proposition 4.1 to derive our main results, the normality, Cohen-Macaulay, and rationality of singularities of Schubert varieties and the Demazure character formula. We emphasize here that the only input from the Frobenius splitting used in Section 5 will be Proposition 4.1.

The material we present here is not original, but rather gathered from a number of different sources, detailed as follows. For general results on algebraic groups, we use [Spr] and [Jan]; in particular, we follow the approach of [Jan] to induced line bundles on quotient schemes. Our general overview of Frobenius splitting is taken from the extensive book [BK], though we follow [MR]'s proof for the splitting criterion of Section 3.2. Our key Theorem 4.1 was originally used in [Kum] in the characteristic 0 context; here we follow its reinterpretation in [LT]. The argument of Section 4.3 is based on [BK]. Finally, we draw from a number of sources for the main results, mostly using [And] and [Ram].

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## 2. THE SCHUBERT AND BOTT-SAMELSON VARIETIES

In this section, we establish the geometric setting for the rest of this thesis. Our main goal will be to show that the Bott-Samelson variety is a desingularization of the Schubert variety and to study the line bundles on these two varieties. In particular, we compute the canonical bundle and Picard group of the Bott-Samelson variety.

The presentation will begin with a brief overview of some standard results on semi-simple algebraic groups and Schubert varieties. We omit some proofs of well-known statements, which may be found in [Spr] or [Jan], for example. We will proceed to a self-contained construction of the Bott-Samelson variety and results on the line bundles on them. All results in this section will be valid in arbitrary characteristic.

2.1. **Notations and preliminaries.** Let $G$ be a connected, simply-connected, semi-simple algebraic group over an algebraically closed field $k$ of arbitrary characteristic. Fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. For $U$ a maximal unipotent subgroup of $B$, we have abstractly that $T \simeq B/U$. Similarly we have the opposite Borel and unipotent subgroups $B^-$ and $U^-$ satisfying $T \simeq B^-/U^-$. Let $X^*(T)$ be the set of characters of $T$ and $X_*(T)$ the set of co-characters of $T$. As usual, denote the perfect pairing $\langle -, - \rangle : X^*(T) \otimes X_*(T) \to \mathbb{Z}$.

Let $\Delta \subset X^*(T)$ be the root system corresponding to $G$. Our choice of $B$ gives rise to a choice of positive roots $\Delta^+ \subset \Delta$. Let the corresponding simple roots and co-roots be $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $\{\alpha_1^\vee, \alpha_2^\vee, \ldots, \alpha_n^\vee\}$, respectively. The dual basis to $\{\alpha_1^\vee, \alpha_2^\vee \ldots, \alpha_n^\vee\}$ in $X^*(T)$ is given by the fundamental weights $\{\chi_1, \chi_2, \ldots, \chi_n\}$. We say that $\lambda \in X^*(T)$ is dominant if $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all $\alpha_i^\vee$, which occurs if and only if $\lambda$ is a non-negative linear combination of fundamental weights. Let $\rho$ denote the element $\chi_1 + \chi_2 + \cdots + \chi_n$, which is half the sum of the positive weights.

Let $W = N(T)/T$ denote the Weyl group of $G$, and let $\{s_1, s_2, \ldots, s_n\}$ to be the set of simple reflections corresponding to $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. For a sequence $\mathfrak{w} = (s_{i_1}, s_{i_2}, \ldots, s_{i_n})$ of simple reflections in $W$, known as a *word*, we write $p(\mathfrak{w})$ for their product and $n = \ell(\mathfrak{w})$ for its length. For a word $\mathfrak{w}$ and $J \subset \{1, 2, \ldots, n\}$, define the subword $\mathfrak{w}_J$ to consist of the simple reflections at indices $j \in J$. In particular, we will often consider the $m^{\text{th}}$ truncation $\mathfrak{w}[m] = (s_{i_1}, s_{i_2}, \ldots, s_{i_m})$ and $m^{\text{th}}$ omission $\mathfrak{w}(m) = (s_{i_1}, s_{i_2}, \ldots, s_{i_{m-1}}, \widehat{s_{i_m}}, s_{i_{m+1}}, \ldots, s_{i_n})$. For any $w \in W$, recall that its length $\ell(w)$ is the minimal $\ell$ such that there exists a decomposition $w = p(\mathfrak{w})$ of $w$ into $\ell$ simple reflections. Such a decomposition is known as reduced. Let $w_0$ denote the longest element in $W$.

2.2. **Flag and Schubert varieties.** We are now ready to introduce the basic geometric backdrop for our study, the flag variety $\mathcal{B} = G/B$, which is equipped with a quotient map $\pi : G \to \mathcal{B}$. Recall the Bruhat decomposition of $G$, the properties of which are summarized in the following proposition.

**Proposition 2.1** (Bruhat decomposition). *We have the following:*

(i) *For $\widetilde{w} \in N(T)$ a representative of $w \in W$, the double coset $B\widetilde{w}B$ depends only on $w$;*

(ii) *For $s_i \in W$ a simple reflection, we have*

$$(Bs_iB) \cdot (BwB) = \begin{cases} Bs_iwB & \ell(s_iw) > \ell(w) \\ (Bs_iwB) \cup (BwB) & otherwise; \end{cases}$$

(iii) *There is a decomposition*

$$G = \bigcup_{w \in W} BwB$$

*of $G$ into the disjoint union of double cosets.*

Recall that parabolic subgroups of $G$ are those which contain some Borel subgroup. All parabolic subgroups $P$ containing $B$ take the form $BW_IB$, where $I \subset \{1, 2, \ldots, n\}$ is a set of indices and $W_I$ is the subgroup of $W$ generated by $s_i$ for $i \in I$. In particular, we may define for each $i$ the standard parabolic subgroup $P_i = B \cup (Bs_iB)$ corresponding to the reflection $s_i \in W$. Note that these double cosets make sense by Proposition 2.1(i).

We now wish to extend this decomposition to $\mathcal{B}$. For $w \in W$, we define $C_w = BwB/B$, the *Bruhat cell* corresponding to $w$. Now take $X_w = \overline{C_w}$ with the reduced subscheme structure in $\mathcal{B}$ to be the *Schubert variety* corresponding to $w$. This will be our fundamental object of study. We now show that we may explicitly describe $X_w$ as the union of Bruhat cells. For $u, v \in W$, write $u \leq v$ in the *Bruhat-Chevalley order* if a reduced decomposition for $u$ may be obtained by deleting some simple reflections for a reduced decomposition of $v$. Then, $X_w$ admits the following description.

**Proposition 2.2.** *For $w \in W$, we have*

$$X_w = \bigcup_{v \leq w} C_w.$$

Therefore, for $v \leq w$, we may consider $X_v$ as a closed subscheme in $X_w$, which we will do in the sequel. Note also that Proposition 2.2 implies that $X_{w_0} = \mathcal{B}$, as the reduced decomposition of any $w \in W$ is a subword of the reduced decomposition of $w_0$.

2.3. **The Bott-Samelson variety.** The geometry of the variety $X_w$ is often highly non-singular, which motivates us to construct a desingularization of $X_w$. Indeed, we will provide a family of smooth varieties $Z_{\mathfrak{w}}$ indexed by words $\mathfrak{w}$ and equipped with closed embeddings $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to \mathcal{B}$. When $\mathfrak{w}$ is a reduced decomposition of some $w \in W$, we will obtain that $\theta_{\mathfrak{w}}(Z_{\mathfrak{w}}) = X_w$ and that $Z_{\mathfrak{w}}$ is a desingularization of $X_w$.

For any word $\mathfrak{w} = (s_{i_1}, s_{i_2}, \ldots, s_{i_n})$, let $P_{\mathfrak{w}}$ be the variety

$$P_{\mathfrak{w}} = P_{i_1} \times P_{i_2} \times \cdots \times P_{i_n},$$

which carries a right action of $B^n$ via

(1) $$(p_1, p_2, \ldots, p_n) \cdot (b_1, b_2, \ldots, b_n) = (p_1b_1, b_1^{-1}p_2b_2^{-1}, \ldots, b_{n-1}^{-1}p_nb_n).$$

This allows us to make the following definition of $Z_{\mathfrak{w}}$.

**Definition 2.3.** The *Bott-Samelson variety* $Z_{\mathfrak{w}}$ is the homogeneous space $P_{\mathfrak{w}}/B^n$ of $P_{\mathfrak{w}}$ with respect to the action of $B^n$ on $P_{\mathfrak{w}}$ defined in (1). Denote the projection by $\pi_{\mathfrak{w}} : P_{\mathfrak{w}} \to Z_{\mathfrak{w}}$.

We first establish some basic properties of $Z_{\mathfrak{w}}$.

**Proposition 2.4.** *We have the following:*

(i) *The Bott-Samelson variety $Z_{\mathfrak{w}}$ is a smooth projective variety and a closed subscheme of $\mathcal{B}^n$;*

(ii) *The map $\pi_{\mathfrak{w}} : P_{\mathfrak{w}} \to Z_{\mathfrak{w}}$ realizes $P_{\mathfrak{w}}$ as a principal $B^n$-bundle over $Z_{\mathfrak{w}}$.*

*Proof.* We first realize $Z_{\mathfrak{w}}$ as a closed subscheme of $\mathcal{B}^n$, from which the remaining assertions will follow quickly. Consider the map $\phi_n : G^n \to G^n$ given by $\phi_n(g_1, \ldots, g_n) = (g_1, g_1 g_2, \ldots, g_1 g_2 \cdots g_n)$. We equip $G^n$ with two different right actions of $B^n$, the standard action by termwise right multiplication and one which restricts to the action of $B^n$ on $P_{\mathfrak{w}}$. Taking the latter action on the source and the former action on the target, $\phi_n$ may be viewed as a $B^n$-equivariant map. Therefore, it induces a map $\widetilde{\phi}_n : G^n/B^n \to \mathcal{B}^n$ between corresponding homogeneous spaces. Notice that $\phi_n$ has inverse map $(g_1, \ldots, g_n) \mapsto (g_1, g_1^{-1} g_2, \ldots, g_{n-1}^{-1} g_n)$, so both $\phi_n$ and $\widetilde{\phi}_n$ are isomorphisms.

Now, notice that we may realize $Z_{\mathfrak{w}}$ as the image of $P_{\mathfrak{w}}$ under the maps $P_{\mathfrak{w}} \to G^n \to G^n/B^n$. This gives the following diagram.

$$
\begin{array}{ccccc}
P_{\mathfrak{w}} & \longrightarrow & G^n & \xrightarrow{\phi_n} & G^n \\
{\scriptstyle \pi_{\mathfrak{w}}}\downarrow & & \downarrow & & \downarrow \\
Z_{\mathfrak{w}} & \longrightarrow & G^n/B^n & \xrightarrow{\widetilde{\phi}_n} & \mathcal{B}^n
\end{array}
$$

Here the left square is Cartesian with both horizontal maps closed embeddings. Further, because $\phi_n$ and $\widetilde{\phi}_n$ are isomorphisms, the right square is also Cartesian. Hence, the combined square is Cartesian and the composed horizontal maps are closed embeddings.

We may now read off the desired assertions. For (ii), $P_{\mathfrak{w}} \to Z_{\mathfrak{w}}$ is a principal $B^n$-bundle as the pullback of $G^n \to \mathcal{B}^n$. For (i), $Z_{\mathfrak{w}}$ is a $P_{\mathfrak{w}}$-homogeneous space by (ii), hence smooth (see [Spr, Theorem 4.3.7]) and a closed subscheme of $\mathcal{B}^n$, hence projective. $\qquad\square$

As with Schubert varieties, we may treat smaller Bott-Samelson varieties as subschemes of larger ones. For any subword $\mathfrak{w}_J$ of $\mathfrak{w}$, let the map $\iota_{\mathfrak{w}_J, \mathfrak{w}} : Z_{\mathfrak{w}_J} \to Z_{\mathfrak{w}}$ be induced by the map $P_{\mathfrak{w}_J} \to P_{\mathfrak{w}}$ given by

$$
(p_{j_1}, \ldots, p_{j_m}) \mapsto (1, \ldots, 1, p_{j_1}, \ldots, p_{j_m}, 1, \ldots, 1),
$$

where we place $p_{j_k}$ in position $j_k$ and 1 elsewhere. Similarly, we have projections $\psi_{\mathfrak{w}, \mathfrak{w}_J} : Z_{\mathfrak{w}} \to Z_{\mathfrak{w}_J}$ induced by the projections $P_{\mathfrak{w}} \to P_{\mathfrak{w}_J}$ given by

$$
(p_1, p_2, \ldots, p_n) \mapsto (p_{j_1}, p_{j_2}, \ldots, p_{j_m}).
$$

Notice that $\iota_{\mathfrak{v}, \mathfrak{w}}$ is a right inverse to $\psi_{\mathfrak{w}, \mathfrak{v}}$. We write $\iota_{\mathfrak{w}, m}$ for $\iota_{\mathfrak{w}(m), \mathfrak{w}}$ and $\psi_{\mathfrak{w}}$ for $\psi_{\mathfrak{w}, \mathfrak{w}[n-1]}$. As the following propositions show, these inclusions provide significant structure on the $Z_{\mathfrak{w}}$ for different words $\mathfrak{w}$.

**Proposition 2.5.** *We have the following:*

*(i) For any subword $\mathfrak{v} = \mathfrak{w}_J$ of $\mathfrak{w}$, the map $\iota_{\mathfrak{v}, \mathfrak{w}}$ is a closed embedding;*

*(ii) As subvarieties of $Z_{\mathfrak{w}}$, we have*

$$
Z_{\mathfrak{w}_J} \simeq \bigcap_{j \notin J} Z_{\mathfrak{w}(j)}.
$$

*(iii) The $Z_{\mathfrak{w}(j)}$ are irreducible, smooth, codimension 1 subvarieties with normal crossings in $Z_{\mathfrak{w}}$;*

*Proof.* Let $\phi_{\mathfrak{w}}$ denote the map $Z_{\mathfrak{w}} \to \mathcal{B}^n$ from Proposition 2.4 that realizes $Z_{\mathfrak{w}}$ as a closed subscheme of $\mathcal{B}^n$. We first use this map to regard each $Z_{\mathfrak{w}_J}$ as a closed subscheme of $Z_{\mathfrak{w}}$. Let $i_{\mathfrak{w}_J, \mathfrak{w}}$ be the map $\mathcal{B}^m \to \mathcal{B}^n$ given by

$$
(g_1, \ldots, g_m) \mapsto (1, \ldots, 1, g_1, \ldots, g_1, g_2, \ldots, g_2, g_3, \ldots, g_{m-1}, g_m, \ldots, g_m),
$$

where $i_{\mathfrak{w}_J, \mathfrak{w}}$ changes values at indices in $J$. Then, this map fits into the following Cartesian square.

$$
\begin{array}{ccc}
Z_{\mathfrak{w}_J} & \xrightarrow{\phi_{\mathfrak{v}}} & \mathcal{B}^m \\
{\scriptstyle \iota_{\mathfrak{w}_J, \mathfrak{w}}}\downarrow & & \downarrow{\scriptstyle i_{\mathfrak{w}_J, \mathfrak{w}}} \\
Z_{\mathfrak{w}} & \xrightarrow{\phi_{\mathfrak{w}}} & \mathcal{B}^n
\end{array}
$$

Write $\mathcal{B}_J^n = \operatorname{Im}(i_{\mathfrak{w}_J, \mathfrak{w}})$, and let $\mathcal{B}_{(j)}^n = \mathcal{B}_{\{1, \ldots, n\} - \{j\}}^j$. We have hence shown that $\iota_{\mathfrak{w}_J, \mathfrak{w}}(Z_{\mathfrak{w}_J}) = \phi_{\mathfrak{w}}^{-1}(\mathcal{B}_J^n)$.

The desired conclusions now follow from this realization of $Z_{\mathfrak{w}_J}$. For (i), $\iota_{\mathfrak{w}_J,\mathfrak{w}}$ is a closed embedding as the pullback of $i_{\mathfrak{w}_J,\mathfrak{w}}$. For (ii), we see that

$$Z_{\mathfrak{w}_J} = \phi_{\mathfrak{w}}^{-1}(\mathcal{B}_J^n) = \phi_{\mathfrak{w}}^{-1}\left(\bigcap_{j \notin J} \mathcal{B}_{(j)}^n\right) = \bigcap_{j \notin J} \phi_{\mathfrak{w}}^{-1}(\mathcal{B}_{(j)}^n) = \bigcap_{j \notin J} Z_{\mathfrak{w}(j)}$$

as subschemes of $Z_{\mathfrak{w}}$. Finally, for (iii), $Z_{\mathfrak{w}(j)}$ is smooth by Proposition 2.4 and irreducible of codimension 1 because its preimage $P_{\mathfrak{w}(j)} \times B$ under the $B^n$-bundle map $P_{\mathfrak{w}} \to Z_{\mathfrak{w}}$ is irreducible of codimension 1. Applying this repeatedly, we see that $Z_{\mathfrak{w}_J}$ is smooth of co-dimension $n - |J|$, which together with (ii) shows that the $Z_{\mathfrak{w}(j)}$ have normal crossings in $Z_{\mathfrak{w}}$. $\qquad\square$

**Proposition 2.6.** *Let $\ell(\mathfrak{w}) = n$. For any $j < n$, we have an isomorphism $\psi_{\mathfrak{w}}^*(\mathcal{O}(Z_{\mathfrak{w}[n-1](j)})) \simeq \mathcal{O}(Z_{\mathfrak{w}(j)})$.*

*Proof.* The map $\psi_{\mathfrak{w}}$ induces the following diagram



where we have that $\mathcal{O}(Z_{\mathfrak{w}[n-1]}(j)) = \mathcal{I}_{Z_{\mathfrak{w}[n-1](j)}}^{-1}$ and $\mathcal{O}(Z_{\mathfrak{w}(j)}) = \mathcal{I}_{Z_{\mathfrak{w}(j)}}^{-1}$. But it is easy to check locally that the maps on the center and the right are isomorphisms because $\psi_{\mathfrak{w}}$ is a $\mathbb{P}^1$-bundle map. Hence, the left map is as well, as needed. $\qquad\square$

We may now relate $Z_{\mathfrak{w}}$ to the rest of our geometric setting as follows. The product map $P_{\mathfrak{w}} \to G \xrightarrow{\pi} \mathcal{B}$ is compatible with the $B^n$ action, hence it factors through $\pi_{\mathfrak{w}}$ to a map $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to \mathcal{B}$. Notice that $\theta_{\mathfrak{w}}$ is given by the composition $Z_{\mathfrak{w}} \xrightarrow{\phi_{\mathfrak{w}}} \mathcal{B}^n \to \mathcal{B}$ of $\phi_{\mathfrak{w}}$ and the projection onto the last coordinate. The following proposition gives an alternate way to define $Z_{\mathfrak{w}}$ in terms of this map.

**Proposition 2.7.** *Denote by $\tau_n : \mathcal{B} \to G/P_{i_n}$ the natural projection. For any $\mathfrak{w}$, we have a Cartesian square*



*where $\psi_{\mathfrak{w}} : Z_{\mathfrak{w}} \to Z_{\mathfrak{w}[n-1]}$ is a $\mathbb{P}^1$-bundle.*

*Proof.* The given square fits into the following diagram.



The right square is evidently Cartesian. Now, because $\theta_{\mathfrak{w}[n-1]}$ is a closed embedding and $\tau_n \circ \phi_{\mathfrak{w}[n-1]}$ is closed as the composition of closed maps, the bottom map in the left square is a closed embedding. Therefore, the fiber product on the left is given by

$$Z_{\mathfrak{w}[n-1]} \times_{\mathcal{B}^{n-1} \times G/P_{i_n}} \mathcal{B}^n = (\mathrm{id} \times \tau_n)^{-1}\Big( \mathrm{Im}(\phi_{\mathfrak{w}[n-1]} \times (\tau_n \circ \phi_{\mathfrak{w}[n-1]})) \Big) = \phi_{\mathfrak{w}}(Z_{\mathfrak{w}}).$$

This shows that the combined square is Cartesian.

It remains now to show that $\tau_n$ is a $\mathbb{P}^1$-bundle, from which it will follow that its pullback $\psi_{\mathfrak{w}}$ is as well. Notice that $\tau_n$ is the projection from $\mathcal{B}$ to its quotient by the right action of $P_{i_n}$, hence is a $P_{i_n}/B \simeq \mathbb{P}^1$-fibration. $\qquad\square$

We would like to treat the $Z_{\mathfrak{w}}$ in analogy with the Schubert varieties, motivating us to define

$$\partial Z_{\mathfrak{w}} = \bigcup_{j=1}^{n} Z_{\mathfrak{w}(j)}$$

with the reduced scheme structure. Then, take $Z_{\mathfrak{w}}^o = Z_{\mathfrak{w}} \setminus \partial Z_{\mathfrak{w}}$. We think of $\partial Z_{\mathfrak{w}}$ as the boundary of $Z_{\mathfrak{w}}$ and of $Z_{\mathfrak{w}}^o$ as the interior. We would therefore like to view $Z_{\mathfrak{w}}^o$ as similar to the big Bruhat cell of a Schubert variety. The following proposition, which is the end goal of this section, makes this analogy precise, showing that the map between them realizes $Z_{\mathfrak{w}}$ as a desingularization of $X_w$.

**Proposition 2.8.** *If $\mathfrak{w}$ is a reduced decomposition of $w$, $\theta_{\mathfrak{w}}$ is a birational map $Z_{\mathfrak{w}} \to X_w$. In particular, it gives an isomorphism $Z_{\mathfrak{w}}^o \to C_w$.*

*Proof.* We first claim that $Z_{\mathfrak{w}}^o$ is the image of $(Bs_{i_1}B) \times \cdots \times (Bs_{i_n}B)$ under the projection $\pi_{\mathfrak{w}} : P_{\mathfrak{w}} \to Z_{\mathfrak{w}}$. For this, it suffices to note that a point $\pi_{\mathfrak{w}}(p_1, \ldots, p_{j-1}, 1, p_{j+1}, \ldots, p_n)$ is in $Z_{\mathfrak{w}(j)}$ (viewed as a subvariety in $Z_{\mathfrak{w}}$) if and only if it is in the image of $P_{i_1} \times \cdots \times P_{i_{j-1}} \times B \times P_{i_{j+1}} \times \cdots \times P_{i_n}$.

Now, consider the following diagrams, where we endow $G$ and $BwB$ with the right $B^n$ action where the first $n-1$ copies of $B$ act trivially.

$$
\begin{array}{ccc}
P_{i_1} \times \cdots \times P_{i_n} & \xrightarrow{\text{mult}} & G \\
\downarrow & & \downarrow \\
Z_{\mathfrak{w}} & \xrightarrow{\theta_{\mathfrak{w}}} & \mathcal{B}
\end{array}
\qquad
\begin{array}{ccc}
(Bs_{i_1}B) \times \cdots \times (Bs_{i_n}B) & \xrightarrow{\text{mult}} & BwB \\
\downarrow & & \downarrow \\
Z_{\mathfrak{w}}^o & \xrightarrow{\theta_{\mathfrak{w}}|_{Z_{\mathfrak{w}}^o}} & C_w
\end{array}
$$

In the left diagram, notice that the multiplication map is compatible with these $B^n$-actions, and that this construction induces the map $\theta_{\mathfrak{w}}$. Because $\mathfrak{w}$ is reduced, by Proposition 2.1, multiplication restricts to an isomorphism $(Bs_{i_1}B) \times \cdots \times (Bs_{i_n}B) \to BwB$ and the $B^n$-action preserves these sub-varieties of $P_{i_1} \times \cdots \times P_{i_n}$ and $G$, hence $\theta_{\mathfrak{w}}$ restricts to an isomorphism of their images $Z_{\mathfrak{w}}^o \to C_w$ (shown on the right). $\qquad\square$

In fact, we may deduce the following lemma, which shows something slightly stronger.

**Lemma 2.9.** *For a reduced decomposition $\mathfrak{w}$ of $w$, the map $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X_w$ induces an isomorphism $(\theta_{\mathfrak{w}})_* \mathcal{O}_{Z_{\mathfrak{w}}} \simeq \mathcal{O}_{X_w}$.*

*Proof.* Let $\mathfrak{w}_0$ be a word of maximal length containing $\mathfrak{w}$. Then, we have the following diagram

$$
\begin{array}{ccc}
Z_{\mathfrak{w}} & \xrightarrow{\theta_{\mathfrak{w}}} & X_w \\
{\scriptstyle \iota_{\mathfrak{w},\mathfrak{w}_0}}\downarrow & & \downarrow{\scriptstyle \iota_{w,w_0}} \\
Z_{\mathfrak{w}_0} & \xrightarrow{\theta_{\mathfrak{w}_0}} & \mathcal{B}
\end{array}
$$

which gives rise to the following map of short exact sequences.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & (\theta_{\mathfrak{w}_0})_* \mathcal{I}_{Z_{\mathfrak{w}}} & \longrightarrow & (\theta_{\mathfrak{w}_0})_* \mathcal{O}_{Z_{\mathfrak{w}_0}} & \longrightarrow & (\theta_{\mathfrak{w}_0})_*(\iota_{\mathfrak{w},\mathfrak{w}_0})_* \mathcal{O}_{Z_{\mathfrak{w}}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{I}_{X_w} & \longrightarrow & \mathcal{O}_{\mathcal{B}} & \longrightarrow & (\iota_{w,w_0})_* \mathcal{O}_{X_w} & \longrightarrow & 0
\end{array}
$$

We have isomorphisms in the left map because $Z_{\mathfrak{w}} \to \mathcal{B}$ is a closed embedding with image $X_w$ and in the center map by the smoothness of $\mathcal{B}$ and Zariski's main theorem (see [Ha77, Corollary 11.4]). These induce the desired isomorphism on the right. $\qquad\square$

2.4. **Line bundles on the Schubert and Bott-Samelson varieties.** We wish to use the Bott-Samelson variety to study line bundles on the Schubert variety. In the sequel, we will need some facts about such bundles, which we collect here. We begin with a fundamental construction which relates the representation theory of $G$ to the geometry of $\mathcal{B}$.

For any representation $V$ of $B$, form the $G$-equivariant vector bundle $\mathcal{L}(V) = G \times^B V$ on $\mathcal{B}$ with sections given by

$$\Gamma(U, \mathcal{L}(V)) = \{f : \pi^{-1}(U) \to V \mid f(g \cdot b) = b^{-1} \cdot f(g)\}.$$

The following theorem shows that this construction is functorial.

**Proposition 2.10.** *We have the following:*

*(i) $V \mapsto \mathcal{L}(V)$ defines an exact functor $G$-mod $\to$ QCoh($\mathcal{B}$);*
*(ii) For each $V$, $\mathcal{L}(V)$ is locally free of rank $\dim V$;*
*(iii) For representations $V, W$ of $G$, we have $\mathcal{L}(V \otimes W) \simeq \mathcal{L}(V) \otimes \mathcal{L}(W)$.*

*Proof.* We will first show (ii). Cover $\mathcal{B}$ by affines $U_i$ such that $\pi^{-1}(U_i) \simeq U_i \times B$ for all $i$. Then, we have that $\mathcal{L}(V)|_{U_i} \simeq \mathcal{O}_{U_i} \otimes_k V$, as needed. The other two conclusions now follow quickly from this.

For (i), $\mathcal{L}$ is evidently functorial with image in QCoh($\mathcal{B}$). On each $U_i$, we just showed the functor $V \mapsto \Gamma(U, \mathcal{L}(V))$ is given by $V \mapsto \mathcal{O}_{U_i} \otimes_k V$, which is evidently exact. So $\mathcal{L}$ is locally exact, hence exact.

For (iii), the multiplication map $\mathcal{L}(V) \otimes \mathcal{L}(W) \to \mathcal{L}(V \otimes W)$ is an isomorphism, as it is given locally on $U_i$ by the isomorphism

$$(\mathcal{O}_{U_i} \underset{k}{\otimes} V) \underset{\mathcal{O}_{U_i}}{\otimes} (\mathcal{O}_{U_i} \underset{k}{\otimes} W) \to \mathcal{O}_{U_i} \underset{k}{\otimes} (V \otimes W). \qquad \square$$

Let us now specialize this construction. Any $\lambda \in X^*(T)$ induces a representation $k_{-\lambda}$ of $B$ via the map $B \to B/U \simeq T \overset{-\lambda}{\to} k$. This gives rise to a family of line bundles $\mathcal{L}(\lambda) = \mathcal{L}(k_{-\lambda})$ on $\mathcal{B}$. Note that Proposition 2.10(iii) implies $\mathcal{L}(\lambda) \otimes \mathcal{L}(\mu) \simeq \mathcal{L}(\lambda + \mu)$. It is well known that all line bundles on $\mathcal{B}$ take this form.

**Proposition 2.11.** *The functor $\mathcal{L}$ defines a surjection $X^*(T) \to \mathrm{Pic}(\mathcal{B})$; that is, every line bundle $\mathcal{L}$ on $\mathcal{B}$ is of the form $\mathcal{L}(\lambda)$ for some $\lambda \in X^*(T)$.*

*Proof.* We only provide a sketch of a concrete proof; see [FI, Section 3] for a more abstract approach. First, any line bundle on $\mathcal{B}$ admits a $G$-equivariant structure on $\mathcal{L}$. To extract the value of $\lambda$ from some $G$-equivariant line bundle $\mathcal{L}$, observe that the left action of $B$ on $G/B$ fixes $\{1\} \in G/B$, hence $B$ acts on the stalk $\mathcal{L}|_{\{1\}}$, which is isomorphic to $k$ as a $k$-module. This action must factor through some $-\lambda \in X^*(T)$. Then, $\lambda$ will be the desired element of $X^*(T)$. $\qquad \square$

The following three propositions allow us to further characterize line bundles of this form. We are in particular interested in when they have many global sections, as the cohomologies of these line bundles acquire a $G$-module structure from the $G$-action on $\mathcal{B}$.

**Proposition 2.12.** *The canonical bundle $\omega_{\mathcal{B}}$ is given by $\mathcal{L}(-2\rho)$.*

**Proposition 2.13** ([Spr, Theorem 8.5.8])**.** *The $G$-module $\Gamma(\mathcal{B}, \mathcal{L}(\lambda))$ is non-zero if and only if $\lambda$ is a dominant weight.*

**Proposition 2.14.** *The line bundle $\mathcal{L}(\lambda)$ on $\mathcal{B}$ is*

*(i) globally generated if $\lambda$ is dominant;*
*(ii) ample if $\lambda - \rho$ is dominant.*

*Proof.* We first address (i). By Proposition 2.13, it suffices for us to show that $\mathcal{L}(\lambda)$ is globally generated when $\lambda$ is dominant. That is, we wish to show that each stalk of $\mathcal{L}(\lambda)$ is generated by the image of some global section. For $x \in \mathcal{B}$, the stalk of $\mathcal{L}(\lambda)$ at $x$ is given by $\mathcal{L}(\lambda)_x \simeq k_{-\lambda}$ because $G \to \mathcal{B}$ is a locally trivial $B$-bundle. In particular, the isomorphism assigns to an element in $\mathcal{L}(\lambda)_x$ represented by $g \in \Gamma(U, \mathcal{L}(\lambda))$ for some $U \ni x$ its value $g(x)$ at $x$. Take such an element $g$.

Now, fix some non-zero global section $f \in \Gamma(\mathcal{B}, \mathcal{L}(\lambda))$. Because $\mathcal{L}(\lambda)$ is $G$-equivariant, we may assume by translation by $G$ that $f(x) \neq 0$ (since $f$ was chosen to be non-zero). But this means that $g \in \mathcal{L}(\lambda)_x$ is in the span of the image of $f$ in $\mathcal{L}(\lambda)_x$, hence $\mathcal{L}(\lambda)$ is globally generated.

We will now obtain (ii) as a consequence of (i). We wish to check that for any coherent sheaf $\mathcal{F}$ on $\mathcal{B}$, $\mathcal{F} \otimes \mathcal{L}(\lambda)^n$ is globally generated for large $n$. Any such $\mathcal{F}$ is a quotient of a direct sum of line bundles, so we

reduce to the case where $\mathcal{F}$ is a line bundle. The conclusion then follows from Proposition 2.11, (i), and the fact that $\lambda - \rho$ is dominant. $\qquad\square$

In the characteristic 0 case, the Borel-Weil theorem allows us to go much further. In fact, it implies that we may recover the irreducible representations of $G$ from the global sections of $\mathcal{L}(\lambda)$.

**Theorem 2.15** (Borel-Weil). *Suppose that $k$ has characteristic $0$. Then for any dominant weight $\lambda$, the $G$-module $\Gamma(\mathcal{B}, \mathcal{L}(\lambda))^*$ is the irreducible representation of $G$ with highest weight $\lambda$.*

Thus, classically the global sections of $\mathcal{L}(\lambda)$ give us significant insight into the structure of the representations of $G$. To obtain a finer understanding, we are therefore motivated to examine the structure of the pullbacks $\mathcal{L}_w(\lambda)$ of $\mathcal{L}(\lambda)$ to $X_w$. As noted earlier, however, the $X_w$ are singular, presenting an obstacle [2] to understanding $\mathcal{L}_w(\lambda)$. Our approach will be instead to study the further pullbacks to $Z_{\mathfrak{w}}$, which was constructed to be smooth. In this vein, for any $B$-module $V$, define $\mathcal{L}_{\mathfrak{w}}(V) = \theta_{\mathfrak{w}}^*(\mathcal{L}(V))$.

Before beginning to study $\mathcal{L}_{\mathfrak{w}}(V)$, we first realize $\mathcal{L}(V)$ and $\mathcal{L}_{\mathfrak{w}}(V)$ as two instances of a more general construction. Let $X$ be a variety equipped with a free right $B$-action such that the quotient $X/B$ exists and $\pi_{X/B} : X \to X/B$ is a $B$-bundle map. For any $B$-module $V$, we may form the vector bundle $\mathcal{L}_{X/B}(V) = X \times^B V$ on $X/B$ so that

$$\Gamma(U, \mathcal{L}(V)) = \{f : \pi_{X/B}^{-1}(U) \to V \mid f(x \cdot b) = b^{-1} \cdot f(x)\}.$$

In this context, we see that $\mathcal{L}(V) = \mathcal{L}_{G/B}(V)$. By the following lemma, we may realize $\mathcal{L}_{\mathfrak{w}}(V)$ in this way.

**Lemma 2.16.** *Viewing $Z_{\mathfrak{w}}$ as the successive quotient of $P_{\mathfrak{w}}$ by $B^{n-1}$ and then $B$ via the inclusion $B^{n-1} \to B^n$ into the first $n-1$ coordinates, we have an isomorphism $\mathcal{L}_{\mathfrak{w}}(V) \simeq \mathcal{L}_{(P_{\mathfrak{w}}/B^{n-1})/B}(V)$.*

*Proof.* The relevant varieties fit into the following diagram, where the multiplication map $P_{\mathfrak{w}}/B^{n-1} \to G$ is $B$-equivariant and induces the map $\theta_{\mathfrak{w}}$.

$$
\begin{array}{ccc}
P_{\mathfrak{w}}/B^{n-1} & \xrightarrow{\text{mult}} & G \\
\downarrow{\scriptstyle \pi_{(P_{\mathfrak{w}}/B^{n-1})/B}} & & \downarrow{\scriptstyle \pi_{G/B}} \\
Z_{\mathfrak{w}} & \xrightarrow{\theta_{\mathfrak{w}}} & \mathcal{B}
\end{array}
$$

Consider the map $\mathcal{L}_{\mathfrak{w}}(V) \to \mathcal{L}_{(P_{\mathfrak{w}}/B^{n-1})/B}(V)$ that on each $U \subset Z_{\mathfrak{w}}$ sends a function $f \in \Gamma(U', \mathcal{L}(V))$ for $U' \supset \theta_{\mathfrak{w}}(U)$ to the restriction of $f \circ \theta_{\mathfrak{w}}$ to $U$. We check that it is an isomorphism locally. Choose an affine $U \subset Z_{\mathfrak{w}}$ such that $\pi_{(P_{\mathfrak{w}}/B^{n-1})/B}$ is trivial above $U$, and $\theta_{\mathfrak{w}}(U)$ is contained in an affine $U'$ above which $\pi_{G/B}$ is trivial. Locally, our map is given by the identifications

$$\Gamma(U, \mathcal{L}_{\mathfrak{w}}(V)) = \mathcal{O}_U \underset{\mathcal{O}_{U'}}{\otimes} (\mathcal{O}_{U'} \otimes_k V) \simeq \mathcal{O}_U \otimes_k V \simeq \Gamma(U, \mathcal{L}_{(P_{\mathfrak{w}}/B^{n-1})/B}(V)),$$

hence it provides the desired isomorphism. $\qquad\square$

The following results and Lemma 2.16 show that $\mathcal{L}_{\mathfrak{w}}(-)$ and our original $\mathcal{L}(-)$ behave in a somewhat similar manner.

**Proposition 2.17.** *We have the following:*

*(i) For any $\mathfrak{w}$, $\mathcal{L}_{X/B}(-)$ is an exact functor $G$-mod $\to \mathrm{QCoh}(Z_{\mathfrak{w}})$;*
*(ii) For each $V$, $\mathcal{L}_{X/B}(V)$ is locally free of rank $\dim V$;*
*(iii) For representations $V, W$ of $G$, we have $\mathcal{L}_{X/B}(V \otimes W) \simeq \mathcal{L}_{X/B}(V) \otimes \mathcal{L}_{X/B}(W)$.*

*Proof.* The proof is identical to that of Proposition 2.10. Cover $X/B$ by affines $U_i$ over which $\pi_{X/B}$ is trivial, obtaining that $\mathcal{L}_{X/B}(V)|_{U_i} \simeq \mathcal{O}_{U_i} \otimes_k V$. As before, this implies the desired claims easily. $\qquad\square$

**Corollary 2.18.** *If $\lambda$ is dominant, then $\mathcal{L}_{\mathfrak{w}}(\lambda)$ is globally generated.*

*Proof.* By Proposition 2.14(i) and the fact that $\mathcal{L}_{\mathfrak{w}}(\lambda)$ is the pullback of $\mathcal{L}(\lambda)$ by $\theta_{\mathfrak{w}}$. $\qquad\square$

---

[2]This obstacle is not intractable; see [Kem] for an approach in this direction.

For the remainder of this section, we will be interested in understanding line bundles on $Z_\mathfrak{w}$. In particular, we would like to compute the canonical bundle and Picard group of $Z_\mathfrak{w}$. For this, our main tool will be the $\mathbb{P}^1$ fibration $Z_\mathfrak{w} \to Z_{\mathfrak{w}[n-1]}$. We begin with the following general lemma on canonical bundles in the presence of such a fibration.

**Lemma 2.19.** *Let $X$ and $Y$ be smooth varieties and $f : X \to Y$ a $\mathbb{P}^1$-bundle with a section $\sigma : Y \to X$ giving a divisor $D = \sigma(Y)$ with corresponding line bundle $\mathcal{O}_X(D)$. Then, for any line bundle $\mathcal{L}$ on $X$ with degree $1$ along the fibers of $f$, we have*

$$\omega_X = f^*(\omega_Y) \otimes \mathcal{O}_X(-D) \otimes \mathcal{L}^{-1} \otimes f^*\sigma^*\mathcal{L}.$$

*Proof.* The key observation is that if a line bundle $\mathcal{F}$ has degree $0$ along the fibers of $f$, then we have that $f^*\sigma^*\mathcal{F} = \mathcal{F}$. Indeed, it is easy to check this locally using $\mathrm{Pic}(X) = \mathrm{Pic}(Y) \times \mathbb{Z}$, which is [Ha77, Exercise II.7.9]. Now, by [Ha77, Proposition II.8.20], we have that

$$\omega_Y = \sigma^*\omega_X \otimes \sigma^*\mathcal{O}_X(D).$$

Now, both $\mathcal{L}$ and $\mathcal{O}_X(D)$ have degree $1$ on the fibers [3], so we find

$$\mathcal{L} \otimes \mathcal{O}_X(-D) \simeq f^*\sigma^*(\mathcal{L} \otimes \mathcal{O}_X(-D)) \simeq f^*\sigma^*\mathcal{L} \otimes f^*\sigma^*\omega_X \otimes f^*\omega_Y^{-1}.$$

Now, notice that $\omega_X$ has degree $-2$ on the fibers of $f$, hence $\omega_X \otimes \mathcal{L}^2$ has degree $0$. This means that

$$\omega_X \otimes \mathcal{L}^2 \simeq f^*\sigma^*\omega_X \otimes f^*\sigma^*\mathcal{L}^2.$$

Combining these two observations and rearranging gives the conclusion. $\qquad\square$

To use Lemma 2.19, however, we first need to understand how to find the degree of line bundles on $Z_\mathfrak{w}$ along the fiber of $\psi_\mathfrak{w}$. The next two results provide this tool and in fact go a little bit further.

**Proposition 2.20.** *Let $i : P_i/B \to \mathcal{B}$ be the natural embedding. Then, the character of the $G$-module $\Gamma(P_i/B, i^*\mathcal{L}(\lambda))$ is given by*

$$\mathrm{ch}\,\Gamma(P_i/B, i^*\mathcal{L}(\lambda)) = \begin{cases} e^\lambda + e^{\lambda - \alpha_i} + \cdots + e^{s_i\lambda} & \text{if } \langle \alpha_i^\vee, \lambda \rangle \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* By an argument similar to the proof of Lemma 2.16, we have that $i^*\mathcal{L}(\lambda) = \mathcal{L}_{P_i/B}(\lambda)$. So we may assume that $G = P_i$ has semi-simple rank 1. By [Spr, Proposition 8.5.8] we see that $\Gamma(P_i/B, \mathcal{L}(\lambda)) = 0$ unless $\langle \alpha_i^\vee, \lambda \rangle \geq 0$. In this case, write $\lambda = \lambda' + n\chi_i$, where $n = \langle \alpha_i^\vee, \lambda \rangle$ and $\langle \alpha_i^\vee, \lambda' \rangle = 0$. By Proposition 2.10(iii), we have that $\mathcal{L}(\lambda) = \mathcal{L}(\lambda') \otimes \mathcal{L}(\chi_i)^n$. Therefore, it suffices to check that $\mathcal{O}_{P_i/B} \simeq \mathcal{L}(\lambda')$ and $\mathcal{O}_{P_i/B}(1) \simeq \mathcal{L}(\chi_i)$ with the appropriate induced $B$-actions.

For the former, because $\langle \alpha_i^\vee, \lambda' \rangle = 0$, [Spr, Proposition 7.3.6] implies that global sections of $\mathcal{L}(\lambda')$ are constant on $(P_i, P_i)$. Thus we have $\mathcal{O}_{P_i/B} \simeq \mathcal{L}(\lambda')$ and note that $B$ may act only through $e^{-\lambda}$, as needed. For the latter, by Proposition 2.11, $\{\mathcal{L}(\chi_j)\}$ is a set of generators for $\mathrm{Pic}(P_i/B)$. But we just showed that neither $\mathcal{L}(-\chi_j)$ nor $\mathcal{L}(\chi_j)$ has non-trivial global sections for $j \neq i$, hence only $\mathcal{L}(\chi_i)$ can generate $\mathrm{Pic}(P_i/B)$. This shows that $\mathcal{L}(\chi_i) \simeq \mathcal{O}_{P_i/B}(1)$. On the other hand, we may compute explicitly that $B$ acts by $e^{\chi_i}$ or $e^{-\chi_i}$ on $\mathcal{L}(\chi_i)$, giving the desired character. $\qquad\square$

**Lemma 2.21.** *The line bundles $\mathcal{L}_\mathfrak{w}(\lambda)$ and $\mathcal{O}(Z_{\mathfrak{w}[n-1]})$ have degree $\langle \alpha_{i_n}^\vee, \lambda \rangle$ and $1$ along the fibers of $\psi_\mathfrak{w} : Z_\mathfrak{w} \to Z_{\mathfrak{w}[n-1]}$.*

*Proof.* Note that $\mathcal{L}_\mathfrak{w}(\lambda)$ is the pullback of the $\mathbb{P}^1$-bundle $\mathcal{B} \to G/P_{i_n}$ by Proposition 2.7, so it suffices to show that $\mathcal{L}(\lambda)$ has degree $\langle \alpha_{i_n}^\vee, \lambda \rangle$ along $P_{i_n}/B \subset \mathcal{B}$. The conclusion for $\mathcal{L}_w(\lambda)$ then follows from Proposition 2.20 and the computation of cohomologies of projective space. For $\mathcal{O}(Z_{\mathfrak{w}[n-1]})$, it suffices to note that the divisor $Z_{\mathfrak{w}[n-1]}$ has codimension 1 in $Z_\mathfrak{w}$ by Proposition 2.5. $\qquad\square$

We are now ready for the main results of this section, the computation of the Picard group and canonical bundle of $Z_\mathfrak{w}$. The proofs of both results will use induction along the $\mathbb{P}^1$-bundles $\psi_\mathfrak{w} : Z_\mathfrak{w} \to Z_{\mathfrak{w}[n-1]}$ in an essential way.

---

[3]The degree of $\mathcal{O}_X(D)$ is 1 on the fibers because codim $D = 1$.

**Proposition 2.22.** *Let $\mathfrak{w}$ be any word. Then, the canonical bundle $\omega_{Z_\mathfrak{w}}$ of $Z_\mathfrak{w}$ is given by*

$$\omega_{Z_\mathfrak{w}} \simeq \mathcal{O}_{Z_\mathfrak{w}}(-\partial Z_\mathfrak{w}) \otimes \mathcal{L}_\mathfrak{w}(-\rho).$$

*Proof.* We induct on $\ell(\mathfrak{w})$. For the base case $n = 1$, we have that $Z_\mathfrak{w} = P_{i_1}/B \simeq \mathbb{P}^1$. Now, we have here that $\partial Z_\mathfrak{w} = \{z_0\}$ for any point $z_0 \in Z_\mathfrak{w}$. Now, recall that $\omega_{\mathbb{P}^1} \simeq \mathcal{O}(-2)$, where we may identify the line bundle $\mathcal{O}(-1)$ on $\mathbb{P}^1$ with $\mathcal{O}_{Z_\mathfrak{w}}(-z_0)$ for $z_0$ the image of the point $\partial Z_\mathfrak{w}$ in $Z$. Now Lemma 2.21 completes the proof.

Now, suppose that $\ell(\mathfrak{w}) = n$ and we have the desired description of the canonical bundle for $Z_{\mathfrak{w}[n-1]}$. Recall from Proposition 2.7 that we had a $\mathbb{P}^1$-bundle $\psi_\mathfrak{w} : Z_\mathfrak{w} \to Z_{\mathfrak{w}[n-1]}$ coming from the pullback of $\mathcal{B} \to G/P_{i_n}$ which has a section $\iota_{\mathfrak{w},n}$. Therefore, applying Lemma 2.19 to this fibration with the bundle $\mathcal{L}_\mathfrak{w}(\rho)$ which has degree 1 along the fibers by Lemma 2.21, we obtain

$$\omega_{Z_\mathfrak{w}} \simeq \psi_\mathfrak{w}^*(\omega_{Z_{\mathfrak{w}[n-1]}}) \otimes \mathcal{O}_{Z_\mathfrak{w}}(-\iota_{\mathfrak{w},n}(Z_{\mathfrak{w}[n-1]})) \otimes \mathcal{L}_\mathfrak{w}(-\rho) \otimes \psi_\mathfrak{w}^* \iota_{\mathfrak{w},n}^* \mathcal{L}_\mathfrak{w}(\rho)$$

$$\simeq \bigotimes_{j=1}^{n-1} \mathcal{O}_{Z_\mathfrak{w}}(-Z_{\mathfrak{w}(j)}) \otimes \psi_\mathfrak{w}^* \iota_{\mathfrak{w},n}^* \mathcal{L}_\mathfrak{w}(-\rho) \otimes \mathcal{O}_{Z_\mathfrak{w}}(-Z_{\mathfrak{w}[n-1]}) \otimes \mathcal{L}_\mathfrak{w}(-\rho) \otimes \psi_\mathfrak{w}^* \iota_{\mathfrak{w},n}^* \mathcal{L}_\mathfrak{w}(\rho)$$

$$\simeq \mathcal{O}_{Z_\mathfrak{w}}(-\partial Z_\mathfrak{w}) \otimes \mathcal{L}_\mathfrak{w}(-\rho),$$

completing the induction. $\qquad\square$

**Remark.** Notice that we may equip both bundles in Proposition 2.22 with a $B$-equivariant structure by right multiplication on the base. However, the $B$-actions do not agree; the action on $\mathcal{O}_{Z_\mathfrak{w}}(-\partial Z_\mathfrak{w}) \otimes \mathcal{L}_\mathfrak{w}(-\rho)$ is twisted by the character $-\rho \in X^*(T)$.

**Proposition 2.23.** *The line bundles $\mathcal{O}_{Z_\mathfrak{w}}(Z_{\mathfrak{w}(j)})$ associated to divisors in $Z_\mathfrak{w}$ form a basis of the Picard group $\mathrm{Pic}(Z_\mathfrak{w})$.*

*Proof.* We proceed by induction on $\ell(\mathfrak{w})$. For $\ell(\mathfrak{w}) = 1$, the claim is simply that $\mathcal{O}(1)$ generates $\mathrm{Pic}(\mathbb{P}^1)$. Now suppose the claim holds for some $n$ and take $\ell(\mathfrak{w}) = n + 1$. We have a $\mathbb{P}^1$-bundle map $\psi_\mathfrak{w} : Z_\mathfrak{w} \to Z_{\mathfrak{w}[n-1]}$, where $\mathcal{O}(Z_{\mathfrak{w}[n-1]})$ is of degree 1 along the fibers of $\psi_\mathfrak{w}$ by Lemma 2.21. By [Ha77, Exercise II.7.9], we have that $\mathrm{Pic}(Z_\mathfrak{w}) \simeq \mathrm{Pic}(Z_{\mathfrak{w}[n-1]}) \oplus \mathbb{Z}$, where the $\mathrm{Pic}(Z_{\mathfrak{w}[n-1]})$ component is given by pullback from $Z_{\mathfrak{w}[n-1]}$ and the $\mathbb{Z}$ component from the fibers. Hence the conclusion follows from Proposition 2.6 and the fact that $\mathcal{O}(Z_{\mathfrak{w}[n-1]})$ is of degree 1 along the fibers. $\qquad\square$

## 3. Frobenius splitting

We now take a pause from our study of the geometry of the flag variety to introduce the characteristic $p$ input of Frobenius splitting, the main technical tool of this thesis. Our goal will be to give a general splitting criterion for smooth projective varieties that we will later apply to the Bott-Samelson variety. In this section, all results will be in characteristic $p > 0$.

3.1. **Basic properties of Frobenius splitting.** Recall that any commutative $k$-algebra $A$ comes equipped with the *Frobenius map* $F : A \to A$ acting by $a \mapsto a^p$. Notice that $F$ is a ring homomorphism but not a map of $k$-algebras, as it may provide a non-trivial automorphism of $k$.

This map admits a natural extension to the algebro-geometric setting. Let $X$ be a scheme (of finite type over $k$). Then, the *absolute Frobenius morphism*

$$F_X : X \to X$$

is the identity on the level of topological spaces and the $p^{\text{th}}$ power map $F_X^\# : \mathcal{O}_X \to (F_X)_* \mathcal{O}_X$ on structure sheaves. By this, we mean that $F_X^\#$ is given on each section by the Frobenius map on the underlying $k$-algebras. When the relevant scheme is clear, we will often omit the subscripts on $F_X$ and $F_X^\#$. Notice that $F$ is a map of schemes, but not a map of schemes over $k$ (precisely because the Frobenius map is not a map of $k$-algebras). We summarize the general properties of $F$ in the following proposition.

**Proposition 3.1.** *We have the following:*
  *(i) For any map of schemes $f : X \to Y$, we have $F_Y \circ f = f \circ F_X$;*
  *(ii) $F$ is a finite map of schemes;*
  *(iii) If $X$ is regular, $F$ is flat.*

*Proof.* For (i), it suffices to check that $F_* f_* \mathcal{O}_X \simeq f_* F_* \mathcal{O}_X$. But they agree on the level of abelian groups, and it is easy to see that the $\mathcal{O}_Y$-module structures are the same, giving the desired.

For (ii), we check on each affine. Because $X$ was of finite type, it suffices to show that the ring $k[t_1, \ldots, t_n]/I$ is finitely generated as a module over itself with action twisted by the Frobenius map $F$. Indeed, an explicit set of generators is given by $t_1^{a_1} \cdots t_n^{a_n}$ for $0 \le a_i < p$, showing that $F$ is finite.

Finally, for (iii), we may check on stalks. Let $A = (\mathcal{O}_X)_x$ and take $t_1, \ldots, t_n$ to be regular elements generating the maximal ideal $\mathfrak{m}$. We now claim that the elements $t_1^{a_1} \cdots t_n^{a_n}$ for $0 \le a_i < p$ form a set of free generators for $A$ as an $A$-module with action twisted by $F$. First, they are free because they are free in the completion $k[[t_1, \ldots, t_n]]$. It remains now to show that they generate $A$. By Nakayama's lemma, it is enough to show they generate $A/F(\mathfrak{m})A$. This follows from the fact that $(t_1^p, \ldots, t_n^p) \subset F(\mathfrak{m})$. Hence, $A$ is locally free, so $F$ is flat. $\qquad\square$

Now, let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. Because $F$ is the identity on the level of topological spaces, we see that $F_*(\mathcal{F}) = \mathcal{F}$ as sheaves of abelian groups on $X$. However, the $\mathcal{O}_X$-module structure on $F_*(\mathcal{F})$ is twisted by the Frobenius map; that is, on each $U \subset X$, the action is given by $a \cdot x = a^p x$ for $a \in \Gamma(U, X)$ and $x \in \Gamma(U, \mathcal{F})$. In particular, the sheaf of $\mathcal{O}_X$-modules $F_* \mathcal{O}_X$ has the same underlying abelian group structure as $\mathcal{O}_X$ with this twisted $\mathcal{O}_X$-action. Similarly, we may characterize the effect of $F^*$ on line bundles in the following lemma.

**Lemma 3.2.** *For any line bundle $\mathcal{L}$, we have that $F^* \mathcal{L} \simeq \mathcal{L}^p$.*

*Proof.* By definition, we have that

$$F^* \mathcal{L} = F^{-1} \mathcal{L} \underset{F^{-1} \mathcal{O}_X}{\otimes} \mathcal{O}_X = \mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X,$$

where the action of $\mathcal{O}_X$ on itself on the right is given by the $p^{\text{th}}$ power map. Therefore, we have a map $\mathcal{L} \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_X \to \mathcal{L}^p$ given by $\sigma \otimes f \mapsto \sigma^p f$. It is surjective, hence defines the desired isomorphism. $\qquad\square$

For every scheme $X$, we have now obtained a map of $\mathcal{O}_X$-modules $\mathcal{O}_X \to F_* \mathcal{O}_X$. It is then natural to consider when this map admits a splitting. We will see that this occurs rarely, but that if such a splitting exists, then $X$ satisfies some very special properties. Let us first define the situation more formally.

**Definition 3.3.** A scheme $X$ is *Frobenius split* (or simply *split*) if the map of sheaves of $\mathcal{O}_X$-modules $\mathcal{O}_X \to F_* \mathcal{O}_X$ admits a splitting. That is, there is a $\mathcal{O}_X$-linear map $\phi : F_* \mathcal{O}_X \to \mathcal{O}_X$, known as a *splitting* of $X$, such that $\phi \circ F^{\#} = \operatorname{id}_{\mathcal{O}_X}$.

**Remark.** If $X$ is a variety then the map $F^{\#}$ is injective, hence the Frobenius morphism fits into a short exact sequence

$$0 \to \mathcal{O}_X \to F_* \mathcal{O}_X \to C \to 0$$

of $\mathcal{O}_X$-modules. In this case, the condition that $X$ is split means that this is a split exact sequence.

While the notion of splitting has been defined rather abstractly here, we may reformulate it in much more concrete terms as follows.

**Proposition 3.4.** *A map $\phi \in \operatorname{Hom}_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X)$ is a splitting of $X$ if and only if $\phi(1) = 1$.*

*Proof.* For any $f \in \mathcal{O}_X$, notice that $(\phi \circ F^{\#})(f) = \phi(f^p) = f\phi(1)$, where in the second equality we note the $\mathcal{O}_X$-module structure on $F_* \mathcal{O}_X$. Hence, the map $\phi \circ F^{\#}$ is simply multiplication by $\phi(1)$, hence $\phi$ is a splitting if and only if $\phi(1) = 1$. $\qquad\square$

In fact, on the level of sheaves of abelian groups, a map $\phi \in \operatorname{Hom}_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X)$ is a map $\mathcal{O}_X \to \mathcal{O}_X$ such that $\phi(f^p g) = f\phi(g)$ for all $f, g$. Thus, to check if such a map is a splitting, it suffices to verify that $\phi(1) = 1$ and that $\phi(f^p g) = f\phi(g)$ for all $f, g$.

By Proposition 3.4, a scheme $X$ is Frobenius split if and only if the evaluation map $\operatorname{Hom}_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X) \to \Gamma(X, \mathcal{O}_X)$ is surjective. But this map naturally extends to a map $\mathcal{H}om_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X) \to \mathcal{O}_X$ of sheaves, giving the following natural condition for Frobenius splitting.

**Corollary 3.5.** *A scheme $X$ is split if and only if the evaluation map $\mathcal{H}om_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X) \to \mathcal{O}_X$ given by $\phi \mapsto \phi(1)$ is surjective on global sections.*

This criterion allows us to show that a large class of schemes are Frobenius split via the following proposition.

**Proposition 3.6.** *Any affine smooth variety is split.*

*Proof.* Fix a variety $X = \operatorname{Spec}(A)$, where $A$ is a regular finitely generated $k$-algebra. Note now that $\operatorname{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ consists of all maps $\phi : A \to A$ such that $\phi(f^p g) = f\phi(g)$ for $f, g \in A$. By Corollary 3.5, we wish to show that the evaluation map $\operatorname{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \to A$ is surjective. But any element of $\operatorname{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ is determined by its value on a finite set of generators for $A$, hence $\operatorname{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ is finitely generated. So it suffices to show surjectivity on the completion of each localization of $A$. On each completed local ring $\widehat{A}_{\mathfrak{p}} \simeq k[[t_1, \ldots, t_n]]$, the map $\phi : k[[t_1, \ldots, t_n]] \to k[[t_1, \ldots, t_n]]$ given by

$$t_1^{a_1} \cdots t_n^{a_n} \mapsto \begin{cases} 0 & p \nmid a_i \text{ for some } i \\ t_1^{a_1/p} \cdots t_n^{a_n/p} & \text{otherwise} \end{cases}$$

is in $\operatorname{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ and satisfies $\phi(1) = 1$. Therefore, the evaluation map is surjective, meaning that $X$ is split by Corollary 3.5. □

In practice, we will use the concept of splitting in conjunction with a refinement. Let $X$ be a Frobenius split scheme with splitting $\phi : F_*\mathcal{O}_X \to \mathcal{O}_X$. If $Y$ is a closed subscheme of $X$, then by Proposition 3.1(i), $F_Y$ is given by the restriction of $F_X$ to $Y$. On the level of structure sheaves, we then have the following diagram, where $\mathcal{I}_Y$ is the sheaf of ideals corresponding to $Y$.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & F_*\mathcal{I}_Y & \longrightarrow & F_*\mathcal{O}_X & \longrightarrow & F_*\mathcal{O}_Y & \longrightarrow & 0 \\
 & & \big\downarrow & & \phi\big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{I}_Y & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_Y & \longrightarrow & 0
\end{array}
$$

We would often like our splitting $\phi$ of $X$ to induce a splitting of $Y$, which will occur if it restricts to a map $F_*\mathcal{I}_Y \to \mathcal{I}_Y$, i.e. if $\phi(F_*\mathcal{I}_Y) \subset \mathcal{I}_Y$. Observe that for any $f \in \mathcal{I}_Y$, we have $f = \phi(f^p)$ with $f^p \in F_*\mathcal{I}_Y$, hence $\mathcal{I}_Y \subset \phi(F_*\mathcal{I}_Y)$. This means that this condition is equivalent to $\phi(F_*\mathcal{I}_Y) = \mathcal{I}_Y$. We formalize this in the following definition.

**Definition 3.7.** Let $X$ be a split scheme and $Y$ a closed subscheme of $X$. We say that a splitting $\phi$ of $X$ is *compatible* with $Y$ if $\phi(F_*\mathcal{I}_Y) \subset \mathcal{I}_Y$. If such a splitting exists, we say that $X$ is *compatibly split* with $Y$.

For multiple closed subschemes $Y_1, \ldots, Y_n$ of $X$, we say that $X$ is compatibly split with $Y_1, \ldots, Y_n$ if there exists some splitting $\phi$ of $X$ compatible with each $Y_i$.

The following proposition relates places our example from Proposition 3.6 in this context.

**Proposition 3.8.** *A affine smooth variety is split compatibly with any smooth subvariety.*

*Proof.* Let $Y$ be a smooth subvariety of the smooth affine variety $X$. We modify the argument for Proposition 3.6 slightly. Instead of $\operatorname{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$, consider the submodule

$$M = \{\phi : F_*\mathcal{O}_X \to \mathcal{O}_X \mid \phi(F_*\mathcal{I}_Y) \subset \mathcal{I}_Y\}.$$

As before, $X$ is split compatibly with $Y$ if and only if the evaluation map $M \to A$ is surjective, and $M$ is finitely generated, so it is enough to check this on the completion of each localization of $A$. Because $Y$ is smooth, we may choose coordinates at each point $x \in X$ so that $(\mathcal{I}_Y)_x = (t_1, \ldots, t_m) \subset k[[t_1, \ldots, t_n]] \simeq (\mathcal{O}_X)_x$ for some $m$. The splitting constructed in the proof of Proposition 3.6 preserves $(\mathcal{I}_Y)_x$, hence is compatible with $Y$. Again, this shows the evaluation map $M \to A$ is surjective, hence $X$ is compatibly split with $Y$. □

We may now understand the behavior of splittings under some basic operations. In particular, the following propositions show that splittings behave nicely under restrictions and images of schemes.

**Proposition 3.9.** *Let $X$ be a scheme and $Y$ a closed subscheme compatibly split by some $\phi$. Then, for any open $U \subset X$, we see that $\phi|_U$ is a compatible splitting of $U$ and $U \cap Y$.*

*Proof.* First, notice that $\phi|_U$ is evidently a map of $\mathcal{O}_U$ modules. Because $\phi(1) = 1$, we see that $\phi|_U(1) = 1$, hence $\phi|_U$ splits $U$ by Corollary 3.5. Now, let $i : U \to Z$ be the inclusion and note that

$$\phi|_U(F_* \mathcal{I}_{U \cap Y}) = \phi|_U(F_* i^* \mathcal{I}_Y) = \phi|_U(i^* F_* \mathcal{I}_Y) = \phi(F_* \mathcal{I}_Y)|_U \subset \mathcal{I}_Y|_U = \mathcal{I}_{U \cap Y},$$

hence this splitting is compatible with $U \cap Y$. Here we note that $F_*$ and $i^*$ commute because $F_*$ is the identity when viewed as a map between sheaves of abelian groups.  □

**Proposition 3.10.** *Let $X$ be a reduced scheme and $Y$ a a reduced closed subscheme. For any map of $\mathcal{O}_X$-modules $\phi : F_* \mathcal{O}_X \to \mathcal{O}_X$, if there exists a dense open subset $U$ of $X$ such that $\phi|_U$ is a splitting of $U$ (resp., compatible with $U \cap Y$), then $\phi$ is a splitting of $X$ (resp., compatible with $Y$).*

*Proof.* For the first assertion, notice that $\phi|_U(1) = 1$, meaning the regular function $\phi(1)$ on $X$ is constant on a dense open subset, hence constant. So we have $\phi(1) = 1$ on all of $X$. Now, if $\phi|_U$ is compatible with $U \cap Y$, we see that $\phi(F_* \mathcal{I}_Y) \supset \mathcal{I}_Y$ is a coherent $\mathcal{O}_X$-module. It agrees with $\mathcal{I}_Y$ on the dense open set $U$, hence it is equal to $\mathcal{I}_Y$ on all of $Y$ because $Y$ is reduced, giving compatibility.  □

**Proposition 3.11.** *Let $X$ be a scheme with a splitting $\phi$ compatible with closed subschemes $Y$ and $Z$. Then, $\phi$ is compatible with their scheme-theoretic intersection $Y \cap Z$.*

*Proof.* Let us recall that $\mathcal{I}_{Y \cap Z} = \mathcal{I}_Y + \mathcal{I}_Z$. We may then compute

$$\phi(F_* \mathcal{I}_{Y \cap Z}) = \phi(F_*(\mathcal{I}_Y + \mathcal{I}_Z)) \subset \phi(F_* \mathcal{I}_Y) + \phi(F_* \mathcal{I}_Z) \subset \mathcal{I}_Y + \mathcal{I}_Z = \mathcal{I}_{Y \cap Z},$$

whence $\phi$ is compatible with $Y \cap Z$.  □

**Proposition 3.12.** *Suppose that $f : X \to Y$ is a map of schemes so that $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is an isomorphism. Let $Z$ be a closed subscheme of $X$, and let $W$ be its scheme theoretic image in $Y$ under $f$. Then, if $X$ is split (resp., compatibly with $Z$), then $Y$ is split (resp., compatibly with $W$).*

*Proof.* Let $\phi : F_* \mathcal{O}_X \to \mathcal{O}_X$ be a splitting of $X$. Then, notice that $f_* \phi$ splits the map $f_* \mathcal{O}_X \to f_*(F_X)_* \mathcal{O}_X \simeq (F_Y)_* f_* \mathcal{O}_X$. The given isomorphism shows this is the map $\mathcal{O}_Y \to F_* \mathcal{O}_Y$, hence $f_* \phi$ gives the desired splitting.

Suppose now that $\phi$ was compatible with some $Z$. Then, we see that

$$f_* \phi(F_* \mathcal{I}_W) = f_* \phi(F_* f_* \mathcal{I}_Z) \subset f_* \mathcal{I}_Z = \mathcal{I}_W,$$

where we have that $\mathcal{I}_W = (f^\#)^{-1}(f_* \mathcal{I}_Z) \simeq f_* \mathcal{I}_Z$ because $W$ is the scheme-theoretic image of $Z$.  □

We may strengthen Proposition 3.12 to the following stronger statement.

**Proposition 3.13.** *Suppose that $f : X \to Y$ is a map of schemes so that $f^\# : \mathcal{O}_Y \to f_* \mathcal{O}_X$ splits as a map of $\mathcal{O}_Y$-modules. Then, if $X$ is split, then $Y$ is split.*

*Proof.* Let $\psi : f_* \mathcal{O}_X \to \mathcal{O}_Y$ be the given splitting. We claim that the composition

$$F_* \mathcal{O}_Y \overset{F_* f^\#}{\to} F_* f_* \mathcal{O}_X \simeq f_* F_* \mathcal{O}_X \overset{f_* \phi}{\to} f_* \mathcal{O}_X \overset{\psi}{\to} \mathcal{O}_Y$$

provides a splitting of $Y$. Indeed, being careful to differentiate between $F_X$ and $F_Y$, we may check that

$$\psi \circ f_* \phi \circ (F_Y)_* f^\# \circ F_Y^\# = \psi \circ f_* \phi \circ f_* F_X^\# \circ f^\# = \psi \circ f^\# = \mathrm{id}_{\mathcal{O}_Y},$$

where we note that $(F_Y)_* f^\# \circ F_Y^\# = f_* F_X^\# \circ f^\#$ because $F_Y \circ f = f \circ F_X$.  □

3.2. **A splitting criterion for smooth projective varieties.** In our application, we would like to consider splittings of $X_w$ via its desingularization $Z_\mathfrak{w}$. Therefore we first restrict our attention to splittings of smooth projective varieties. In this case, by Proposition 3.1, we see that $F : X \to X$ is a finite flat morphism, suggesting that we should apply finite flat duality (see Theorem A.1) to $F$. However, our situation is much much nicer than the general case of this duality, so we will be able to use a simplified and more explicit presentation. We will suppress the more technically involved details to Appendix A.

Let $X$ be a smooth projective variety of dimension $n$, and let $\omega_X$ denote its canonical bundle. Let us now recall the situation of Corollary 3.5 for $X$. Because $X$ is projective, the evaluation map $F_* \mathcal{O}_X \to \mathcal{O}_X$ gives rise to a regular, hence constant function on $X$. Therefore, to check whether a global function $\phi : F_* \mathcal{O}_X \to \mathcal{O}_X$ gives a splitting, it suffices to check that it evaluates to 1 at a single point $x \in X$. Therefore, we wish to examine the local properties of such maps.

We first introduce some notation on local rings. Fix some $x \in X$; we have a isomorphism of regular local rings $\bigwedge^n \Omega^1_{X/k} \simeq \omega_{X,x} \simeq \mathcal{O}_{X,x}$. We will generically take $t_1, \ldots, t_n$ to be a system of local coordinates at $x$ and fix isomorphisms $\widehat{\mathcal{O}}_{X,x} \simeq k[[t_1, \ldots, t_n]]$ and $\widehat{\omega}_{X,x} \simeq k[[t_1, \ldots, t_n]] \cdot (dt_1 \wedge \cdots \wedge dt_n)$ by the Cohen structure theorem. Write $\omega = dt_1 \wedge \cdots \wedge dt_n$ for the generator of $\widehat{\omega}_{X,x}$. We denote the monomial $t_1^{a_1} \cdots t_n^{a_n}$ in $k[[t_1, \ldots, t_n]]$ by $t^{\mathbf{a}}$ with $\mathbf{a} = (a_1, \ldots, a_n)$. Similarly, for an integer $m$, $\mathbf{m}$ denotes the $n$-long vector $(m, \ldots, m)$. Finally, we define the function

$$\delta_p(\mathbf{a}) = \begin{cases} 1 & p \mid a_i \text{ for all } i \\ 0 & \text{otherwise.} \end{cases}$$

We can now characterize evaluation locally via the following proposition, our version of finite flat duality.

**Proposition 3.14.** *Let $X$ be a smooth projective variety of dimension $n$. We have an isomorphism*

$$F_* \omega_X^{1-p} \to \mathcal{H}om_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X)$$

*which in each system of local coordinates is given by*

$$t^{\mathbf{a}} \cdot \omega^{1-p} \mapsto \left( t^{\mathbf{b}} \mapsto \delta_p(\mathbf{a} + \mathbf{b} + \mathbf{1}) t^{\frac{\mathbf{a}+\mathbf{b}+\mathbf{1}}{p} - \mathbf{1}} \right).$$

*Proof.* See Appendix A. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Denote the composition $\mathrm{Tr}_X : F_* \omega_X^{1-p} \to \mathcal{H}om_{\mathcal{O}_X}(F_* \mathcal{O}_X, \mathcal{O}_X) \xrightarrow{\mathrm{ev}} \mathcal{O}_X$. By Proposition 3.14 and our previous discussion, we see that $X$ is split if and only if there is a global section $\sigma \in \Gamma(X, F_* \omega_X^{1-p})$ such that $\mathrm{Tr}_{X,x}(\sigma_x) = 1$ locally at some $x \in X$. This observation leads to the following criterion for splitting.

**Proposition 3.15.** *Let $X$ be a smooth projective variety of dimension $n$. Then $X$ is Frobenius split if and only if there exists $\sigma \in \Gamma(X, F_* \omega_X^{1-p})$ and $x \in X$ such that its local expansion $\sigma_x$ at $x$ is given by*

$$\sigma_x = \sum_{\mathbf{a} \in \mathbb{N}^n} c_{\mathbf{a}} t^{\mathbf{a}} \cdot \omega^{1-p}$$

*with $\alpha_{\mathbf{p}-\mathbf{1}} \neq 0$ for some choice of coordinates $t_1, \ldots, t_n$.*

*Proof.* As we just noted, $X$ is Frobenius split if and only if there is some $\sigma$ such that $\mathrm{Tr}_{X,x}(\sigma_x) = 1 \in \widehat{\mathcal{O}}_{X,x}$ locally at $x \in X$. But this occurs if and only if $\mathrm{Tr}_{X,x}(\sigma_x)$ lies outside the maximal ideal for some $\sigma$ and $x$. Now, taking the local expansion $\sigma_x = \sum_{\mathbf{a} \in \mathbb{N}^n} c_{\mathbf{a}} t^{\mathbf{a}} \cdot \omega^{1-p}$ in coordinates so that $\mathrm{Tr}_{X,x}$ takes the form of Proposition 3.14, we find that

$$\mathrm{Tr}_{X,x}(\sigma_x) = \sum_{\alpha \in \mathbb{N}^n} c_{\mathbf{a}} \delta_p(\mathbf{a} + \mathbf{1}) t^{\frac{\mathbf{a}+\mathbf{1}}{p} - \mathbf{1}},$$

hence $\mathrm{Tr}_{X,x}(\sigma_x)$ lies outside the maximal ideal if and only if its constant coefficient $\alpha_{\mathbf{p}-\mathbf{1}}$ is non-zero. $\quad\square$

We may now obtain the following criterion for splitting in terms of prime divisors which is designed specifically to apply to the Bott-Samelson variety.

**Proposition 3.16.** *Let $X$ be a smooth projective variety of dimension $n$. If there exists some $\sigma \in \Gamma(X, \omega_X^{-1})$ whose divisor of zeros is*

$$\sigma_0 = Y_1 + \cdots + Y_n + Z$$

*with $Y_i$ prime divisors intersecting transversely at some $x$ and $Z$ an effective divisor not containing $x$, then $\sigma^{p-1}$ splits $X$ compatibly with $Y_1, \ldots, Y_n$.*

*Proof.* We will apply the criterion of Proposition 3.15. Because $Y_1, \ldots, Y_n$ intersect transversely at $x$ and are prime, we may find a system of local coordinates $t_1, \ldots, t_n$ at $x$ where $Y_i$ is given by the equation $t_i = 0$. In these coordinates then, we may write

$$\sigma_x = t_1 \cdots t_n f(t_1, \ldots, t_n) \cdot \omega^{-1}$$

with $f(0, \ldots, 0) \neq 0$. Now consider the section $\sigma^{p-1} \in \Gamma(X, \omega_X^{1-p})$. We have that

$$\sigma_x^{p-1} = t^{\mathbf{p}-\mathbf{1}} f(t_1, \ldots, t_n)^{p-1} \cdot \omega^{1-p},$$

which has $f(0, \ldots, 0)^{p-1} \neq 0$. By Proposition 3.15, $\sigma^{p-1}$ splits $X$. Let $\phi$ be the splitting induced by $\sigma^{p-1}$.

Now, let us check that this splitting is compatible with each $Y_i$. Let $\mathcal{I}_{Y_i}$ be the ideal sheaf of $Y_i$. We wish to check that $\phi(F_* \mathcal{I}_{Y_i}) \subset \mathcal{I}_{Y_i}$ which, as we showed before, is equivalent to $\phi(F_* \mathcal{I}_{Y_i}) = \mathcal{I}_{Y_i}$ as quasi-coherent subsheaves of $\mathcal{O}_X$. It suffices by Lemma 3.10 to check this statement locally at $x$ (as it will then hold on some open, hence dense, neighborhood of $x$). But at $x$, $Y_i$ has equation given by $t_i = 0$. Let us write out the expansion of $\sigma_x^{p-1}$ as

$$\sigma_x^{p-1} = t^{\mathbf{p-1}} \sum_{\mathbf{a}} c_{\mathbf{a}} t^{\mathbf{a}} \cdot \omega^{1-p}$$

with $c_{\mathbf{0}} \neq 0$. Then, the splitting map $\widehat{\phi}_x$ is given in these local coordinates by

$$t^{\mathbf{b}} \mapsto \sum_{\mathbf{a}} c_{\mathbf{a}} \delta(\mathbf{a} + \mathbf{b} + \mathbf{1}) t^{\frac{\mathbf{a+b}}{p}}.$$

Any $h(t_1, \ldots, t_n) \in \widehat{(\mathcal{I}_{Y_i})_x}$ is the linear combination of monomials $t^{\mathbf{b}}$ with $b_i > 0$; these are mapped by $\widehat{\phi}_x(h(t_1, \ldots, t_n))$ to either zero or monomials $t^{\mathbf{a}}$ with $a_i > 0$. Thus, we find that $\widehat{\phi}_x(h(t_1, \ldots, t_n))$ is the linear combination of monomials $t^{\mathbf{b}}$ with $b_i > 0$, meaning that $\widehat{\phi}_x(h(t_1, \ldots, t_n)) \in \widehat{(\mathcal{I}_{Y_i})_x}$. This shows the desired inclusion locally at $x$, meaning that $\phi$ is compatible with $Y_i$. $\qquad\square$

3.3. **Splitting and cohomology vanishing.** We begin by giving a general cohomology vanishing consequence of Frobenius splitting. This provides a general prototype for the vanishing theorems that hold on a split scheme and will serve as a model for our later more specific extensions.

**Proposition 3.17.** *Let $X$ be a split scheme proper over an affine scheme. Let $\mathcal{L}$ be an ample line bundle on $X$. Then we have the following:*

*(i) For $i > 0$, we have $H^i(X, \mathcal{L}) = 0$;*

*(ii) If $X$ is compatibly split with some closed subscheme $Y$, the map $H^0(X, \mathcal{L}) \to H^0(Y, \mathcal{L})$ is surjective and $H^i(Y, \mathcal{L}) = 0$ for $i > 0$.*

*Proof.* We first address (i). Tensoring the split map $\mathcal{O}_X \to F_* \mathcal{O}_X$ by the line bundle $\mathcal{L}$, we obtain a split map $\mathcal{L} \to F_* F^* \mathcal{L} \simeq F_* \mathcal{L}^p$ by Lemma 3.2. On the level of cohomology, then, this gives rise to an injection $H^i(X, \mathcal{L}) \hookrightarrow H^i(X, F_* \mathcal{L}^p) \simeq H^i(X, \mathcal{L}^p)$. Iterating this argument, we obtain a sequence of embeddings

$$H^i(X, \mathcal{L}) \hookrightarrow H^i(X, \mathcal{L}^p) \hookrightarrow H^i(X, \mathcal{L}^{p^2}) \hookrightarrow \cdots \hookrightarrow H^i(X, \mathcal{L}^{p^N}) \hookrightarrow \cdots$$

which vanishes for $N$ large by ampleness. This shows that $H^i(X, \mathcal{L}) = 0$ for $i > 0$.

For (ii), let $i : Y \to X$ be the embedding. Tensoring $\mathcal{L}^n$ with the ideal exact sequence associated to $Y$, we obtain the short exact sequence

$$0 \to \mathcal{L}^n \otimes \mathcal{I}_Y \to \mathcal{L}^n \to \mathcal{L}^n|_Y \to 0,$$

which gives a long exact sequence

$$0 \to H^0(X, \mathcal{L}^n \otimes \mathcal{I}_Y) \to H^0(X, \mathcal{L}^n) \to H^0(Y, \mathcal{L}^n) \to H^1(X, \mathcal{L}^n \otimes \mathcal{I}_Y) \to \cdots$$

where $H^i(X, \mathcal{L}^n \otimes \mathcal{I}_Y) = 0$ for $i > 0$ and large $n$ because $\mathcal{L}$ is ample. Hence, for $i > 0$ and large $n$ we have a surjection $H^i(X, \mathcal{L}^n) \twoheadrightarrow H^i(Y, \mathcal{L}^n)$. Now because $Y$ was compatibly split, we have the following diagram

$$
\begin{array}{ccccccc}
\mathcal{L} & \longrightarrow & F_* \mathcal{L}^p & \longrightarrow & F_*(F_* \mathcal{L}^p)^p & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
i_* \mathcal{L} & \longrightarrow & i_* F_* \mathcal{L}^p & \longrightarrow & i_* F_*(F_* \mathcal{L}^p)^p & \longrightarrow & \cdots
\end{array}
$$

where the split horizontal maps were constructed in (i). This induces the following diagram in cohomology

$$
\begin{array}{ccccccc}
H^i(X, \mathcal{L}) & \longrightarrow & H^i(X, \mathcal{L}^p) & \longrightarrow & H^i(X, \mathcal{L}^{p^2}) & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \\
H^i(Y, \mathcal{L}) & \longrightarrow & H^i(Y, \mathcal{L}^p) & \longrightarrow & H^i(Y, \mathcal{L}^{p^2}) & \longrightarrow & \cdots
\end{array}
$$

where the vertical maps are eventually surjective by what we just showed. But the horizontal maps are split, hence we have a surjection $H^i(X, \mathcal{L}) \twoheadrightarrow H^i(Y, \mathcal{L})$. The result follows from this and (i). $\qquad \square$

Let us now specialize a bit more. In particular, we restrict to the case of smooth split varieties with splittings of the form produced by Proposition 3.16. In this case, we obtain the following strengthening of the cohomology embedding theorem used in the proof of Proposition 3.17.

**Proposition 3.18.** *Let $X$ be a smooth variety split by $\sigma^{p-1}$ for some $\sigma \in \Gamma(X, \omega_X^{-1})$. If $\sigma$ has zero divisor $\sigma_0 = X_1 + \cdots + X_n$ with $X_i$ irreducible of codimension $1$, then for any line bundle $\mathcal{L}$ on $X$, $i > 0$, and $0 \leq c_j < p$, we have the embedding*

$$H^i(X, \mathcal{L}) \hookrightarrow H^i\Big(X, \mathcal{L}^p \otimes \mathcal{O}\Big(\sum_{j=1}^n c_j X_j\Big)\Big).$$

*Proof.* We will construct a split map of line bundles $\mathcal{L} \to \mathcal{L}^p \otimes \mathcal{O}\left(\sum_{j=1}^n c_j X_j\right)$. Unwinding the proof of Proposition 3.15, we see that the splitting map $\phi : F_*\mathcal{O}_X \to \mathcal{O}_X$ was constructed as the composition

$$F_*\mathcal{O}_X \overset{F_*\sigma^{p-1}}{\to} F_*\omega_X^{1-p} \to F_*\omega_X \otimes \omega_X^{-1} \overset{C \otimes \mathrm{id}}{\to} \omega_X \otimes \omega_X^{-1} \to \mathcal{O}_X.$$

This gives rise to a splitting of the map $F_*\sigma^{p-1} \circ F^\# : \mathcal{O}_X \to F_*\omega_X^{1-p}$. Now, because $\sigma$ has zero divisor $X_1 + \cdots + X_n$, we may write $\omega_X^{1-p} = \mathcal{O}\left(\sum_{j=1}^n c_j X_j\right) \otimes \mathcal{L}'$ for some line bundle $\mathcal{L}'$, where $\sigma^{p-1} = \tau_1 \otimes \tau_2$ with $\tau_1 \in \Gamma\left(X, \mathcal{O}\left(\sum_{j=1}^n c_j X_j\right)\right)$ and $\tau_2 \in \Gamma(X, \mathcal{L}')$. With these notations, we see that $F_*\sigma^{p-1} \circ F^\#$ is the composition

$$\mathcal{O}_X \overset{F_*\tau_1}{\to} F_*\mathcal{O}\Big(\sum_{j=1}^n c_j X_j\Big) \overset{F_*\tau_2}{\to} F_*\omega_X^{1-p},$$

which means that the first map $\mathcal{O}_X \to F_*\mathcal{O}\left(\sum_{j=1}^n c_j X_j\right)$ splits.

Tensoring this map with the line bundle $\mathcal{L}$, we obtain a split map $\mathcal{L} \to \mathcal{L} \otimes F_*\mathcal{O}\left(\sum_{j=1}^n c_j X_j\right)$, which induces desired injective map in cohomology

$$H^i(X, \mathcal{L}) \hookrightarrow H^i\Big(X, \mathcal{L} \otimes \mathcal{O}\Big(\sum_{j=1}^n c_j X_j\Big)\Big) \simeq H^i\Big(X, F_*\Big(F^*\mathcal{L} \otimes \mathcal{O}\Big(\sum_{j=1}^n c_j X_j\Big)\Big)\Big) \simeq H^i\Big(X, \mathcal{L}^p \otimes \mathcal{O}\Big(\sum_{j=1}^n c_j X_j\Big)\Big).$$

Here, the last isomorphism follows from the projection formula and the identification $F^*\mathcal{L} \simeq \mathcal{L}^p$ given by Lemma 3.2. $\qquad \square$

## 4. A KEY COHOMOLOGY VANISHING RESULT ON THE BOTT-SAMELSON VARIETY

After working so hard to develop the basic theory of Frobenius splitting, it is time for us to make some figurative withdrawals from the bank. These withdrawals will come in two forms. First, we will obtain that $Z_{\mathfrak{w}}$ and $X_w$ are split, and, second, we will find cohomology vanishing theorems for line bundles on $Z_{\mathfrak{w}}$. Our main goal is to obtain the following vanishing result for specific line bundles on the Bott-Samelson variety.

**Proposition 4.1.** *Let $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$ be a reduced word. Then, for $1 \leq k \leq l \leq n$ and any dominant weight $\lambda$, we have*

$$H^i\Big(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)^h \otimes \mathcal{O}\Big(-\sum_{j=k}^l Z_{\mathfrak{w}(j)}\Big)\Big) = 0 \text{ for } i, h > 0.$$

*Further, we have $H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)^h) = 0$ for all $i, h > 0$.*

In later sections, we derive a number of surprising consequences from this without significant further input from Frobenius splitting methods.

We will initially obtain all vanishing results in characteristic $p$, and then transfer them to characteristic $0$ using semi-continuity. Therefore in Sections 4.1 and 4.2 we work over characteristic $p$ and in Section 4.3 in mixed characteristic. Perhaps most importantly, Proposition 4.1, our main result, continues to hold in characteristic $0$, though we initially prove it in positive characteristic.

4.1. **Splitting of the Bott-Samelson and Schubert varieties.** Of course, we would first like to show that the Bott-Samelson and Schubert varieties are Frobenius split. More precisely, we show that $Z_\mathfrak{w}$ is split compatibly with the divisors $Z_{\mathfrak{w}(j)}$ and that $X_w$ is split compatibly with $X_v$ for $v \leq w$.

Our approach will be as follows. We use the computation of $\omega_{Z_\mathfrak{w}}$ in Proposition 2.22 to find a section of $\omega_{Z_\mathfrak{w}}$ satisfying the conditions necessary to apply our splitting criterion. This will give the Frobenius splitting of the Bott-Samelson varieties. We then use Proposition 3.12 to extend this to a splitting of Schubert varieties, a result which is the culmination of all of our efforts until this point.

**Theorem 4.2.** *There exists some $\sigma \in \Gamma(Z_\mathfrak{w}, \omega_{Z_\mathfrak{w}}^{-1})$ vanishing on $Z_{\mathfrak{w}(j)}$ for all $j$ such that $\sigma^{p-1}$ splits $Z_\mathfrak{w}$ compatibly with $Z_{\mathfrak{w}_J}$ for all $J$.*

*Proof.* Let us first produce such a section and then check that it fulfills the prerequisites of Proposition 3.16. Because $Z_\mathfrak{w}$ is a smooth projective variety, we may find a canonical section $\sigma_1 \in \Gamma(Z_\mathfrak{w}, \omega_{Z_\mathfrak{w}})$ such that its divisor of zeros is $(\sigma_1)_0 = \partial Z_\mathfrak{w}$. Now, the line bundle $\mathcal{L}_\mathfrak{w}(\rho) = \theta_\mathfrak{w}^*(\mathcal{L}(\rho))$ is the pullback of the canonical bundle $\omega_\mathcal{B} \simeq \mathcal{L}(-2\rho)$, which has global sections by Proposition 2.14.[4] Take such a section $\sigma_2 \in \Gamma(\mathcal{B}, \mathcal{L}(\rho))$; because $\mathcal{L}(\rho)$ is $G$-equivariant and not everywhere zero, we may choose $\sigma_2$ so that $1 \notin (\sigma_2)_0$.

Now take $\sigma = \sigma_1 \otimes \theta_\mathfrak{w}^*(\sigma_2) \in \Gamma(Z_\mathfrak{w}, \omega_{Z_\mathfrak{w}}^{-1})$. It is clear that $\sigma$ vanishes on each $Z_{\mathfrak{w}(j)}$. By construction, we see that

$$\sigma_0 = Z_{\mathfrak{w}(1)} + \cdots + Z_{\mathfrak{w}(n)} + W$$

for some effective divisor $W$ not containing $\pi_\mathfrak{w}(1, \ldots, 1) \in Z_\mathfrak{w}$. By Proposition 2.5, these divisors satisfy the conditions of Proposition 3.16. We thus obtain that $\sigma^{p-1}$ splits $Z_\mathfrak{w}$ compatibly with $Z_{\mathfrak{w}(j)}$ for each $j$. The slightly stronger compatibility with $Z_{\mathfrak{w}_J}$ for all $J$ follows from this by Proposition 3.11.                      $\square$

**Theorem 4.3.** *For $w \in W$, the Schubert variety $X_w$ is split compatibly with the subvarieties $X_v$ for $v \leq w$.*

*Proof.* Let $\mathfrak{w}$ be a reduced decomposition of $w$; then, for any $v \leq w$, we may find a subword $\mathfrak{v}$ of $\mathfrak{w}$ that is a reduced decomposition of $v$. In this case, notice that $X_v$ is the scheme theoretic image of $Z_\mathfrak{v}$ for all $v \leq w$. Now, by Lemma 2.9, we have $(\theta_\mathfrak{w})_* \mathcal{O}_{Z_\mathfrak{w}} \simeq \mathcal{O}_{X_w}$, hence the desired follows by applying Proposition 3.12 to the result of Theorem 4.2.                      $\square$

**Remark.** While we prove Theorem 4.3 here for completeness, we will not make further use of it. We note that it is possible to derive most of our results by using Theorem 4.3 to obtain Corollary 5.5 in positive characteristic and passing to characteristic 0, but we choose to rely on Proposition 4.1 instead to emphasize the role of the Bott-Samelson variety.

4.2. **Proof of the vanishing result.** As suggested by Proposition 3.17, we have good vanishing theorems for ample line bundles on Frobenius split schemes. To establish Proposition 4.1, then, our general strategy will be as follows. We first show that for the Bott-Samelson variety, we may replace the condition of ampleness with global generation. We then use the embedding theorems we have just obtained to show that the relevant line bundle embeds in a series of successively higher powers. We then show that these higher powers will eventually be globally generated, giving the theorem.

The additional strength of our results in this case come from the fact that we have a great deal of control over the line bundles $\mathcal{O}_{Z_\mathfrak{w}}(Z_{\mathfrak{w}(j)})$ associated to the prime divisors of $\partial Z_\mathfrak{w}$. In particular, they form a source of ample and globally generated line bundles from which to extract vanishing.

**Proposition 4.4.** *Let $\ell(\mathfrak{w}) = n$. There exist positive integers $c_j > 0$ such that $\mathcal{O}\left(\sum_{j=1}^n c_j Z_{\mathfrak{w}(j)}\right)$ is ample.*

*Proof.* We induct on $\ell(\mathfrak{w})$. For the base case, the statement reduces to the fact that $\mathcal{O}(1)$ is ample on $\mathbb{P}^1$. Now, take $\mathcal{L}$ ample on $Z_{\mathfrak{w}[n-1]}$ of this form and for each $m$ consider the line bundle

$$\mathcal{L}_m' = \psi_\mathfrak{w}^*(\mathcal{L})^m \otimes \mathcal{O}(Z_{\mathfrak{w}(n)}).$$

Now, notice that $\mathcal{O}(Z_{\mathfrak{w}(n)})$ has degree 1 along the $\mathbb{P}^1$-bundle map $\psi_\mathfrak{w} : Z_\mathfrak{w} \to Z_{\mathfrak{w}[n-1]}$, so by [Ha77, Theorem II.7.10] we see that $\mathcal{L}_m'$ is very ample for large $m$. Hence, $\mathcal{L}_m'$ is an ample bundle of the desired form. [5]   $\square$

**Corollary 4.5.** *Let $\ell(\mathfrak{w}) = n$. For any $k \leq n$, there exist positive integers $c_j > 0$ such that $\mathcal{O}\left(\sum_{j=1}^k c_j Z_{\mathfrak{w}(j)}\right)$ is globally generated.*

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[4]Here the fact that $\omega_\mathcal{B} \simeq \mathcal{L}(-2\rho)$ comes from Proposition 2.12.

[5]Here we implicitly use Proposition 2.6.

*Proof.* By Proposition 4.4, we may pick $c_j$ so that $\mathcal{O}\left(\sum_{j=1}^{k} c_j Z_{\mathfrak{w}[k](j)}\right)$ is ample, hence $\mathcal{O}\left(N \sum_{j=1}^{k} c_j Z_{\mathfrak{w}[k](j)}\right)$ is globally generated on $Z_{\mathfrak{w}[k]}$ for some $N$. The conclusion follows from Proposition 2.6. □

We may now extend the vanishing that we previously had for ample line bundles to globally generated line bundles and set up a sufficient supply of globally generated line bundles on $Z_{\mathfrak{w}}$. Note that the globally generated line bundles provided by Lemma 4.7 are designed specifically to prove Proposition 4.1.

**Lemma 4.6.** *Let $\mathcal{L}$ be a globally generated line bundle on $Z_{\mathfrak{w}}$. Then, we have $H^i(Z_{\mathfrak{w}}, \mathcal{L}) = 0$ for $i > 0$.*

*Proof.* Pick $c_j > 0$ so that $\mathcal{L}' = \mathcal{O}\left(\sum_{j=1}^{n} c_j Z_{\mathfrak{w}(j)}\right)$ is ample by Proposition 4.4. Let $c_j = \sum_{l=0}^{N} c_{j,l} p^l$ with $0 \le c_{j,l} < p$ be the base $p$ expansion of each $c_j$ for some large $N$ that does not depend on $j$. By Theorem 4.2, we may apply Proposition 3.18 with the bundles $\mathcal{O}\left(\sum_{j=1}^{n} c_{j,l} Z_{\mathfrak{w}(j)}\right)$ for $l = N, N-1, \ldots, 0$ to obtain embeddings

$$H^i(Z_{\mathfrak{w}}, \mathcal{L}) \hookrightarrow H^i\left(Z_{\mathfrak{w}}, \mathcal{L}^p \otimes \mathcal{O}\left(\sum_{j=1}^{n} c_{j,N} Z_{\mathfrak{w}(j)}\right)\right)$$

$$\hookrightarrow H^i\left(Z_{\mathfrak{w}}, \mathcal{L}^{p^2} \otimes \mathcal{O}\left(\sum_{j=1}^{n} (c_{j,N} p + c_{j,N-1}) Z_{\mathfrak{w}(j)}\right)\right) \hookrightarrow \cdots \hookrightarrow H^i\left(Z_{\mathfrak{w}}, \mathcal{L}^{p^{N+1}} \otimes \mathcal{O}\left(\sum_{j=1}^{n} c_j Z_{\mathfrak{w}(j)}\right)\right).$$

But now notice that the final line bundle

$$\mathcal{L}^{p^{N+1}} \otimes \mathcal{O}\left(\sum_{j=1}^{n} c_j Z_{\mathfrak{w}(j)}\right)$$

in the sequence is ample because $\mathcal{L}^{p^{N+1}}$ is globally generated and $\mathcal{O}\left(\sum_{j=1}^{n} c_j Z_{\mathfrak{w}(j)}\right)$ is ample. So by Proposition 3.17, it has vanishing cohomology, hence our embeddings give that $H^i(Z_{\mathfrak{w}}, \mathcal{L}) = 0$. □

**Lemma 4.7.** *Let $\ell(\mathfrak{w}) = n$ with $\mathfrak{w}$ reduced. For any $1 \le k, l \le n$, we may find integers $c_1, \ldots, c_n$ and $m$ with $c_j \ge 0$ for $1 \le j < k$ and $j > l$, $c_j \le 0$ for $k \le j < l$, and $c_l = 0$ such that the line bundle*

$$\mathcal{O}\left(\sum_{j=1}^{n} c_j Z_{\mathfrak{w}(j)} - Z_{\mathfrak{w}(l)}\right) \otimes \mathcal{L}_{\mathfrak{w}}(\rho)^m$$

*is globally generated.*

*Proof.* We argue by induction on $\ell(\mathfrak{w})$. The base case $\ell(\mathfrak{w}) = 1$ follows from the fact that $\mathcal{L}_{\mathfrak{w}}(\rho) \simeq \mathcal{O}(1)$ under the identification $Z_{\mathfrak{w}} \simeq \mathbb{P}^1$. Now suppose $\ell(\mathfrak{w}) = n$ and the claim holds for $\ell(\mathfrak{w}) < n$. We now have two cases. First, suppose that $l < n$. In this case, find integers $c_1, \ldots, c_{n-1}, m$ with $c_j = 0$ for $k \le j \le l$ such that

$$\mathcal{O}\left(\sum_{j=1}^{n-1} c_j Z_{\mathfrak{w}[n-1](j)} - Z_{\mathfrak{w}[n-1](l)}\right) \otimes \mathcal{L}_{\mathfrak{w}[n-1]}(\rho)^m$$

is globally generated. Thus its pullback

$$\psi_{\mathfrak{w}}^*\left(\mathcal{O}\left(\sum_{j=1}^{n-1} c_j Z_{\mathfrak{w}[n-1](j)} - Z_{\mathfrak{w}[n-1](l)}\right) \otimes \mathcal{L}_{\mathfrak{w}[n-1]}(\rho)^m\right) = \mathcal{O}\left(\sum_{j=1}^{n-1} c_j Z_{\mathfrak{w}(j)} - Z_{\mathfrak{w}(l)}\right) \otimes \psi_{\mathfrak{w}}^*(\mathcal{L}_{\mathfrak{w}[n-1]}(\rho))^m$$

is globally generated. Now, by Propositions 2.7 and 2.22, we see that

$$\omega_{Z_{\mathfrak{w}}/Z_{\mathfrak{w}[n-1]}}^{-1} \otimes \mathcal{L}_{\mathfrak{w}}(\rho)^h = \omega_{Z_{\mathfrak{w}}}^{-1} \otimes \psi_{\mathfrak{w}}^*(\omega_{Z_{\mathfrak{w}[n-1]}}) \otimes \mathcal{L}_{\mathfrak{w}}(\rho)^h$$

$$= \mathcal{O}(\partial Z_{\mathfrak{w}}) \otimes \mathcal{L}_{\mathfrak{w}}(\rho)^{h+1} \otimes \mathcal{O}\left(-\sum_{j=1}^{n-1} Z_{\mathfrak{w}(j)}\right) \otimes \psi_{\mathfrak{w}}^* \mathcal{L}_{\mathfrak{w}[n-1]}(-\rho)$$

$$= \mathcal{O}(Z_{\mathfrak{w}[n-1]}) \otimes \mathcal{L}_{\mathfrak{w}}(\rho)^{h+1} \otimes \psi_{\mathfrak{w}}^* \mathcal{L}_{\mathfrak{w}[n-1]}(-\rho)$$

is the pullback of $\omega^{-1}_{\mathcal{B}/(G/P_{i_n})} \otimes \mathcal{L}(\rho)^h = \mathcal{L}(\alpha_{i_n} + h\rho)$. Thus, we see that, for large $h$, $\alpha_{i_n} + h\rho$ is dominant, hence the line bundle is globally generated by Corollary 2.18. Tensoring together the two line bundles we have shown to be globally generated, we see that

$$\mathcal{O}\Big( \sum_{j=1}^{n-1} c_j Z_{\mathfrak{w}(j)} - Z_{\mathfrak{w}(l)} \Big) \otimes \mathcal{O}(m Z_{\mathfrak{w}(n)}) \otimes \mathcal{L}_{\mathfrak{w}}(\rho)^{m(h+1)}$$

is globally generated, as needed.

Now consider the case $l = n$. In this case, consider the line bundle $\mathcal{L}_{\mathfrak{w}}(\chi_{i_n})$ on $Z_{\mathfrak{w}}$ for $\chi_{i_n}$ the fundamental weight corresponding to $s_{i_n}$. By Lemma 2.21, $\chi_{i_n}$ has degree 1 along the fibers of $\psi_{\mathfrak{w}}$, hence by Proposition 2.23 we have in $\mathrm{Pic}(Z_{\mathfrak{w}})$ the identity

$$\mathcal{L}_{\mathfrak{w}}(\chi_{i_n}) = \mathcal{O}(Z_{\mathfrak{w}(n)}) + \mathcal{O}\Big( \sum_{j=1}^{n-1} d_j Z_{\mathfrak{w}(j)} \Big)$$

for some $d_j$. We claim that $d_j \geq 0$ for all $j$. For this, we will need to use the fact that $\mathfrak{w}$ is a reduced word. By Corollary 2.18, we see that $\mathcal{L}_{\mathfrak{w}}(\chi_{i_n})$ is globally generated, hence $\mathcal{O}\left( Z_{\mathfrak{w}(n)} + \sum_{j=1}^{n-1} d_j Z_{\mathfrak{w}(j)} \right)$ is an effective divisor. Now, because $\mathcal{L}_{\mathfrak{w}}(\chi_{i_n})$ is $B$-equivariant, we may find a global section $\sigma \in \Gamma(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\chi_{i_n}))$ which is $B$-invariant up to scaling. Such a $\sigma$ has a $B$-invariant zero divisor $\sigma_0$, and in addition $\mathcal{L}_{\mathfrak{w}}(\chi_{i_n}) = \mathcal{O}_{Z_{\mathfrak{w}}}(\sigma_0)$. Now, because $\mathfrak{w}$ is reduced, $Z_{\mathfrak{w}} - \partial Z_{\mathfrak{w}} \to C_w$ is an isomorphism by Proposition 2.8, hence it is a dense orbit of the $B$-action on $Z_{\mathfrak{w}}$. This shows that $\sigma_0 \subset \partial Z_{\mathfrak{w}}$, so we obtain that $\mathcal{L}_{\mathfrak{w}} = \mathcal{O}(\sigma_0)$ is in the non-negative span of $\mathcal{O}(Z_{\mathfrak{w}(j)})$, as needed.

Now, by Corollary 2.18, we see that

$$\mathcal{L}_{\mathfrak{w}}(\rho - \chi_{i_n}) = \mathcal{L}_{\mathfrak{w}}(\rho) \otimes \mathcal{O}(-Z_{\mathfrak{w}(n)}) \otimes \mathcal{O}\Big( -\sum_{j=1}^{n-1} d_j Z_{\mathfrak{w}(j)} \Big)$$

is globally generated. By Corollary 4.5, we may find $c'_1, \ldots, c'_{k-1}$ large and positive such that $\mathcal{O}\left( \sum_{j=1}^{k-1} c'_j Z_{\mathfrak{w}(j)} \right)$ is globally generated. Tensoring these two line bundles, we obtain that

$$\mathcal{L}_{\mathfrak{w}}(\rho) \otimes \mathcal{O}(-Z_{\mathfrak{w}(n)}) \otimes \mathcal{O}\Big( \sum_{j=1}^{k-1} c'_j Z_{\mathfrak{w}(j)} - \sum_{j=1}^{n-1} d_j Z_{\mathfrak{w}(j)} \Big)$$

is globally generated, as needed. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

We are now ready to prove Proposition 4.1. Our strategy will be to embed the desired cohomology into the cohomology of a line bundle which is the product of a line bundles of the form given in Lemma 4.7 and other globally generated line bundles. The conclusion will then follow from Lemma 4.6.

*Proof of Proposition 4.1.* First, if $\lambda$ is dominant, $\mathcal{L}_{\mathfrak{w}}(\lambda)$ is globally generated as the pullback of the globally generated line bundle $\mathcal{L}(\lambda)$ by Lemma 2.14. The second assertion then follows from Lemma 4.6. For the rest of the proof, replace $\mathcal{L}_{\mathfrak{w}}(\lambda)$ by any globally generated line bundle $\mathcal{L}$ on $Z_{\mathfrak{w}}$.

We will emulate the proof of Proposition 4.6. However, our sequence of embeddings will be a bit more delicate here. First, let us use Lemma 4.7 to construct a special globally generated line bundle. For $k \leq t \leq l$, we may find globally generated line bundles $\mathcal{L}_t = \mathcal{O}\left( \sum_{j=1}^n c'_{j,t} Z_{\mathfrak{w}(j)} - Z_{\mathfrak{w}(t)} \right) \otimes \mathcal{L}_{\mathfrak{w}}(\rho)^{m_j}$ with $c_{j,t} \geq 0$ for $1 \leq j < k$ and $j > t$, $c_{j,t} \leq 0$ for $k \leq j < t$, and $c_{t,t} = 0$. Therefore, we may inductively construct a globally generated line bundle of the form

$$\mathcal{L}' = \mathcal{L}_k^{a_k} \otimes \cdots \otimes \mathcal{L}_l^{a_l} = \mathcal{O}\Big( \sum_{j=1}^n d_j Z_{\mathfrak{w}(j)} \Big) \otimes \mathcal{L}_{\mathfrak{w}}(\rho)^m$$

for some positive integers $a_k, \ldots, a_l$ such that $d_j < 0$ for $k \leq j \leq t$ and $d_j \geq 0$ otherwise.

We now embed $H^i\left( Z_{\mathfrak{w}}, \mathcal{L}^h \otimes \mathcal{O}\left( -\sum_{j=k}^l Z_{\mathfrak{w}(j)} \right) \right)$ into $H^i(Z_{\mathfrak{w}}, \mathcal{L}')$. We may rewrite the form of $\mathcal{L}'$ as

$$\mathcal{L}' = \mathcal{O}\Big( \sum_{j=1}^{k-1} d_j Z_{\mathfrak{w}(j)} + \sum_{j=l+1}^n d_j Z_{\mathfrak{w}(j)} - \sum_{j=k}^l e_j Z_{\mathfrak{w}(j)} \Big) \otimes \mathcal{L}_{\mathfrak{w}}(\rho)^m$$

for $e_j = -d_j > 0$. Fix some $N$ such that $p^{N+1} > \max_j\{d_j, e_j\}$, and take base $p$ expansions $d_j = \sum_{r=0}^N d_{j,r} p^r$, $m = \sum_{r=0}^N m_r p^r$, and $p^{N+1} - e_j = \sum_{r=0}^N e_{j,r} p^r$, for some $0 \le d_{j,r}, m_r, e_{j,r} < p$. We will now apply Proposition 3.18 a total of $N+1$ times, where at step $N+1-r$, we will apply it to multiply by the bundle

$$\mathcal{O}\Big( \sum_{j \notin [k,l]} d_{j,r} Z_{\mathfrak{w}(j)} + \sum_{j=k}^l e_{j,r} Z_{\mathfrak{w}(j)} \Big) \otimes \mathcal{L}_{\mathfrak{w}}(\rho)^{m_r}.$$

This gives rise to the sequence of embeddings

$$H^i\Big( Z_{\mathfrak{w}}, \mathcal{L}^h \otimes \mathcal{O}\Big( -\sum_{j=k}^l Z_{\mathfrak{w}(j)} \Big) \Big)$$

$$\hookrightarrow H^i\Big( Z_{\mathfrak{w}}, \mathcal{L}^{hp} \otimes \mathcal{O}\Big( \sum_{j \notin [k,l]} d_{j,N} Z_{\mathfrak{w}(j)} + \sum_{j=k}^l (e_{j,N} - p) Z_{\mathfrak{w}(j)} \Big) \otimes \mathcal{L}_{\mathfrak{w}}(\rho)^{m_N} \Big)$$

$$\hookrightarrow H^i\Big( Z_{\mathfrak{w}}, \mathcal{L}^{hp^2} \otimes \mathcal{O}\Big( \sum_{j \notin [k,l]} (d_{j,N-1} + d_{j,N} p) Z_{\mathfrak{w}(j)} + \sum_{j=k}^l (e_{j,N-1} + e_{j,N} p - p^2) Z_{\mathfrak{w}(j)} \Big) \otimes \mathcal{L}_{\mathfrak{w}}(\rho)^{m_{N-1} + m_N p} \Big)$$

$$\hookrightarrow \cdots$$

$$\hookrightarrow H^i\Big( Z_{\mathfrak{w}}, \mathcal{L}^{hp^{N+1}} \otimes \mathcal{O}\Big( \sum_{j \notin [k,l]} \Big( \sum_{r=0}^N d_{j,r} p^r \Big) Z_{\mathfrak{w}(j)} + \sum_{j=k}^l \Big( -p^{N+1} + \sum_{r=0}^N e_{j,r} p^r \Big) Z_{\mathfrak{w}(j)} \Big) \otimes \mathcal{L}_{\mathfrak{w}}(\rho)^{\sum_{r=0}^N m_r p^r} \Big)$$

$$= H^i(Z_{\mathfrak{w}}, \mathcal{L}^{hp^{N+1}} \otimes \mathcal{L}').$$

But recall that $\mathcal{L}'$ was constructed to be globally generated, hence $\mathcal{L}^{hp^{N+1}} \otimes \mathcal{L}'$ is globally generated. Therefore, by Proposition 4.6, the final cohomology vanishes, so all the cohomologies in the chain of embeddings vanish, as needed.                                                                                           $\square$

4.3. **Passing to characteristic** 0. We now transfer Proposition 4.1 to the characteristic 0 case. Let us now consider all schemes to be over $\mathrm{Spec}(\mathbb{Z})$ instead of $\mathrm{Spec}(k)$ (as we were implicitly assuming before). We would like to apply the semi-continuity theorem, stated below.

**Theorem 4.8** ([Ha77, Theorem 12.8])**.** *Let $X \to Y$ be a projective morphism of Noetherian schemes, and let $\mathcal{F}$ be a coherent sheaf on $X$, flat over $Y$. Then, for each $i \ge 0$, the function*

$$h^i(y, \mathcal{F}) = \dim_{k(y)} H^i(X_y, \mathcal{F}_y)$$

*is an upper semicontinuous function on $Y$.*

For this, we need coherent sheaves, flat over some base. In this case, they will be provided by generic flatness.

**Lemma 4.9** ([Eis, Theorem 14.4])**.** *If $\mathcal{F}$ is a coherent sheaf on a scheme $X$ defined over $\mathbb{Z}$, then there is a finite set $S$ of primes such that the sheaf $\mathcal{F}_{\mathbb{Z}[S^{-1}]}$ given by base change from $\mathbb{Z}$ to $\mathbb{Z}[S^{-1}]$ is flat over $\mathbb{Z}[S^{-1}]$.*

We may apply semi-continuity and Lemma 4.9 to obtain the following.

**Proposition 4.10.** *Let $X$ be a projective scheme over $\mathbb{Z}$. Let $\mathcal{F}$ be a coherent sheaf on $X$, and let $i \ge 0$. If $H^i(X_{\bar{p}}, \mathcal{F}_{\bar{p}}) = 0$ for any $p$, then $H^i(X_{\bar{\mathbb{Q}}}, \mathcal{F}_{\bar{\mathbb{Q}}}) = 0$.*

*Proof.* By Lemma 4.9, we may find some finite set of primes $S$ such that $\mathcal{F}_{\mathbb{Z}[S^{-1}]}$ is flat over $\mathbb{Z}[S^{-1}]$. So in particular, we may take $S = \mathrm{Spec}(\mathbb{Z}) - \{(0), (p)\}$ for some prime $p$.

Now, suppose otherwise for contradiction. By [Ha77, Proposition 9.3], we have that

$$H^i(X_{\bar{\mathbb{Q}}}, \mathcal{F}_{\bar{\mathbb{Q}}}) = H^i(X_{\mathbb{Q}}, \mathcal{F}_{\mathbb{Q}}) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}},$$

meaning that $h^i((0), \mathcal{F}) > 0$. Now, the set $\{x \mid h^i((x), \mathcal{F})\}$ is closed in $\mathrm{Spec}(\mathbb{Z}[S^{-1}])$, but the closure of $(0)$ contains all of $\mathrm{Spec}(\mathbb{Z}[S^{-1}])$, which implies that $h^i((p), \mathcal{F}) > 0$. Again applying [Ha77, Proposition 9.3], we have that

$$H^i(X_{\bar{p}}, \mathcal{F}_{\bar{p}}) = H^i(X_p, \mathcal{F}_p) \otimes_{\mathbb{F}_p} \overline{\mathbb{F}}_p,$$

meaning that $h^i((p), \mathcal{F}) = 0$, a contradiction. So we must have $H^i(X_{\bar{\mathbb{Q}}}, \mathcal{F}_{\bar{\mathbb{Q}}}) = 0$. $\qquad\square$

The relevant sheaves in Proposition 4.1 are line bundles over $Z_\mathfrak{w}$, hence coherent. Furthermore, $Z_\mathfrak{w}$ is projective, so Proposition 4.10 implies that the result holds for $\bar{\mathbb{Q}}$. Finally, for any other algebraically closed field $k$ of characteristic 0, we may again use [Ha77, Proposition 9.3] to obtain that

$$H^i((Z_\mathfrak{w})_k, \mathcal{F}_k) = H^i((Z_\mathfrak{w})_{\bar{\mathbb{Q}}}, \mathcal{F}_{\mathbb{Q}}) \underset{\bar{\mathbb{Q}}}{\otimes} k = 0$$

for any coherent sheaf $\mathcal{F}$ on $Z_\mathfrak{w}$. Hence Proposition 4.1 remains valid in characteristic 0.

## 5. Proofs of the main results

Having condensed the technical input of Frobenius splitting into Proposition 4.1, let us recall our original purpose. We began with the construction of the flag variety $\mathcal{B}$ and Schubert varieties $X_w \subset \mathcal{B}$. To understand the geometry of $X_w$, we constructed their desingularizations $Z_\mathfrak{w} \to X_w$ via the Bott-Samelson varieties. Armed with our newly developed cohomology vanishing result on $Z_\mathfrak{w}$, we will now retrace our steps and use it to show that the geometry of $X_w$ is remarkably nice.

We will show that the Schubert varieties $X_w$ are normal, Cohen-Macaulay, and have rational resolutions. Further, line bundles $\mathcal{L}_w(\lambda)$ on them will satisfy a certain character formula known as the Demazure character formula, which we view as a refinement of the usual Weyl character formula. Our results in this section will hold over arbitrary characteristic, as the only result we will draw from our Frobenius splitting methods is Proposition 4.1.

5.1. **Schubert varieties are normal and Cohen-Macaulay.** In this subsection, we first draw some additional technical consequences of Proposition 4.1. We would like to emphasize that these are more or less formal consequences and do not require significant geometric insight to prove.

**Corollary 5.1.** *For any $\mathfrak{w}$, $1 \leq j \leq n$ and dominant weight $\lambda$, the map*

$$H^0(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(\lambda)) \to H^0(Z_{\mathfrak{w}(j)}, \mathcal{L}_{\mathfrak{w}(j)}(\lambda))$$

*induced by the inclusion $\iota_{\mathfrak{w},j} : Z_{\mathfrak{w}(j)} \to Z_\mathfrak{w}$ is surjective.*

*Proof.* Viewing $Z_{\mathfrak{w}(j)}$ as a closed subvariety in $Z_\mathfrak{w}$, we have the exact sequence

$$0 \to \mathcal{O}_{Z_\mathfrak{w}}(-Z_{\mathfrak{w}(j)}) \to \mathcal{O}_{Z_\mathfrak{w}} \to (\iota_{\mathfrak{w},j})_* \mathcal{O}_{Z_{\mathfrak{w}(j)}} \to 0.$$

Now, we may tensor with the line bundle $\mathcal{L}_\mathfrak{w}(\lambda)$ to obtain

$$0 \to \mathcal{L}_\mathfrak{w}(\lambda) \otimes \mathcal{O}_{Z_\mathfrak{w}}(-Z_{\mathfrak{w}(j)}) \to \mathcal{L}_\mathfrak{w}(\lambda) \to (\iota_{\mathfrak{w},j})_* \mathcal{L}_{\mathfrak{w}(j)}(\lambda) \to 0,$$

where we note that $\iota_{\mathfrak{w},j}^* \mathcal{L}_\mathfrak{w}(\lambda) = \mathcal{L}_{\mathfrak{w}(j)}(\lambda)$. The result follows upon applying the long exact sequence in cohomology and obtaining that $H^1(Z_\mathfrak{w}, \mathcal{L}_\mathfrak{w}(\lambda) \otimes \mathcal{O}_{Z_\mathfrak{w}}(-Z_{\mathfrak{w}(j)})) = 0$ from Proposition 4.1. $\qquad\square$

We now introduce a lemma from homological algebra in order to obtain the next consequence, Corollary 5.3, which will be the form in which we would like to view our technical input.

**Lemma 5.2.** *Let $\mathcal{F}$ be a line bundle on $Z_\mathfrak{w}$ such that for any $i > 0$ and $m$ large, we have $H^i(Z_\mathfrak{w}, \mathcal{F} \otimes \mathcal{L}_\mathfrak{w}(\rho)^m) = 0$. We have $R^i(\theta_\mathfrak{w})_* \mathcal{F} = 0$ for $i > 0$.*

*Proof.* Note that $\rho$ is dominant, hence $\mathcal{L}_w(\rho)$ is ample on $X_w$. For any $m$, we have the Leray spectral sequence associated to $\theta_\mathfrak{w} : Z_\mathfrak{w} \to X_w$ given by

$$H^j(X_w, R^i(\theta_\mathfrak{w})_*(\mathcal{F} \otimes \mathcal{L}_\mathfrak{w}(\lambda)^m)) \implies H^{j+i}(Z_\mathfrak{w}, \mathcal{F} \otimes (\theta_\mathfrak{w})^* \mathcal{L}_w(\lambda)^m).$$

Now, for $j > 0$, we have

$$H^j(X_w, R^i(\theta_\mathfrak{w})_*(\mathcal{F} \otimes \mathcal{L}_\mathfrak{w}(\lambda)^m)) \simeq H^j(X_w, R^i(\theta_\mathfrak{w})_* \mathcal{F} \otimes \mathcal{L}_w(\lambda)^m) = 0$$

for $m$ large because $\mathcal{L}_w(\lambda)$ is ample. For such an $m$, the spectral sequence degenerates and we have

$$H^0(X_w, R^i(\theta_\mathfrak{w})_* \mathcal{F} \otimes \mathcal{L}_w(\lambda)^m) \simeq H^i(Z_\mathfrak{w}, \mathcal{F} \otimes \mathcal{L}_\mathfrak{w}(\lambda)^m) = 0$$

for any $i > 0$ by our assumption on $\mathcal{F}$. But for large $m$, $R^i(\theta_\mathfrak{w})_* \mathcal{F} \otimes \mathcal{L}_w(\lambda)^m$ is globally generated because $\mathcal{L}_w(\lambda)$ is ample, hence $R^i(\theta_\mathfrak{w})_* \mathcal{F} = 0$. $\qquad\square$

**Corollary 5.3.** *Let $\mathfrak{w}$ be a reduced word with $p(\mathfrak{w}) = w$. We have the following:*

(i) For any $i > 0$, we have that $R^i(\theta_\mathfrak{w})_* \mathcal{O}_{Z_\mathfrak{w}} = R^i(\theta_\mathfrak{w})_* \omega_{Z_\mathfrak{w}} = 0$;

(ii) $\theta_\mathfrak{w}$ induces an isomorphism of line bundles $(\theta_\mathfrak{w})_* \mathcal{O}_{Z_\mathfrak{w}} \simeq \mathcal{O}_{X_w}$;

(iii) For any locally free sheaf $\mathcal{F}$ on $X_w$, we have $H^i(X_w, \mathcal{F}) \simeq H^i(Z_\mathfrak{w}, (\theta_\mathfrak{w})^* \mathcal{F})$ for $i \geq 0$.

*Proof.* For (i), note that $\mathcal{O}_{Z_\mathfrak{w}}$ and $\omega_{Z_\mathfrak{w}}$ satisfy the conditions of Lemma 5.2 by Proposition 4.1. [6] Statement (ii) is exactly Lemma 2.9; we state it here for completeness only. For (iii), the Leray spectral sequence

$$H^j(X_w, R^i(\theta_\mathfrak{w})_* \mathcal{O}_{Z_\mathfrak{w}} \otimes \mathcal{F}) \implies H^{i+j}(Z_\mathfrak{w}, (\theta_\mathfrak{w})^* \mathcal{F})$$

degenerates by (i), hence in the $j = 0$ case we obtain $H^i(X_w, \mathcal{F}) = H^i(Z_\mathfrak{w}, (\theta_\mathfrak{w})^* \mathcal{F})$.            $\square$

Results (i) and (ii) of Corollary 5.3 together make up the fact that $\theta_\mathfrak{w} : Z_\mathfrak{w} \to X_w$ is a *rational resolution*. Using these formal properties, we may now prove the following, our first main result.

**Theorem 5.4.** *For any $w \in W$, the Schubert variety $X_w$ is normal and Cohen-Macaulay.*

*Proof.* We first show that $X_w$ is normal. This follows immediately from Corollary 5.3(ii) because $\theta_\mathfrak{w}$ is a birational map to $X_w$ from the smooth variety $Z_\mathfrak{w}$.

We now show that $X_w$ is Cohen-Macaulay. Because $X_w$ is projective of dimension $n = \ell(w)$, it suffices by [Ha77, Theorem 7.6] to show that for any ample line bundle $\mathcal{L}$ on $X_w$, we have $H^i(X_w, \mathcal{L}^{-m}) = 0$ for $i < n$ and $m$ sufficiently large. [7] We now have

$$H^i(X_w, \mathcal{L}^{-m}) \simeq H^i(Z_\mathfrak{w}, (\theta_\mathfrak{w})^* \mathcal{L}^{-m}) \simeq H^{n-i}(Z_\mathfrak{w}, \omega_{Z_\mathfrak{w}} \otimes (\theta_\mathfrak{w})^* \mathcal{L}^m) \simeq H^{n-i}(X_w, (\theta_\mathfrak{w})_*(\omega_{Z_\mathfrak{w}}) \otimes \mathcal{L}^m) = 0$$

for $m$ large because $\mathcal{L}$ is ample. Here, the first isomorphism is by Corollary 5.3(iii), the second by Serre duality on the smooth projective variety $Z_\mathfrak{w}$, and the last by the Leray spectral sequence and the projection formula.            $\square$

5.2. **Demazure character formula.** We now come to our original motivation for studying the geometry of Schubert varieties. Recall that the Borel-Weil theorem related the irreducible representations of $G$ to the global sections of the line bundles $\mathcal{L}(\lambda)$ on $\mathcal{B}$. We would like to obtain an analogue of this result for $\mathcal{L}_w(\lambda)$. Notice, however, that $X_w$ has only an action of $B$ and not of $G$, so the cohomologies $H^i(X_w, \mathcal{L}_w(\lambda))$ are only $B$-modules, not $G$-modules. Thus, we can at most hope for a character formula for these modules, and, indeed, one exists. First, however, we note that in fact we do not need to consider the higher cohomologies.

**Corollary 5.5.** *For any Schubert variety $X_w$ and any dominant weight $\lambda$, we have that $H^i(X_w, \mathcal{L}_w(\lambda)) = 0$ for $i > 0$ and the restriction map $H^0(\mathcal{B}, \mathcal{L}(\lambda)) \to H^0(X_w, \mathcal{L}_w(\lambda))$ is surjective.*

*Proof.* This follows immediately from Corollaries 5.1 and 5.3(iii).            $\square$

We are now ready to state the Demazure character formula. Recall that any finite dimensional $T$-module $M$ splits uniquely as the sum of its weight spaces $M_\lambda$ for $\lambda \in X^*(T)$. Then, we may define the *character* $\operatorname{ch} M$ of $M$ as the formal sum

$$\operatorname{ch} M = \sum_{\lambda \in X^*(T)} \dim M_\lambda \, e^\lambda,$$

where we interpret $\operatorname{ch} M$ in $\mathbb{Z}[X^*(T)]$. Now, for any simple reflection $s_i \in W$, define the *Demazure operator* $D_{s_i} : \mathbb{Z}[X^*(T)] \to \mathbb{Z}[X^*(T)]$ associated to $s_i$ by

$$D_{s_i}(e^\lambda) = \frac{e^\lambda - e^{s_i \lambda - \alpha_i}}{1 - e^{-\alpha_i}}.$$

It is easy to check that $D_{s_i}$ is a linear operator $\mathbb{Z}[X^*(T)] \to \mathbb{Z}[X^*(T)]$ and is given explicitly by

$$(2) \qquad D_{s_i}(e^\lambda) = \begin{cases} e^\lambda + e^{\lambda - \alpha_i} + \cdots + e^{s_i \lambda} & \text{if } \langle \lambda, \alpha_i^\vee \rangle \geq 0 \\ 0 & \text{if } \langle \lambda, \alpha_i^\vee \rangle = -1 \\ -(e^{\lambda + \alpha_i} + \cdots + e^{s_i \lambda - \alpha_i}) & \text{if } \langle \lambda, \alpha_i^\vee \rangle < -1. \end{cases}$$

For any word $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$, we write $D_\mathfrak{w} = D_{s_{i_1}} \circ \cdots \circ D_{s_{i_n}}$. We now have the character formula.

---

[6] Here we use implicitly that $\omega_{Z_\mathfrak{w}} \otimes \mathcal{L}_\mathfrak{w}(\rho)^2 = \mathcal{O}(-\partial Z_\mathfrak{w}) \otimes \mathcal{L}_\mathfrak{w}(\rho)$ by Proposition 2.22.

[7] Applying [Ha77, Theorem 7.6] as stated requires that $H^i(X_w, \mathcal{F} \otimes \mathcal{L}^{-m}) = 0$ for $i < n$ and $m$ sufficiently large for all locally free sheaves $\mathcal{F}$. From the proof in [Ha77], however, only the case $\mathcal{F} = \mathcal{O}_{X_w}$ is used.

**Theorem 5.6** (Demazure character formula). *Let $\mathfrak{w}$ be any reduced decomposition of $w$ and $\lambda$ a dominant weight. Then, we have that*

$$\operatorname{ch}\Gamma(X_w, \mathcal{L}_w(\lambda)) = \overline{D_{\mathfrak{w}}(e^\lambda)},$$

*where $\overline{e^\lambda} = e^{-\lambda}$ is an involution on $\mathbb{Z}[X^*(T)]$.*

*Proof.* We will write $\bar{D}_{\mathfrak{w}}$ instead of $\overline{D_{\mathfrak{w}}}$. By Corollary 5.3 and Proposition 5.5, it suffices to show that $\chi(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) = \bar{D}_{\mathfrak{w}}(e^\lambda)$, where $\chi(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))$ denotes the characteristic

$$\chi(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) = \sum_i (-1)^i \operatorname{ch} H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)).$$

We will instead show the stronger statement that $\chi(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(M)) = \bar{D}_{\mathfrak{w}}(\overline{\operatorname{ch} M})$ for any $T$-module $M$. This statement will follow directly from the geometry of $Z_{\mathfrak{w}}$, requiring no input from our Frobenius splitting methods. The desired is then the case $M = k_{-\lambda}$.

First, for any short exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

of $T$-modules, we have $\bar{D}_{\mathfrak{w}}(\overline{\operatorname{ch} M_2}) = \bar{D}_{\mathfrak{w}}(\overline{\operatorname{ch} M_1}) + \bar{D}_{\mathfrak{w}}(\overline{\operatorname{ch} M_3})$ by the definition of ch. Further, by Lemma 2.17, the functor $\mathcal{L}_{\mathfrak{w}}(-)$ is exact, meaning that the functor $\chi(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(-))$ is exact, so we have $\chi(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(M_2)) = \chi(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(M_1)) + \chi(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(M_3))$. Thus, because any $T$-module is completely reducible, for a fixed $\ell(\mathfrak{w})$ it is enough for us to show that $\chi(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) = \sum_i (-1)^i \operatorname{ch} H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda))$ for all $\lambda$.

We now proceed by induction on $\ell(\mathfrak{w})$. [8] For $\ell(\mathfrak{w}) = 0$, we have $w = 1$. Hence, we see that

$$\Gamma(X_{\{1\}}, \mathcal{L}_w(\lambda)) = \mathcal{L}(\lambda)_{\{1\}} = (G \times^B k_{-\lambda})_{\{1\}} \simeq k_{-\lambda},$$

as needed. Now, for $\ell(\mathfrak{w}) = 1$, we find that

$$\chi(P_{s_i}/B, \mathcal{L}_{s_i}(\lambda)) = D_{s_i}(e^{-\lambda})$$

using Lemma 2.20, Serre duality on $\mathbb{P}^1$, and the explicit form of $D_{s_i}(e^\lambda)$ given in Equation (2).

In the general case, for $\ell(\mathfrak{w}) = n$, consider the map $\psi_{\mathfrak{w}} : Z_{\mathfrak{w}} \to Z_{\mathfrak{w}[n-1]}$, which gives the Leray spectral sequence

$$E_2^{i,j} = H^i(Z_{\mathfrak{w}[n-1]}, R^j(\psi_{\mathfrak{w}})_* \mathcal{L}_{\mathfrak{w}}(\lambda)) \implies H^{i+j}(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)).$$

Now, we claim that

$$(3) \qquad R^j(\psi_{\mathfrak{w}})_* \mathcal{L}_{\mathfrak{w}}(\lambda) \simeq \mathcal{L}_{\mathfrak{w}[n-1]}(H^j(P_{i_n}/B, \mathcal{L}_{s_{i_n}}(\lambda))) = \mathcal{L}_{\mathfrak{w}[n-1]}(H^j(Z_{s_{i_n}}, \mathcal{L}_{s_{i_n}}(\lambda))).$$

The proof of this will be extremely technical, so we postpone it in favor of the conclusion of our argument. Because the spectral sequence converges, we have

$$\begin{aligned}
\chi(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda)) &= \sum_j (-1)^j \chi(Z_{\mathfrak{w}[n-1]}, \mathcal{L}_{\mathfrak{w}[n-1]}(H^j(Z_{s_{i_n}}, \mathcal{L}_{s_{i_n}}(\lambda)))) \\
&= \sum_j (-1)^j \bar{D}_{\mathfrak{w}[n-1]}(\overline{\operatorname{ch} H^j(Z_{s_{i_n}}, \mathcal{L}_{s_{i_n}}(\lambda))}) \\
&= \bar{D}_{\mathfrak{w}[n-1]}(\overline{\chi(Z_{s_{i_n}}, \mathcal{L}_{s_{i_n}}(\lambda))}) \\
&= \bar{D}_{\mathfrak{w}[n-1]}(D_{s_{i_n}}(e^{-\lambda})) \\
&= \bar{D}_{\mathfrak{w}}(e^{-\lambda}))
\end{aligned}$$

using the inductive step for $Z_{\mathfrak{w}[n-1]}$ and $Z_{s_{i_n}}$. This completes the induction.

We now prove (3) modulo one technical claim for which we will appeal to [Jan]. We first claim that it suffices to give an isomorphism of functors [9]

$$(4) \qquad (\psi_{\mathfrak{w}})_* \mathcal{L}_{\mathfrak{w}}(-) \simeq \mathcal{L}_{\mathfrak{w}[n-1]}(\Gamma(Z_{s_{i_n}}, \mathcal{L}_{s_{i_n}}(-))).$$

Indeed, it would then suffice to show that $R^j(\psi_{\mathfrak{w}})_* \mathcal{L}_{Z_{\mathfrak{w}}}$ vanishes on injective $T$-modules, as the desired claim would follow by taking the $j^{\text{th}}$ right derived functor of both sides and noting that $\mathcal{L}_{\mathfrak{w}}(-)$, $\mathcal{L}_{\mathfrak{w}[n-1]}(-)$, and $\mathcal{L}_{s_{i_n}}(-)$ are exact by Lemma 2.17. This fact is checked in [Jan, II.5.19].

---

[8]It is important to note that our induction is still for general $M$.

[9]Here, we are using Lemma 2.16 to interpret the functors $\mathcal{L}_{\mathfrak{w}}$, $\mathcal{L}_{\mathfrak{w}[n-1]}$, and $\mathcal{L}_{s_{i_n}}$.

We now construct the isomorphism of (3) explicitly. Consider the following commutative diagram, where both vertical maps are $B$-bundles via the obvious right $B$-actions.

$$
\begin{array}{ccc}
P_{\mathfrak{w}[n-1]}/B^{n-2} \times^B P_{i_n} & \xrightarrow{\eta_{\mathfrak{w}}} & P_{\mathfrak{w}[n-1]}/B^{n-2} \\
\Big\downarrow{\scriptstyle \pi_1} & & \Big\downarrow{\scriptstyle \pi_2} \\
Z_{\mathfrak{w}} & \xrightarrow{\psi_{\mathfrak{w}}} & Z_{\mathfrak{w}[n-1]}
\end{array}
$$

Here $\eta_{\mathfrak{w}}$ is the projection onto the first $n-1$ coordinates. For any $\lambda$, consider the map

$$
\mathcal{L}_{\mathfrak{w}[n-1]}(\Gamma(Z_{s_{i_n}}, \mathcal{L}_{s_{i_n}}(\lambda))) \to (\psi_{\mathfrak{w}})_* \mathcal{L}_{\mathfrak{w}}(\lambda)
$$

that on $U \subset Z_{\mathfrak{w}[n-1]}$ sends $f : \pi_2^{-1}(U) \to \Gamma(Z_{s_{i_n}}, \mathcal{L}_{s_{i_n}}(\lambda))$ to the map $\pi_1^{-1}(\psi_{\mathfrak{w}}^{-1}(U)) \to k_{-\lambda}$ given by

$$
(p_1, \ldots, p_n) \mapsto f(\eta_{\mathfrak{w}}(p_1, \ldots, p_n))(p_n).
$$

It is easy to check that the resulting map is compatible with the $B$-actions. We now check locally that this map of line bundles is an isomorphism. Indeed, choose some affine $U \subset Z_{\mathfrak{w}[n-1]}$ such that $\pi_2$ and $\psi_{\mathfrak{w}}$ are trivial above $U$. Then, locally our map is given by the identifications

$$
\begin{aligned}
\Gamma(U, \mathcal{L}_{\mathfrak{w}[n-1]}(\Gamma(Z_{s_{i_n}}, \mathcal{L}_{s_{i_n}}(\lambda)))) &\simeq \mathcal{O}_{\pi_2^{-1}(U)} \otimes_k \Gamma(Z_{s_{i_n}}, \mathcal{L}_{s_{i_n}}(\lambda)) \\
&\simeq \mathcal{O}_{\pi_2^{-1}(U)} \otimes_k (\mathcal{O}_{P_{i_n}} \otimes_k k_{-\lambda})^B \\
&\simeq ((\mathcal{O}_{\pi_2^{-1}(U)} \otimes_k \mathcal{O}_{P_{i_n}}) \otimes_k k_{-\lambda})^B \\
&\simeq (\mathcal{O}_{\eta_{\mathfrak{w}}^{-1}(\pi_2^{-1}(U))} \otimes_k k_{-\lambda})^B \\
&\simeq \Gamma(\psi_{\mathfrak{w}}^{-1}(U), \mathcal{L}_{\mathfrak{w}}(\lambda)) \\
&\simeq \Gamma(U, (\psi_{\mathfrak{w}})_* \mathcal{L}_{\mathfrak{w}}(\lambda)),
\end{aligned}
$$

where for a sheaf $\mathcal{F}$ with a $B$-action, $\mathcal{F}^B$ denotes the $B$-invariants. Here, all identifications are by definition except the third, which follows because $B$ acts on $\mathcal{O}_{\pi_2^{-1}(U)} \otimes_k \mathcal{O}_{P_{i_n}}$ via $\mathcal{O}_{P_{i_n}}$ only. In addition, we are using that $P_{i_n}$ and $\mathcal{O}_{\pi_2^{-1}}$ are affine in the first two identifications. One may check that this corresponds to our earlier defined map; in particular, the evaluation map is done in the fourth identification. Thus, we have the desired isomorphism. $\qquad\square$

Notice that for $w = w_0$, Theorem 5.6 allows us to find the character of the $G$-module $\Gamma(\mathcal{B}, \mathcal{L}(\lambda))^*$. In characteristic 0, this corresponds to the irreducible $G$-module of highest weight $\lambda$ by Theorem 2.15. Let us check that Theorem 5.6 allows us to recover the usual Weyl character formula in this case.

**Corollary 5.7** (Weyl character formula). *Let $\lambda$ be a dominant weight. Then, the $G$-module $\Gamma(\mathcal{B}, \mathcal{L}(\lambda))^*$ has character*

$$
\operatorname{ch} \Gamma(\mathcal{B}, \mathcal{L}(\lambda))^* = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)}}.
$$

*Proof.* It suffices to compute $\overline{D_{\mathfrak{w}}(e^\lambda)}$ for a reduced decomposition $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$ of $w_0$. Suppose that we have

$$
\operatorname{ch} \Gamma(Z_w, \mathcal{L}_w(\lambda)) = \sum_\mu c_{w,\mu} e^\mu
$$

for some $c_{w,\mu}$, where we realize each $w$ as a subword of $w_0$, and let $c_{w_0,\mu} = c_\mu$. Then, we see that

$$
\begin{aligned}
(5) \qquad \operatorname{ch} \Gamma(\mathcal{B}, \mathcal{L}(\lambda))^* \cdot \sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)} &= \sum_{w,\mu} (-1)^{\ell(w)} c_\mu e^{w(\rho)+\mu} = \sum_{w,\mu} (-1)^{\ell(w)} c_{\mu-w(\rho)} e^\mu \\
&= \sum_{w,\mu} (-1)^{\ell(w)} c_{w^{-1}(\mu)-\rho} e^\mu = \sum_{w,\mu} (-1)^{\ell(w)} c_{w(\mu+\rho)-\rho} e^{\mu+\rho},
\end{aligned}
$$

where the third equality follows from the $W$-action on $\Gamma(\mathcal{B}, \mathcal{L}(\lambda))$ and the others reindex the sum.

We now claim that

$$(6) \qquad \sum_w (-1)^{\ell(w)} c_{v,w(\mu+\rho)-\rho} = \begin{cases} (-1)^{\ell(w)} & \mu = w(\lambda+\rho) - \rho \text{ for some } w \in W \\ 0 & \text{otherwise} \end{cases}$$

for all $v \in W$. Let us first show how this implies the desired and defer the proof until afterward. Specializing to $v = w_0$ and substituting into Equation (5), we see that

$$\operatorname{ch}\Gamma(\mathcal{B}, \mathcal{L}(\lambda))^* \cdot \sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)} = \sum_\mu e^{\mu+\rho} \sum_w (-1)^{\ell(w)} c_{w(\mu+\rho)-\rho} = \sum_w (-1)^{\ell(w)} e^{w(\lambda+\rho)},$$

which gives the desired character formula.

It suffices now to establish (6). We argue by induction on $\ell(v)$. For the base case $\ell(v) = 0$, the conclusion is clear. Now, write $v = s_i u$ with $\ell(u) = \ell(v) - 1$ and notice that

$$\sum_\mu c_{v,\mu} e^\mu = D_{s_i}\Big(\sum_\mu c_{u,\mu} e^\mu\Big) = \sum_\mu c_{u,\mu}\Big(e^\mu \sum_{k=0}^\infty e^{-k\alpha_i} - e^{s_i\mu} \sum_{k=1}^\infty e^{-k\alpha_i}\Big),$$

where we take $D_{s_i}$ instead of $\bar{D}_{s_i}$ because we are taking the dual of $\Gamma(\mathcal{B}, \mathcal{L}(\lambda))^*$. Taking the coefficients of $e^\mu$ on both sides, we obtain

$$c_{v,\mu} = \sum_{k=0}^\infty c_{u,\mu+k\alpha_i} - \sum_{k=1}^\infty c_{u,s_i\mu-k\alpha_i}.$$

Substituting into the desired sum, we obtain

$$\sum_w (-1)^{\ell(w)} c_{v,w(\mu+\rho)-\rho} = \sum_w (-1)^{\ell(w)} \sum_{k=0}^\infty c_{u,w(\mu+\rho)+k\alpha_i-\rho} - \sum_w (-1)^{\ell(w)} \sum_{k=1}^\infty c_{u,s_i w(\mu+\rho)-k\alpha_i-s_i\rho}$$

$$= \sum_w (-1)^{\ell(w)} \sum_{k=0}^\infty c_{u,w(\mu+\rho)+k\alpha_i-\rho} + \sum_w (-1)^{\ell(w)} \sum_{k=1}^\infty c_{u,w(\mu+\rho)-(k-1)\alpha_i-\rho}$$

$$= \sum_w \sum_{k=-\infty}^\infty (-1)^{\ell(w)} c_{u,w(\mu+\rho)+k\alpha_i-\rho} + \sum_w (-1)^{\ell(w)} c_{u,w(\mu+\rho)-\rho},$$

where the second equality follows by reindexing the summation over $s_i w$ and noting that $s_i\rho = \rho - \alpha_i$. By a similar reindexing, we see that

$$\sum_w \sum_{k=-\infty}^\infty (-1)^{\ell(w)} c_{u,w(\mu+\rho)+k\alpha_i-\rho} = -\sum_w \sum_{k=-\infty}^\infty (-1)^{\ell(w)} c_{u,s_i w(\mu+\rho)+k\alpha_i-\rho}$$

$$= -\sum_w \sum_{k=-\infty}^\infty (-1)^{\ell(w)} c_{u,w(\mu+\rho)+(k-\langle\alpha_i^\vee,w(\mu+\rho)\rangle)\alpha_i-\rho}$$

$$= -\sum_w \sum_{k=-\infty}^\infty (-1)^{\ell(w)} c_{u,w(\mu+\rho)+k\alpha_i-\rho},$$

which means that $\sum_w \sum_{k=-\infty}^\infty (-1)^{\ell(w)} c_{u,w(\mu+\rho)+k\alpha_i-\rho} = 0$. So we may conclude that

$$\sum_w (-1)^{\ell(w)} c_{v,w(\mu+\rho)-\rho} = \sum_w (-1)^{\ell(w)} c_{u,w(\mu+\rho)-\rho} = \begin{cases} (-1)^{\ell(w)} & \mu = w(\lambda+\rho) - \rho \text{ for some } w \in W \\ 0 & \text{otherwise,} \end{cases}$$

completing the induction. $\qquad\square$

**Remark.** In the proof of Corollary 5.7, the expression $w(\mu+\rho) - \rho$ occurs several times. This is known as the *dotted action* of $W$ on $X^*(T)$.

## Appendix A. Duality for the Frobenius morphism on smooth varieties

In this appendix we provide the technical details necessary to provide an explicit version of finite flat duality for the Frobenius morphism $F : X \to X$ on a smooth variety. Our model will be Theorem A.1 below, which gives the more general duality that holds in this situation.

**Theorem A.1** ([Ha66, III.6-10]). *Let $X, Y$ be Noetherian schemes and $f : X \to Y$ a finite flat morphism with fibers of dimension $0$. Then, there exists a pair $(\omega_f, \mathrm{Tr}_f)$ of a line bundle $\omega_f$ on $X$ and a map $\mathrm{Tr}_f : f_*\omega_f \to \mathcal{O}_Y$ given by evaluation at $1$ such that for any $\mathcal{F} \in \mathrm{QCoh}(X)$, the bilinear pairing*

$$f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_f) \times f_*\mathcal{F} \to f_*\omega_f \overset{\mathrm{Tr}_f}{\to} \mathcal{O}_Y$$

*induces an isomorphism*

$$f_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \omega_f) \simeq \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{F}, \mathcal{O}_Y).$$

*In this situation, we may take $\omega_f = \mathcal{H}om_{\mathcal{O}_Y}(f_*\mathcal{O}_X, \mathcal{O}_Y) \simeq \omega_{X/Y}$.*

While it is in theory possible to extract the trace map $\mathrm{Tr}_f : f_*\omega_f \to \mathcal{O}_Y$ from the general nonsense used in the proof of Theorem A.1, in practice this is extremely involved. In our case, we know that the morphism is in fact the Frobenius map $F : X \to X$ on a smooth variety $X$, allowing us to provide an alternate description of this duality in Proposition 3.14. Note that the trace map of Theorem A.1 becomes the composition $F_*\omega_X^{1-p} \to \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{O}_X$ in this new formulation.

Our strategy will be as follows. We first explicitly construct a map $C : F_*\omega_X \to \omega_X$ with a nice form in local coordinates. We then use $C$ to construct a map $F_*\omega_X^{1-p} \to \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$, which we show is an isomorphism by looking at its local properties. Let us begin by constructing this map.

**Proposition A.2.** *Let $X$ be a smooth variety. There is a map $C : F_*\omega_X \to \omega_X$ of $\mathcal{O}_X$-modules whose local expansion takes the form*

$$C(t^{\mathbf{a}} \cdot \omega) = \delta_p(\mathbf{a} + \mathbf{1}) t^{\frac{\mathbf{a}+\mathbf{1}}{p} - \mathbf{1}} \cdot \omega.$$

*Proof.* Let us first construct a map $f : \omega_X \to F_*(\omega_X/d\Omega_X^{n-1})$. We claim that there is a map on affines $f_A : \Omega_A^1 \to F_*(\Omega_A^1/d\Omega_A^0)$ given by $adb \mapsto a^p b^{p-1} db$. We first check that $f_A$ is a valid map. Indeed, we find directly that

$$f_A(adb) = a^p f_A(db) = a \cdot f_A(db)$$

and

$$f_A(d(ab)) = a^{p-1} b^{p-1} d(ab) = a^p b^{p-1} db + a^{p-1} b^p da = f_A(adb + bda).$$

For the final verification, notice that

$$p(a+b)^{p-1} d(a+b) = d((a+b)^p) = \sum_{i=0}^{p} \binom{p}{i} d(a^i b^{p-i}) = pa^{p-1} da + pb^{p-1} db + \sum_{i=1}^{p-1} \binom{p}{i} d(a^i b^{p-i})$$

as integer polynomials. Therefore, dividing by $p$, we see that

$$f_A(d(a+b)) - f_A(da) - f_A(db) = (a+b)^{p-1} d(a+b) - a^{p-1} da - b^{p-1} db = \sum_{i=1}^{p-1} \frac{1}{p} \binom{p}{i} d(a^i b^{p-i}) \in \mathrm{Im}(\Omega_A^0)$$

because $p \mid \binom{p}{i}$ for $1 \le i \le p-1$. So we see that $f_A$ gives a valid map $\Omega_A^1 \to F_*(\Omega_A^1/\Omega_A^0)$.

Taking this map on each affine and applying the $n^{\mathrm{th}}$ exterior power, we obtain a map $f : \omega_X \to F_*(\omega_X/\Omega_X^{n-1})$. Locally, $f$ is given on each $\mathrm{Spec}(A)$ by

$$adb_1 \wedge \cdots \wedge db_n \mapsto a^p b_1^{p-1} \cdots b_n^{p-1} db_1 \wedge \cdots \wedge db_n.$$

We now claim that $f$ is an isomorphism. It suffices to check this in local coordinates $t_1, \ldots, t_n$ at all points $x \in X$. In this case, the map becomes

$$g(t_1, \ldots, t_n) \cdot \omega \mapsto g(t_1, \ldots, t_n)^p t_1^{p-1} \cdots t_n^{p-1} \cdot \omega.$$

Let us first examine the space $d\Omega_{k[[t_1, \ldots, t_n]]}^{n-1}$. Notice that for $\mathbf{a}$ with $\delta_p(\mathbf{a} + \mathbf{1}) = 0$, we have

$$t^{\mathbf{a}} \cdot \omega = \partial_{t_i} \left( \frac{1}{a_i + 1} t_1^{a_1} \cdots t_{i-1}^{a_{i-1}} t_i^{a_i+1} t_{i+1}^{a_{i+1}} \cdots t_n^{a_n} dt_1 \wedge \cdots \wedge dt_{i-1} \wedge dt_{i+1} \cdots \wedge dt_n \right) \in d\Omega_{k[[t_1, \ldots, t_n]]}^{n-1},$$

where $i$ is chosen so that $a_i + 1 \not\equiv 0 \pmod{p}$. On the other hand, if $\delta_p(\mathbf{a} + \mathbf{1}) = 1$, we see that $\partial_{t_i}(t^{\mathbf{a}} \cdot dt_1 \wedge \cdots \wedge \widehat{dt_i} \wedge \cdots \wedge dt_n) = 0$ for all $i$, hence $d\Omega^{n-1}_{k[[t_1,\ldots,t_n]]}$ is spanned over $k[[t_1,\ldots,t_n]]$ by monomials $t^{\mathbf{a}} \cdot \omega$ with $\delta_p(\mathbf{a} + \mathbf{1}) = 0$. But notice that the action is twisted by the Frobenius map $F : k[[t_1,\ldots,t_n]] \to k[[t_1,\ldots,t_n]]$, meaning that the action of $k[[t_1,\ldots,t_n]]$ on a monomial produces only monomials with exponents of the same residue modulo $p$. Thus we conclude that $d\Omega^{n-1}_{k[[t_1,\ldots,t_n]]}$ consists of formal power series containing only monomials of the form $t^{\mathbf{a}}$, where $\delta_p(\mathbf{a} + \mathbf{1}) = 0$.

Deducing that $f$ is an isomorphism is now straightforward. Any $g(t_1,\ldots,t_n) \cdot \omega \in \Omega^n_{k[[t_1,\ldots,t_n]]}/d\Omega^{n-1}_{k[[t_1,\ldots,t_n]]}$ is equal to the sum of its monomials $c_{\mathbf{a}} t^{\mathbf{a}}$ with $\delta_p(\mathbf{a}+\mathbf{1}) = 1$. But these are evidently in the image of $f$, hence $f$ is surjective. Now, suppose that $f$ kills some $\sum_{\mathbf{a}} c_{\mathbf{a}} t^{\mathbf{a}}$. This means that

$$f\Big(\sum_{\mathbf{a}} c_{\mathbf{a}} t^{\mathbf{a}}\Big) = \sum_{\mathbf{a}} c_{\mathbf{a}}^p t^{p\mathbf{a}+\mathbf{p}-\mathbf{1}} \in d\Omega^{n-1}_{k[[t_1,\ldots,t_n]]},$$

which is impossible, since elements $d\Omega^{n-1}_{k[[t_1,\ldots,t_n]]}$ contain no monomials of the form $t^{p\mathbf{a}+\mathbf{p}-\mathbf{1}}$. Hence $f$ is also injective, completing the proof that $f$ is an isomorphism.

Now, we claim that $C : F_*\omega_X \to F_*(\omega_X/d\Omega^{n-1}_X) \xrightarrow{f^{-1}} \omega_X$ is the desired map. Notice that $C(t^{\mathbf{a}} \cdot \omega) = g(t_1,\ldots,t_n) \cdot \omega$ satisfies

$$f(g(t_1,\ldots,t_n) \cdot \omega) - t^{\mathbf{a}} \cdot \omega = g(t_1,\ldots,t_n)^p t^{\mathbf{p}-\mathbf{1}} \cdot \omega - t^{\mathbf{a}} \cdot \omega \in d\Omega^{n-1}_{k[[t_1,\ldots,t_n]]}$$

and that $C(t^{\mathbf{a}} \cdot \omega)$ is uniquely defined by this. Thus we see immediately that when $\delta_p(\mathbf{a} + \mathbf{1}) = 1$, we have $C(t^{\mathbf{a}} \cdot \omega) = t^{\frac{\mathbf{a}+\mathbf{1}}{p}-\mathbf{1}}$. On the other hand, for $\delta_p(\mathbf{a}+\mathbf{1}) = 0$, we showed above that $t^{\mathbf{a}} \cdot \omega \in d\Omega^{n-1}_{k[[t_1,\ldots,t_n]]}$, hence we have that $C(t^{\mathbf{a}} \cdot \omega) = 0$ in this case. So $C$ has the desired local form. $\qquad \square$

Now that we know the local behavior of the map $C$, we may use it to construct the desired isomorphism. In particular, this will allow us to check for isomorphism locally.

*Proof of Proposition 3.14.* We first define the map abstractly and then check that it is an isomorphism from its local description. Consider the composition of identifications

$$F_*\omega_X^{1-p} \to F_*(\omega_X^{-p} \otimes \omega_X) \to \omega_X^{-1} \otimes F_*\omega_X \to \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X),$$

where the first is clear, the second results from the identification $\omega_X^{-1} \otimes F_*\mathcal{O}_X \simeq F^*\omega_X^{-1} \simeq \omega_X^{-p}$ given by $w \otimes f \mapsto w^p f$ and the projection formula, and the last by tensoring $\omega_X^{-1}$ with

$$F_*\mathcal{O}_X \otimes F_*\omega_X \xrightarrow{\text{mult}} F_*\omega_X \xrightarrow{C} \omega_X.$$

Now, tracing the map in local coordinates $t_1,\ldots,t_n$ at some $x \in X$, it maps

$$t^{\mathbf{a}} \cdot \omega^{1-p} \mapsto \omega^{-p} \otimes t^{\mathbf{a}} \cdot \omega \mapsto \omega^{-1} \otimes t^{\mathbf{a}} \cdot \omega \mapsto \Big(t^{\mathbf{b}} \mapsto C(t^{\mathbf{a}+\mathbf{b}} \cdot \omega) \cdot \omega^{-1}\Big) = \Big(t^{\mathbf{b}} \mapsto \delta_p(\mathbf{a}+\mathbf{b}+\mathbf{1}) t^{\frac{\mathbf{a}+\mathbf{b}+\mathbf{1}}{p}-\mathbf{1}}\Big),$$

where we've used the local expansion of $C$. Finally, to check that the map is an isomorphism, we may again check at each point $x \in X$ in local coordinates. But in this case, we have an explicit inverse

$$\text{Hom}_{k[[t_1,\ldots,t_n]]}(F_*k[[t_1,\ldots,t_n]], k[[t_1,\ldots,t_n]]) \to F_*k[[t_1,\ldots,t_n]]$$

given by

$$f \mapsto \sum_{\mathbf{a} < \mathbf{p}-\mathbf{1}} f(t^{\mathbf{a}})^p,$$

where by $\mathbf{a} < \mathbf{p}-\mathbf{1}$ we mean that $0 \leq a_i < p$. Verifying that these maps are inverse is straightforward after noting that a map $F_*k[[t_1,\ldots,t_n]] \to k[[t_1,\ldots,t_n]]$ is specified uniquely by its values on $t^{\mathbf{a}}$ for $\mathbf{a} < \mathbf{p}-\mathbf{1}$. $\qquad \square$

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206 Cabot Mail Center, Cambridge, MA 02138
*E-mail address*: yisun@fas.harvard.edu