Smoothing, Fudging, and Ordering

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MOP 2010

1 Warmup

Problem 1 (China 2004)**.** Find the largest positive real k, such that for any positive reals a, b, c, d, we have

$$(a+b+c)\left[81(a+b+c+d)^5 + 16(a+b+c+2d)^5\right] \geq kabcd^3.$$

Problem 2 (MOP 2002)**.** For a, b, c positive reals, prove that

$$\left(\frac{2a}{b+c}\right)^{\frac{2}{3}} + \left(\frac{2b}{c+a}\right)^{\frac{2}{3}} + \left(\frac{2c}{a+b}\right)^{\frac{2}{3}} \geq 3.$$

2 Useful Techniques

In this lecture we discuss several more ad-hoc methods of attacking inequalities:

- **Smoothing:** Suppose you wish to prove an inequality of the form

  $$f(x_1, x_2, \ldots, x_n) \geq C$$

  with the constraint $x_1 + \cdots + x_n = k$. If equality holds when all x_i are equal, then, heuristically, you can try to make $f(x_1, x_2, \ldots, x_n)$ smaller by "moving the x_i together." Rigorously, this means showing inequalities of the form

  $$f(x_1, x_2, \ldots, x_n) \geq f(k/n, x_1 + x_2 - k/n, \ldots, x_n).$$

- **Isolated Fudging:** Given an inequality of the form

  $$f(a, b, c) + f(b, c, a) + f(c, a, b) \geq k,$$

  we can try to bound each term individually by

  $$f(a, b, c) \geq \frac{k\, a^r}{a^r + b^r + c^r}$$

  for some r. More generally, it is useful to attempt to modify each portion of an inequality separately.

- **Ordering:** It can be useful to assume an order on the variables in an inequality. Suppose it is possible to write an inequality in the form

  $$S_a(b-c)^2 + S_b(c-a)^2 + S_c(a-b)^2 \geq 0$$

  for some S_a, S_b, S_c. In this case, if the ordering $a \geq b \geq c$ induces an appropriate ordering on some linear functions of S_a, S_b, S_c, we may imitate the proof of Schur to obtain the result.

3 Useful Facts

Throughout this section, we refer to *convex* functions. We say that f is convex if $f''(x) \geq 0$ for all x or if

$$\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$$

for all $0 \leq \lambda \leq 1$ and $x < y$. We will also refer to the concept of *majorization*; we say that the sequence $a_1, \ldots, a_n$ majorizes the sequence $b_1, \ldots, b_n$ if $a_1 + \cdots + a_i \geq b_1 + \cdots + b_i$ for $i < n$ and $a_1 + \cdots + a_n = b_1 + \cdots + b_n$.

Theorem 1 (Weighted Jensen)**.** Let f be a convex function, $x_1, \ldots, x_n$ real numbers, and $a_1, \ldots, a_n$ non-negative reals with $a_1 + \cdots + a_n = 1$. Then, we have

$$a_1 f(x_1) + \cdots + a_n f(x_n) \geq f\left(a_1 x_1 + \cdots + a_n x_n\right).$$

Theorem 2 (Karamata)**.** Let f be convex. Then, for $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ such that the $\{x_i\}$ majorize the $\{y_i\}$, we have

$$f(x_1) + \cdots + f(x_n) \geq f(y_1) + \cdots + f(y_n).$$

Theorem 3 (Schur)**.** For non-negative reals x, y, z and $r > 0$, we have

$$x^r(x-y)(x-z) + y^r(y-x)(y-z) + z^r(z-x)(z-y) \geq 0$$

where equality holds if $x = y = z$ or $\{x, y, z\} = \{0, a, a\}$ for some a.

4 Problems

4.1 Smoothing

Problem 3 (USAMO 1980)**.** Show that for all non-negative reals $a, b, c \leq 1$,

$$\frac{a}{b+c+1} + \frac{b}{c+a+1} + \frac{c}{a+b+1} + (1-a)(1-b)(1-c) \leq 1.$$

Problem 4 (USAMO 1999)**.** Let $a_1, a_2, \ldots, a_n$ $(n > 3)$ be real numbers such that

$$a_1 + a_2 + \cdots + a_n \geq n \qquad \text{and} \qquad a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2.$$

Prove that $\max(a_1, a_2, \ldots, a_n) \geq 2$.

Problem 5 (USAMO 1998)**.** Let $a_1, \ldots, a_n$ be real numbers in the interval $\left(0, \frac{\pi}{2}\right)$ such that

$$\tan\left(a_0 - \frac{\pi}{4}\right) + \tan\left(a_1 - \frac{\pi}{4}\right) + \cdots + \tan\left(a_n - \frac{\pi}{4}\right) \geq n - 1.$$

Prove that

$$\tan(a_0) \cdot \tan(a_1) \cdot \cdots \cdot \tan(a_n) \geq n^{n+1}.$$

Problem 6 (IMO 1974)**.** Determine all possible values of

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

where a, b, c, d are arbitrary positive numbers.

Problem 7 (Vietnam 1998)**.** Let $x_1, \ldots, x_n$ be positive numbers satisfying

$$\frac{1}{x_1 + 1998} + \frac{1}{x_2 + 1998} + \cdots + \frac{1}{x_n + 1998} = \frac{1}{1998}.$$

Prove that

$$\frac{\sqrt[n]{x_1 x_2 \cdots x_n}}{n-1} \geq 1998.$$

4.2 Fudging

Problem 8 (IMO 2001)**.** Prove that for all positive real numbers a, b, c,

$$\frac{a}{\sqrt{a^2 + 8bc}} + \frac{b}{\sqrt{b^2 + 8ca}} + \frac{c}{\sqrt{c^2 + 8ab}} \geq 1.$$

Problem 9 (USAMO 2004)**.** Let $a, b, c > 0$. Prove that

$$(a^5 - a^2 + 3)(b^5 - b^2 + 3)(c^5 - c^2 + 3) \geq (a + b + c)^3.$$

Problem 10 (USAMO 2003)**.** Let a, b, c be positive real numbers. Prove that

$$\frac{(2a + b + c)^2}{2a^2 + (b + c)^2} + \frac{(2b + c + a)^2}{2b^2 + (c + a)^2} + \frac{(2c + a + b)^2}{2c^2 + (a + b)^2} \leq 8.$$

Problem 11 (IMO 2005)**.** Let x, y, z be three positive reals such that $xyz \geq 1$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{x^2 + y^5 + z^2} + \frac{z^5 - z^2}{x^2 + y^2 + z^5} \geq 0.$$

Problem 12 (Japan 1997)**.** Show that for all positive reals a, b, c,

$$\frac{(a + b - c)^2}{(a + b)^2 + c^2} + \frac{(b + c - a)^2}{(b + c)^2 + a^2} + \frac{(c + a - b)^2}{(c + a)^2 + b^2} \geq \frac{3}{5}.$$

4.3 Ordering

Problem 13. Prove Schur's Inequality.

Problem 14 (USAMO 2001)**.** Let a, b, c be non-negative reals such that

$$a^2 + b^2 + c^2 + abc = 4.$$

Prove that

$$0 \leq ab + bc + ca - abc \leq 2.$$

Problem 15. Prove that for any positive reals a, b, c,

$$(a^2 + b^2 + c^2)^2 \geq 3(a^3 b + b^3 c + c^3 a).$$

Problem 16 (TST 2009)**.** Prove that for positive real numbers x, y, z, we have

$$x^3(y^2 + z^2)^2 + y^3(z^2 + x^2)^2 + z^3(x^2 + y^2)^2 \geq xyz[xy(x + y)^2 + yz(y + z)^2 + zx(z + x)^2].$$

4.4 Bonus Weird Inequalities

Problem 17 (ISL 2001)**.** Let $x_1, x_2, \ldots, x_n$ be real numbers. Prove that

$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

Problem 18 (IMO 2004)**.** Let $n \geq 3$ be an integer. Let $t_1, t_2, \ldots, t_n$ be positive real numbers such that

$$n^2 + 1 > (t_1 + t_2 + \cdots + t_n)\left(\frac{1}{t_1} + \frac{1}{t_2} + \cdots + \frac{1}{t_n}\right).$$

Show that t_i, t_j, t_k are side lengths of a triangle for all i, j, k with $1 \leq i < j < k \leq n$.

Problem 19 (Russia 2004)**.** Let $n > 3$ be an integer and let $x_1, x_2, \ldots, x_n$ be positive reals with product 1. Prove that

$$\frac{1}{1+x_1+x_1x_2} + \frac{1}{1+x_2+x_2x_3} + \cdots + \frac{1}{1+x_n+x_nx_1} > 1.$$

Problem 20 (Romania 2004)**.** Let $n \geq 2$ be an integer and let $a_1, a_2, \ldots, a_n$ be real numbers. Prove that for any non-empty subset $S \subset \{1, 2, \ldots, n\}$, we have

$$\left(\sum_{i \in S} a_i\right)^2 \leq \sum_{1 \leq i \leq j \leq n} (a_i + \cdots + a_j)^2.$$