PERVERSE SHEAVES AND THE KAZHDAN-LUSZTIG CONJECTURES

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Contents

1. Introduction

1.1. **Motivation.** Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. The finite-dimensional representation theory of $\mathfrak{g}$ is well-studied; the category of representations is semisimple, and irreducible representations are classified by their positive integral highest weights. Further, the Weyl character formula gives an explicit expression for the characters of these irreducibles.

A natural next step is then to expand the class of representations under consideration. We would like such a class to contain the Verma modules M_λ, which are perhaps the prototypical example of highest weight modules. It turns out that a nice category $\mathcal{O}$ of representations exists which contains the Verma modules and all extensions and quotients of them. In particular, the category $\mathcal{O}$ is Artinian, and we may again parametrize the simple objects L_λ of $\mathcal{O}$ by their highest weight. We would like then to understand:

- the decomposition of $[M_\lambda]$ into $[L_\mu]$ in the Grothendieck group of $\mathcal{O}$, and
- the characters of the irreducible representations L_λ.

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The characters of the Verma modules are easily determined, so the second goal follows from the first, which is known as the Kazhdan-Lusztig conjectures. These two goals were achieved in the early 1980's, and the goal of this paper is to give an expository account of the relevance of perverse sheaves in their realization.

The solution proceeds by using geometric methods which successively transformed the original algebraic problem into different contexts. We summarize these steps in the following diagram, where the arrows mean that an equivalence of categories exists under some appropriate conditions.

$$\boxed{\text{Category } \mathcal{O}} \overset{\text{Beilinson-Bernstein}}{\rightarrow} \boxed{\text{Regular holonomic } \mathcal{D}\text{-modules}} \overset{\text{Riemann-Hilbert}}{\rightarrow} \boxed{\text{Perverse sheaves}}.$$

After translation into the context of perverse sheaves, the problem becomes to understand the composition series of some intersection cohomology sheaves on flag varieties. Applying the Decomposition Theorem to desingularizations of the Schubert varieties allows us to compute stalks of these sheaves and solve the original problem.

1.2. **Organization and references.** Let us now discuss the specific structure of this paper. In Section 2, we introduce the geometric backdrop for the paper. The main goals are to introduce Schubert varieties and to discuss the decomposition of category $\mathcal{O}$ into blocks via the Harish-Chandra isomorphism. In Section 3, we define and discuss the Beilinson-Bernstein and Riemann-Hilbert correspondences, which allow us to pass from representations of a Lie algebra to perverse sheaves. The main objective is understand the image of the simple modules and the Verma modules under these correspondences. In Section 4, our main work takes place; we formulate the Kazhdan-Lusztig conjectures and proof them modulo the major work of Section 3. The key step will be the computation of the stalks of intersection cohomology sheaves on Schubert varieties. This will involve an application of the Decomposition Theorem to the Bott-Samelson resolution of the Schubert varieties.

The material we present here builds on a large body of mathematics and is drawn from a number of different sources. As the list of necessary inputs for our goal is quite large, we have chosen to emphasize those aspects which involve perverse sheaves and the Kazhdan-Lusztig result directly. As such, we have omitted many proofs which do not pertain directly to these topics, and we will assume knowledge of the theory of $\mathcal{D}$-modules and the geometry of the flag variety of a semisimple algebraic group. However, we have attempted to include sufficient background material to connect the final computation using perverse sheaves on the flag variety coherently to the entire story.

We now detail the sources we used. Throughout the paper, we have made extensive reference to [HTT08], which gives quite a comprehensive exposition. For the theory of $\mathcal{D}$-modules and the Riemann-Hilbert correspondence, we have also consulted [Ber83]. For our main work, we followed primarily [Ric10a], [Ric10b], and [Spr81].

1.3. **Conventions and notation.** We collect here some notations which we will use throughout this essay. Unless explicitly specified otherwise, when we write functors $f_*, f^*, f_!, f^!$, we will always mean the derived versions. We will always work over the field $\mathbb{C}$ of complex numbers. By $\mathcal{RH}om$ and $\mathcal{H}om$, we mean the derived and underived sheaf Hom's.

2. The geometric setting

In this section, we establish the geometric setting for the rest of this essay will take place. We begin by describing the flag and Schubert varieties associated to a semisimple algebraic group G and the corresponding Lie algebra $\mathfrak{g}$. We then define the category $\mathcal{O}$ of representations of $\mathfrak{g}$ and describe its decomposition into blocks via the Harish-Chandra isomorphism.

2.1. **Preliminaries on semisimple algebraic groups.** Let G be a connected, simply connected, semisimple algebraic group over $\mathbb{C}$. Pick a Borel subgroup B of G and a maximal torus $T \subset B$. Let U be a maximal unipotent subgroup of B, and recall that $T \simeq B/U$. Let $\Phi(G,T) = (X^*, R, X_*, R^\vee)$ be the root datum associated to G. For $\lambda \in X^*$, denote by e^λ the corresponding character $e^\lambda : T \to \mathbb{G}_m$. Our choice of B defines a choice of positive roots $R^+ \subset R$. Let the corresponding sets of simple roots be $\{\alpha_1, \ldots, \alpha_n\}$, and let $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$ be half the sum of the positive roots, which is also the longest positive root.

Let $W = N(T)/T$ be the Weyl group of G, and let s_i be the simple reflection corresponding to α_i. For $w \in W$, let $\ell(w)$ denote its length, which is the length of the shortest reduced decomposition for w. Denote by w_0 the longest element in W.

2.2. **Flag varieties, the Bruhat decomposition and Schubert varieties.** We are now ready to introduce our basic geometric setting, the flag variety $X = G/B$. For any Borel subgroup B of G, let $X = G/B$ be the corresponding flag variety of G, interpreted as a quotient under the action of B by right translation. We recall the following characterization of X which is independent of choice of B.

Proposition 2.1. *The flag variety $X = G/B$ is projective and its k-points are given by*

$$X(k) = \{\textit{Borel subgroups of } G\}.$$

Recall now the Bruhat decomposition of G, the properties of which we summarize below.

Proposition 2.2 ([Spr98, Lemmas 8.3.6 and 8.3.7, Theorem 8.3.8])**.** *We have the following:*

(i) *for $\widetilde{w} \in N(T)$ a representative of $w \in W$, the double coset $B\widetilde{w}B$ depends only on w;*
(ii) *we have an isomorphism $\mathbb{A}^{\ell(w)} \times B \to BwB$;*
(iii) *for $s_i \in W$ a simple reflection, we have*

$$(Bs_iB) \cdot (BwB) = \begin{cases} Bs_iwB & \ell(s_iw) > \ell(w) \\ (Bs_iwB) \cup (BwB) & \ell(s_iw) < \ell(w); \end{cases}$$

(iv) *there is a decomposition*

$$G = \bigcup_{w \in W} BwB$$

of G into the disjoint union of double cosets under the left and right B-actions.

The double cosets BwB are called *Bruhat cells* and are locally closed. This decomposition may be extended to the flag variety X. For each $w \in W$, let the *Schubert cell* $X_w^o = BwB/B$ be the quotient of the corresponding Bruhat cell corresponding to w. Then, the *Schubert variety* X_w corresponding to w is defined to be its closure $X_w = \overline{X}_w^o$. For $u, v \in W$, write $u \leq v$ in the *Bruhat-Chevalley order* if it is possible to form a reduced decomposition of u by deleting some simple reflections from a reduced decomposition of v. Then, X_w admits the following decomposition.

Proposition 2.3 ([Spr98, Corollary 8.5.5])**.** *For $w \in W$, we have*

$$X_w = \bigcup_{v \leq w} X_v^o.$$

For the longest element w_0 of W, we see that $X_{w_0} = X$, as a reduced decomposition of any $w \in W$ occurs as a subword of the reduced decomposition of w_0. Therefore, the Schubert varieties provide a stratification of the flag variety X with affine open strata (since $X_w^o = BwB/B \simeq \mathbb{A}^{\ell(w)}$).

2.3. **The Lie algebra of a semisimple algebraic group.** Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{h} \subset \mathfrak{g}$ and $\mathfrak{b} \subset \mathfrak{g}$ denote the Cartan and Borel subalgebras corresponding to T and B, respectively. Let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ so that $\mathfrak{h} \simeq \mathfrak{b}/\mathfrak{n}$. Let $\mathfrak{b}^-$ be the opposite Borel subalgebra and set $\mathfrak{n}^- = [\mathfrak{b}^-, \mathfrak{b}^-]$ so that $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$. We can make these constructions for all points $x \in X$ under the correspondence between points of X and Borel subgroups of G; for any such x, we will denote by $\mathfrak{b}_x, \mathfrak{b}_x^-, \mathfrak{n}_x, \mathfrak{n}_x^-$ the subalgebras corresponding to the Borel subgroup of x in the sense of Corollary 2.1. Let $U(\mathfrak{g})$ denote the universal enveloping algebra of $\mathfrak{g}$.

Fix now a choice of Borel subalgebra $\mathfrak{b}$ corresponding to a choice of Borel subgroup B. For a weight λ, denote by

$$M_\lambda := U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} k_\lambda$$

the *Verma module of highest weight* λ.[1] It is known that M_λ has a unique simple quotient, which we denote by L_λ. A basic problem in the study of $\mathfrak{g}$-representations is to find the multiplicities of the L_λ in the Jordan-Holder series of M_λ.

[1]This construction depends crucially on our choice of Borel $\mathfrak{b}$.

2.4. **Harish-Chandra isomorphism.** We now describe in some detail the Harish-Chandra isomorphism, which characterizes the action of the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ on a representation of $\mathfrak{g}$. By Schur's lemma, the action of $Z(\mathfrak{g})$ on any finite-dimensional irreducible $U(\mathfrak{g})$-module factors through k, meaning that $Z(\mathfrak{g})$ acts via a map of k-algebras $\chi : Z(\mathfrak{g}) \to k$, which we call a *central character*. For any central character χ, define

$$U(\mathfrak{g})_\chi = U(\mathfrak{g})/U(\mathfrak{g}) \cdot \ker \chi$$

to be the quotient of $U(\mathfrak{g})$ by the (two-sided) ideal generated by $\ker \chi$. Note that $U(\mathfrak{g})_\chi$-modules are simply $U(\mathfrak{g})$-modules where $Z(\mathfrak{g})$ acts by χ.

We now classify the central characters χ by relating them to characters $\lambda \in \mathfrak{h}^*$. By the PBW Theorem, given a choice of Cartan and Borel subalgebras $\mathfrak{h} \subset \mathfrak{b}$, $U(\mathfrak{g})$ splits as a direct sum

$$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- \cdot U(\mathfrak{g}) + U(\mathfrak{g}) \cdot \mathfrak{n}), \tag{1}$$

where we note that any pure tensor in $U(\mathfrak{n}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n})$ which does not lie in $U(\mathfrak{h})$ lies in at least one of $\mathfrak{n}^- \cdot U(\mathfrak{g})$ or $U(\mathfrak{g}) \cdot \mathfrak{n}$. Let the map

$$\psi_\mathfrak{b} : U(\mathfrak{g}) \to U(\mathfrak{h})$$

to be the projection onto $U(\mathfrak{h})$ under this direct sum. We call the restriction of $\psi_\mathfrak{b}$ to $Z(\mathfrak{g})$ the *Harish-Chandra homomorphism* relative to $\mathfrak{b}$.[2] One may check that this map is a map of algebras.

Viewing $\mathrm{Sym}(\mathfrak{h})$ as the space of polynomial functions on $\mathfrak{h}^*$, we may interpret $\psi_\mathfrak{b}$ as a map

$$\mathfrak{h}^* \to \mathrm{MaxSpec}(Z(\mathfrak{g})), \tag{2}$$

where the central characters χ are in bijection with $\mathrm{MaxSpec}(Z(\mathfrak{g}))$. For $\lambda \in \mathfrak{h}^*$, denote by $\chi_\lambda^\mathfrak{b}$ the central character corresponding to λ relative to $\mathfrak{b}$. We may immediately understand the following important concrete example of the action of $Z(\mathfrak{g})$ through a central character.

Corollary 2.4. *For any weight $\lambda \in \mathfrak{h}^*$, $Z(\mathfrak{g})$ acts via $\chi_\lambda^\mathfrak{b}$ on the Verma module $M_\lambda^\mathfrak{b}$.*

Thus far, we have considered what we call the Harish-Chandra homomorphism *relative to* $\mathfrak{b}$, a map $\psi_\mathfrak{b} : Z(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{h})$ which may (and does) depend on the embedding $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ we chose when applying the PBW decomposition. As we have hinted in our nomenclature thus far, it is possible to modify the construction of $\psi_\mathfrak{b}$ to obtain a map $\psi : Z(\mathfrak{g}) \to U(\mathfrak{h})$ that is independent of the choice of Cartan subalgebra. Indeed, define the map ψ, which we will call the *Harish-Chandra homomorphism*, to be the composition

$$Z(\mathfrak{g}) \overset{\psi_\mathfrak{b}}{\to} U(\mathfrak{h}) \overset{\xi \mapsto \xi - \rho_\mathfrak{b}(\xi)}{\to} U(\mathfrak{h}),$$

where the map $U(\mathfrak{h}) \to U(\mathfrak{h})$ is the one induced by mapping $\mathfrak{h} \to U(\mathfrak{h})$ via $\xi \mapsto \xi - \rho_\mathfrak{b}(\xi)$. Here, we write $\rho_\mathfrak{b}$ for the longest positive root relative to $\mathfrak{b}$ to emphasize that the choice of $\rho_\mathfrak{b} \in \mathfrak{g}$ depends on the choice of $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. We see that ψ then admits the following nice characterization, which is known as the *Harish-Chandra isomorphism.*

Theorem 2.5 ([Dix77, Theorem 7.4.5])**.** *The map ψ is an isomorphism $Z(\mathfrak{g}) \to \mathrm{Sym}(\mathfrak{h})^W$ which is independent of the choice of $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$.*

Notice that ψ gives rise to a different map

$$\mathfrak{h}^* \to \mathrm{MaxSpec}(Z(\mathfrak{g})). \tag{3}$$

For $\lambda \in \mathfrak{h}^*$, denote now by χ'_λ the central character corresponding to λ. We then have the following characterization of the space of central characters, which follows formally from the Harish-Chandra isomorphism.

Corollary 2.6. *We have the following:*

(i) *every central character χ' lies in the image of the map (3), and*
(ii) *$\chi'_\lambda = \chi'_\mu$ if and only if there is some $w \in W$ such that $\lambda = w(\mu)$.*

Translating Corollary 2.6 into our original situation of a fixed chosen Borel subalgebra $\mathfrak{b}$ yields the following. Define the *dotted action* of W on $\mathfrak{h}^*$ by

$$w \cdot \lambda = w(\lambda + \rho_\mathfrak{b}) - \rho_\mathfrak{b}$$

for any $\lambda \in \mathfrak{h}^*$. Then, $\chi_\lambda^\mathfrak{b} = \chi_\mu^\mathfrak{b}$ if and only if λ and μ are in the same W-orbit under the dotted action.

[2]While the definition of $\psi_\mathfrak{b}$ depends on the choice of both $\mathfrak{h}$ and $\mathfrak{b}$, we write $\psi_\mathfrak{b}$ instead of $\psi_{\mathfrak{h},\mathfrak{b}}$ because the latter is somewhat cumbersome.

Remark. For the remainder of this paper, we will work with respect to a fixed Borel subalgebra $\mathfrak{b}$. For notational convenience, we will write χ_λ instead of $\chi_\lambda^{\mathfrak{b}}$. By this, we will always mean the map of (2).

2.5. **Category $\mathcal{O}$.** While finite-dimensional $\mathfrak{g}$-representations are well-understood classically, the category of all $\mathfrak{g}$-representations is quite large, meaning that an analogous treatment is unlikely. Instead, we choose to study the following full subcategory.

Definition 2.7. Let $\mathfrak{g}$ be a semisimple Lie algebra. The *category* $\mathcal{O}$ of $\mathfrak{g}$-representations is the full subcategory of representations V such that
- (i) $\mathfrak{n}$ acts locally finitely on V,
- (ii) $\mathfrak{h}$ acts locally finitely and semisimply on V, and
- (iii) V is finitely generated over $\mathfrak{g}$.

Evidently, all finite dimensional modules lie in this category $\mathcal{O}$. By construction, it also contains all Verma modules. In fact, it is known that the objects of category $\mathcal{O}$ are exactly quotients of finite extensions of Verma modules. In particular, this means that the L_λ parametrize the simple objects in category $\mathcal{O}$.

Now, for any representation V of $\mathfrak{g}$ lying in category $\mathcal{O}$, the Harish-Chandra homomorphism allows us to decompose V as a direct sum

$$V = \bigoplus_{\chi \in \mathrm{MaxSpec}(Z(\mathfrak{g}))} V_\chi$$

of submodules upon which $Z(\mathfrak{g})$ acts by central character χ. If we define $\mathcal{O}_\chi$ to be the full subcategory of $\mathcal{O}$ on which $Z(\mathfrak{g})$ acts by χ, then this decomposition implies that

$$\mathcal{O} \simeq \bigoplus_{\chi \in \mathrm{MaxSpec}(Z(\mathfrak{g}))} \mathcal{O}_\chi. \tag{4}$$

The $\mathcal{O}_\chi$ are called the *blocks of category* $\mathcal{O}$.

Example 2.8. Recall by Corollary 2.4 that $Z(\mathfrak{g})$ acts by χ_λ on M_λ and hence also on L_λ, meaning that M_λ and L_λ lie in $\mathcal{O}_{\chi_\lambda}$.

By (4), L_ν can occur in the Jordan-Holder series of M_λ if and only if $\chi_\nu = \chi_\lambda$. Therefore, to study these Jordan-Holder series, it suffices for us to restrict our attention to a single block $\mathcal{O}_\chi$ at a time. By Corollary 2.6, $\chi_\nu = \chi_\lambda$ if and only λ and ν lie in the same orbit under the dotted action of W. In particular, notice that the simple objects of $\mathcal{O}_{-\rho}$ take the form $L_{-w\rho-\rho}$ for $w \in W$.

Remark. In this paper, in order to avoid dealing with twisted $\mathcal{D}$-modules under the Beilinson-Bernstein correspondence, we will restrict ourselves to the analysis of the block $\mathcal{O}_{-\rho}$, meaning that we will only consider the simple objects $L_{-w\rho-\rho}$ for $w \in W$. However, similar techniques will provide analogous results for the other blocks.

3. From $\mathfrak{g}$-modules to $\mathcal{D}$-modules to perverse sheaves

In this section we describe two successive equivalences of categories which allow us to pass from representations of $\mathfrak{g}$ to perverse sheaves. The first, the Beilinson-Bernstein localization theorem, associates to each $\mathfrak{g}$-module a corresponding $\mathcal{D}$-module on the flag variety. We will then show that the $\mathcal{D}$-modules we are interested in are regular holonomic by Lemma 3.2 and Proposition 3.11. This allows us to apply the second equivalence, the Riemann-Hilbert correspondence, which yields perverse sheaves on the flag variety. Our main goal is Theorem 3.13, which identifies the image of the simple modules and Verma modules under these two equivalences. The main sources for this section are [Ber83] and [HTT08].

3.1. **Beilinson-Bernstein localization.** Recall that the flag variety X comes naturally equipped with a left G-action $G \times X \to X$. Differentiating this action gives rise to a natural map

$$\mathfrak{g} \to \mathrm{Der}_k(\mathcal{O}_X, \mathcal{O}_X), \tag{5}$$

hence a map $U(\mathfrak{g}) \to \Gamma(X, \mathcal{D}_X)$. Given a $\mathcal{D}_X$-module $\mathcal{M}$ on X, the map (5) endows its global sections $\Gamma(X, \mathcal{M})$ with the structure of a $U(\mathfrak{g})$-module. Conversely, given a $U(\mathfrak{g})$-module M, (5) allows us to construct the $\mathcal{D}_X$-module

$$\mathrm{Loc}(M) := \mathcal{D}_X \underset{U(\mathfrak{g})}{\otimes} M,$$

where $U(\mathfrak{g})$ is viewed as a (locally) constant sheaf of algebras on X, M is viewed as the corresponding (locally) constant sheaf of modules over $U(\mathfrak{g})$, and the map $U(\mathfrak{g}) \to \mathcal{D}_X$ is induced by (5). We call $\mathrm{Loc}(M)$ the *localization* of M and Loc a *localization functor*. Some careful computation (which we omit here) shows that the map of (5) factors through $U(\mathfrak{g})_{\chi_{-\rho}}$, meaning that the operations of localization and taking global sections can be viewed as functors $U(\mathfrak{g})_{\chi_{-\rho}} - \mathrm{mod} \rightleftharpoons \mathrm{DMod}(X)$. In [BB81], A. Beilinson and J. Bernstein showed the following localization theorem, which implies that these functors give an equivalence.

Theorem 3.1 (Beilinson-Bernstein localization). *The pair*

$$\mathrm{Loc} : U(\mathfrak{g})_{\chi_{-\rho}} - mod \rightleftharpoons \mathrm{DMod}(X) : \Gamma$$

is an equivalence of categories.

3.2. **B-equivariant $\mathcal{D}_X$-modules and perverse sheaves.** To study composition series of Verma modules in $\mathcal{O}_{-\rho}$, it suffices to consider the corresponding $\mathcal{D}_X$-modules. For this, we must consider some extra structure coming from the condition of lying in $\mathcal{O}_{-\rho}$. On Lie algebra side, such a condition comes from the action of B. Indeed, we say that a representation V of $\mathfrak{g}$ is B-equivariant if the action of $\mathfrak{b}$ lifts to an action of the corresponding Borel subgroup B. Denote by

$$U(\mathfrak{g})_{\chi_\lambda, B} - \mathrm{mod}$$

the category of B-equivariant $U(\mathfrak{g})$-modules upon which $Z(\mathfrak{g})$ acts by χ_λ. As the following lemma shows, all $\mathfrak{g}$-modules we are concerned with are B-equivariant.

Lemma 3.2. *Any representation in category $\mathcal{O}$ is B-equivariant. In particular, $\mathcal{O}_{-\rho} \subset U(\mathfrak{g})_{\chi_{-\rho}, B}$.*

Proof. By Property (i) in Definition 2.7 and the fact that the action of $\mathfrak{n}$ increases weight, we see that the action of $\mathfrak{n}$ on any object of category $\mathcal{O}$ is nilpotent. Further, the weights of $\mathfrak{h}$ on such an object are integral because the highest weight of such an object is $-w\rho - \rho$ for some $w \in W$. Together, these imply that the action of $\mathfrak{b}$ lifts to a compatible action of B. $\square$

On the $\mathcal{D}$-module side, consider the left B-action $\sigma : B \times X \to X$ on the the flag variety X, and let $\pi_2 : B \times X \to X$ be the projection. For a $\mathcal{D}_X$-module $\mathcal{M}$, we say that $\mathcal{M}$ is *a B-equivariant $\mathcal{D}_X$-module* if it is B-equivariant as an $\mathcal{O}_X$-module and the corresponding isomorphism

$$\pi_2^* \mathcal{M} \to \sigma^* \mathcal{M}$$

is an isomorphism of $\mathcal{D}_{B \times X}$-modules. Denote by $\mathrm{DMod}(X)^B$ the category of B-equivariant $\mathcal{D}_X$-modules. The correspondence of Theorem 3.1 restricts to the following localization of B-equivariant representations.

Theorem 3.3 ([HTT08, Theorem 11.5.3]). *The restriction of the equivalence of Theorem 3.1 to the pair*

$$\mathrm{Loc} : U(\mathfrak{g})_{\chi_{-\rho}, B} - mod \rightleftharpoons \mathrm{DMod}(X)^B : \Gamma$$

is an equivalence of categories.

It therefore suffices for us to restrict our attention to B-equivariant $\mathcal{D}_X$-modules. It will turn out that such $\mathcal{D}_X$-modules are automatically regular holonomic, which will allow us to understand them in terms of perverse sheaves. Such perverse sheaves will also be B-equivariant in the following sense.

Consider again the left B-action $\sigma : B \times X \to X$ on the flag variety. Then we say a perverse sheaf $\mathcal{M} \in \mathrm{Perv}(X)$ is *B-equivariant* if it is equipped with an isomorphism

$$\pi_2^* \mathcal{M} \to \sigma^* \mathcal{M}$$

in $D^b_c(B \times X)$ which satisfies the same cocycle conditions as hold for a B-equivariant $\mathcal{O}_X$-module. Denote the category of B-equivariant perverse sheaves on X by

$$\mathrm{Perv}(X)^B.$$

In the next subsection, we discuss the transformation between $\mathcal{D}_X$-modules and perverse sheaves in general.

3.3. **Regular holonomic $\mathcal{D}$-modules and the Riemann-Hilbert correspondence.** In this subsection, we give an overview of the theory of regular holonomic $\mathcal{D}$-modules and their link to perverse sheaves via the Riemann-Hilbert correspondence. As this is rather deep and technical subject, we will restrict ourselves mainly to statements of the main results.

First, we recall a fact about holonomic $\mathcal{D}$-modules. Let $i : Z \hookrightarrow X$ be a locally closed affine embedding of a smooth subvariety. Then, for any $\mathcal{O}_Z$-coherent $\mathcal{D}_Z$-module $\mathcal{F}$, we define its *minimal extension* to be

$$i_{!\star}\mathcal{F} := \mathrm{Im}(i_!\mathcal{F} \to i_\star\mathcal{F})$$

as a $\mathcal{D}_X$-module. Notice that $i_{!\star}\mathcal{F}$ is a holonomic $\mathcal{D}_X$-module (and not a complex) because $\mathcal{D}_X$, $\mathcal{D}_Z$, and $i_\star$ all send holonomic $\mathcal{D}$-modules to holonomic $\mathcal{D}$-modules and the property of being holonomic is preserved under taking submodules. When $\mathcal{F}$ is furthermore irreducible, this construction yields all irreducible holonomic $\mathcal{D}_X$-modules by the following theorem.

Theorem 3.4 ([HTT08, Theorem 3.4.2])**.** *Suppose that $\mathcal{F}$ is an irreducible $\mathcal{O}_Z$-coherent $\mathcal{D}_Z$-module. Then its minimal extension $i_{!\star}\mathcal{F}$ is an irreducible holonomic $\mathcal{D}_X$-module. Moreover, all irreducible holonomic $\mathcal{D}_X$-modules take this form.*

We are now ready to define the class of regular holonomic $\mathcal{D}$-modules. We will do this in four stages.

Definition 3.5. Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then, we say that $\mathcal{M}$ is *regular holonomic* if one of the following conditions holds for the corresponding case.

- **If X is a curve C equipped with a smooth completion $i : C \to C'$ and $\mathcal{M}$ is a $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module:** For any point $c \in C' - C$, consider a local coordinate t at c and the subalgebra $\mathcal{D}'_{C'} := \mathcal{O}_{C'}[t\partial_t] \subset \mathcal{D}_{C'}$. Then, we say that $\mathcal{M}$ has a *regular singularity* at c if $i_*\mathcal{M}$ is the union of its $\mathcal{D}'_{C'}$-submodules which are coherent as $\mathcal{O}_{C'}$-modules.[3] We say that $\mathcal{M}$ is *regular holonomic* if it has a regular singularity at each point of $C' - C$.
- **If X is a curve C and $\mathcal{M}$ is a $\mathcal{O}_X$-coherent $\mathcal{D}_X$-module:** We say that $\mathcal{M}$ is *regular holonomic* if the restriction of $\mathcal{M}$ to any open $U \hookrightarrow C$ is regular holonomic.

  In the preceding two cases, we say also that $\mathcal{M}$ has *regular singularities.*
- **If $\mathcal{M}$ is $\mathcal{O}_X$-coherent:** We say that $\mathcal{M}$ is *regular holonomic* if the restriction of $\mathcal{M}$ to any curve $C \hookrightarrow X$ is regular holonomic.
- **The general case:** We say that $\mathcal{M}$ is *regular holonomic* if any composition factor of $\mathcal{M}$ takes the form $i_{!\star}\mathcal{F}$ for some locally closed affine embedding $i : Y \to X$ and some $\mathcal{O}_Y$-coherent regular holonomic $\mathcal{D}_Y$-module $\mathcal{F}$.

Definition 3.5 will play an essentially formal role for us, as we restrict ourselves to applying the Riemann-Hilbert correspondence (Theorems 3.8 and 3.9) to regular holonomic $\mathcal{D}$-modules and do not mention its proof. Denote the derived category of $\mathcal{D}_X$-modules with regular holonomic cohomology by $D^b_{rh}(\mathcal{D}_X)$. Intuitively speaking, these correspond to systems of differential equations.

Remark. For the case $X = \mathbb{A}^1 - \{0\}$, the original motivation for this definition stems from the study of differential equations on the punctured complex plane $\mathbb{C} - \{0\}$ of the form

$$\frac{\partial}{\partial z} f(z) = C(z) f(z),$$

where $C(z)$ is a meromorphic function with poles only at 0. Such a system corresponds to the $\mathcal{D}_X$-module

$$\mathcal{D}_X/(\partial_z - C(z))\mathcal{D}_X,$$

and it will turn out that this module is regular if and only if $C(z)$ has a pole of order at most 1 at $z = 0$. We refer the reader to [BGK$^+$87, Section 3] for more details.

One might therefore expect some corresponding concept of solution. This is provided by the de Rham functor DR_X. Before defining DR_X, we must first briefly discuss some issues relating to analytification.

We have thus far been working implicitly in the algebraic category of varieties over $\mathbb{C}$. For a variety X over $\mathbb{C}$, denote by

$$-^{\mathrm{an}} : X \mapsto X^{\mathrm{an}}$$

[3]Because $\mathcal{M}$ is holonomic, $i_*\mathcal{M}$ is actually a $\mathcal{D}_{C'}$-module here rather than a complex of $\mathcal{D}_{C'}$-modules.

the analytification functor. By abuse of notation, we will also use $\mathcal{M}^{\mathrm{an}}$ to denote the analytification of a $\mathcal{O}_X$ or $\mathcal{D}_X$-module. Denote by $D^b_c(X^{\mathrm{an}})$ the derived category of constructible sheaves on X^{an} with the analytic topology. We are now ready to define the de Rham functor.

Definition 3.6. The *de Rham functor* $DR_X : D^b_{rh}(\mathcal{D}_X) \to D^b_c(X)$ is given by

$$DR_X(\mathcal{M}) = \Omega_{X^{\mathrm{an}}} \underset{\mathcal{D}_{X^{\mathrm{an}}}}{\otimes} \mathcal{M}^{\mathrm{an}},$$

where the canonical sheaf $\Omega_{X^{\mathrm{an}}}$ is a right $\mathcal{D}_{X^{\mathrm{an}}}$-module.

Remark. While DR_X may be defined in the same way on $D^b_{coh}(\mathcal{D}_X)$, we only consider it on $D^b_{rh}(\mathcal{D}_X)$ because it possesses good properties there. By this, we mean that DR_X commutes with the operations $\mathbb{D}_X$, f^*, f_*, $f^!$, and $f_!$ of Verdier duality (see [Ber83, Theorem 5.C] for a discussion).

Remark. The de Rham functor of Definition 3.6 is closely related to the perhaps more intuitive *solution functor*, which is given by

$$\mathrm{Sol}_X(\mathcal{M}) = \mathcal{RH}om_{\mathcal{D}_{X^{\mathrm{an}}}}(\mathcal{M}^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}})$$

and models the idea of solving a system of differential equations. The two satisfy the relation

$$DR_X(\mathcal{M}) = \mathrm{Sol}_X(\mathbb{D}_X\mathcal{M})[\dim X],$$

which allows us to interpret Theorems 3.7 and 3.8 in terms of Sol_X as well.

Though we have been somewhat careful about distinguishing between X and X^{an} until this point, for convenience we will omit the use of $-^{\mathrm{an}}$ in the future and use X to denote both what we have thus far called X and X^{an}. Which one is meant should be clear from context.

We are now ready to state the Riemann-Hilbert correspondence, a powerful technical result which relates the algebraically defined category of regular holonomic $\mathcal{D}$-modules to the more analytic category of perverse sheaves. We will state three versions, one at the derived level and two on the heart.

Theorem 3.7 ([HTT08, Theorem 7.2.1])**.** *The de Rham functor provides an equivalence of categories*

$$DR_X : D^b_{rh}(\mathcal{D}_X) \to D^b_c(X).$$

Theorem 3.8 ([HTT08, Theorem 7.2.5])**.** *The equivalence of Theorem 3.7 restricts to an equivalence of categories*

$$DR_X : \mathrm{DMod}_{rh}(X) \to \mathrm{Perv}(X),$$

where $\mathrm{Perv}(X)$ *is the category of perverse sheaves on* X*.*

In analogy to Theorem 3.3, we may restrict this equivalence to $\mathrm{DMod}_{rh}(X)^B$ and $\mathrm{Perv}(X)^B$.

Theorem 3.9 ([HTT08, Theorem 11.6.1])**.** *The equivalence of Theorem 3.8 restricts to an equivalence of categories*

$$DR_X : \mathrm{DMod}_{rh}(X)^B \to \mathrm{Perv}(X)^B.$$

We do not comment on the proofs of Theorems 3.8 and 3.9. However, we give here an example to provide some intuition for what the de Rham functor does by considering the closely related solution functor, which is more intuitive to compute.

Example 3.10. Take $X = \mathbb{A}^1 = \mathrm{Spec}(k[t])$, so that $\mathcal{D}_X = k[z, \partial_z]$. Consider $\mathcal{D}_X$-modules $\mathcal{M}$ of the form $\mathcal{D}_X/D \cdot \mathcal{D}_X$ for some $D \in \mathcal{D}_X$. Then

$$H^0(\mathrm{Sol}_X(\mathcal{M})) = R^0\,\mathrm{Hom}(\mathcal{D}_X/D \cdot \mathcal{D}_X, \mathcal{O}_X) = \Big(U \mapsto \{f \in \mathcal{O}_U \mid Df = 0\}\Big),$$

meaning that $H^0(\mathrm{Sol}_X(\mathcal{M}))$ is exactly the sheaf of local solutions to the differential equation $Df = 0$. The fact that the cohomologies of $\mathrm{Sol}_X(\mathcal{M})$ are local systems then corresponds to the local uniqueness of solutions to differential equations. For instance, if $D = \partial_z - 1$, then $\Gamma(H^0(\mathrm{Sol}_X(\mathcal{M}))) = \mathbb{C} \cdot e^z$, and if $D = z\partial_z - n$, then $\Gamma(H^0(\mathrm{Sol}_X(\mathcal{M}))) = \mathbb{C} \cdot z^n$.

3.4. **The image of category $\mathcal{O}$.** We now specialize the discussion of Subsection 3.3 to those $\mathcal{D}_X$-modules coming from $\mathfrak{g}$-modules in $\mathcal{O}_{-\rho}$. The key tool which allows us to apply these constructions is the following proposition.

Proposition 3.11 ([HTT08, Theorem 11.6.1(i)])**.** *If $\mathcal{M}$ is a coherent B-equivariant $\mathcal{D}_X$-module with respect to the left B-action on the flag variety, then $\mathcal{M}$ is regular holonomic.*

Sketch of proof. We restrict ourselves to a discussion of the main ideas. The key observation is that there are only finitely many left B-orbits on X. The proof is then by induction on the number of B-orbits. In the base case of a single orbit, we see that $\pi : B \to X$ is a smooth surjective map. The proof uses the B-equivariance to show that $\pi^*\mathcal{M}$ is regular holonomic, which implies by a general criterion that $\mathcal{M}$ is regular holonomic. We refer the reader to [HTT08] for the details.

For the inductive step, let $i : Y \to X$ be the inclusion of a closed B-orbit and j the inclusion of the complement. This decomposition gives rise to the distinguished triangle

$$i_*i^!\mathcal{M} \to \mathcal{M} \to j_*j^!\mathcal{M} \overset{+1}{\to},$$

where $i^!\mathcal{M}$ and $i^!\mathcal{M}$ are coherent $\mathcal{D}$-modules on $X - Y$ and Y, each of which consists of a smaller number of B-orbits. By the inductive hypothesis, they have regular holonomic cohomology, hence $i_*i^!\mathcal{M}$ and $j_*j^!\mathcal{M}$ do as well. Because the property of being regular holonomic is by definition closed under subquotients, the long exact sequence in cohomology shows that $\mathcal{M}$ is regular holonomic. □

For any $\mathfrak{g}$-module V in $\mathcal{O}_{-\rho}$, conditions (i) and (ii) in Definition 2.7 imply precisely that V is B-equivariant. Therefore, by Theorem 3.3, the corresponding $\mathcal{D}_X$-module $\mathrm{Loc}(V)$ is also B-equivariant. Further, condition (iii) of Definition 2.7 implies that it is also coherent, hence regular holonomic by Proposition 3.11. Finally, applying the de Rham functor DR_X, the Riemann-Hilbert correspondence of Theorem 3.8 implies that

$$\mathcal{V} := DR_X(\mathrm{Loc}(V))$$

is a perverse sheaf on X. In particular, for $w \in W$, define the perverse sheaves

$$\mathcal{M}_w := DR_X(\mathrm{Loc}(M_{-w\rho-\rho}))$$

and

$$\mathcal{L}_w := DR_X(\mathrm{Loc}(L_{-w\rho-\rho})).$$

Our goal for the remainder of this section will be to derive the second of the following two theorems from the first. This will identify $\mathcal{M}_{-w\rho-\rho}$ and $\mathcal{L}_{-w\rho-\rho}$ as more familiar perverse sheaves on X. These two results reduce the computation of composition series of the Verma modules $M_{-w\rho-\rho}$ to the computation of cohomologies of the perverse sheaves $IC(X_w)$.

Theorem 3.12 ([HTT08, Proposition 12.3.2])**.** *Denote by $i_w : X_w \to X$ and $i_w^o : X_w^o \to X$ the inclusions of the Schubert variety X_w and its open cell X_w^o. In $\mathrm{DMod}_{rh}(X)^B$, we have (i) $\mathrm{Loc}(L_{-w\rho-\rho}) = (i_w)_{!\star}(X_w, \mathcal{O}_{X_w})$ and (ii) $\mathrm{Loc}(M_{-w\rho-\rho}) = \mathbb{D}_X(i_w^o)_\star(\mathcal{O}_{X_w^o})$.*

Theorem 3.13. *In $\mathrm{Perv}(X)^B$, we have (i) $\mathcal{L}_w = IC(X_w)$ and (ii) $\mathcal{M}_w = \underline{\mathbb{C}}_{X_w^o}[\dim X_w]$.*

Proof. For (i), the $\mathcal{L}_w$ must be the simple objects of $\mathrm{Perv}(X)^B$ by Theorem 3.9 and Theorem 3.12(i). These all take the form $IC(X_v)$ because X_v are the B-equivariant strata of X, so it remains only to check that $\mathcal{L}_w$ restricts to the constant sheaf on X_w^o. Indeed, this follows because the restriction of $\mathrm{Loc}(L_{-w\rho-\rho}) = (i_w)_{!*}(X_w, \mathcal{O}_{X_w})$ to X_w^o is $\mathcal{O}_{X_w}^o$, which is mapped to $\underline{\mathbb{C}}_{X_w^o}$ under $DR_{X_w^o}$.

For (ii), we compute using Theorem 3.12 and the fact that DR_X commutes with $\mathbb{D}_X$ and $(i_w^o)_*$ to see

$$\mathcal{M}_w = DR_X(\mathbb{D}_X(i_w^o)_*(\mathcal{O}_{X_w^o})) = \mathbb{D}_X((i_w^o)_*(DR_X(\mathcal{O}_{X_w^o})).$$

Now, we have $DR_X(\mathcal{O}_{X_w^o}) = \underline{\mathbb{C}}_{X_w^o}[\dim X_w]$, which allows us to conclude

$$\mathcal{M}_w = \mathbb{D}_X((i_w^o)_*(\underline{\mathbb{C}}_{X_w^o}[\dim X_w])) = \underline{\mathbb{C}}_{X_w^o}[2\dim X_w - \dim X_w] = \underline{\mathbb{C}}_{X_w^o}[\dim X_w].$$ □

4. The Kazhdan-Lusztig conjectures

In this section, we compute the intersection cohomologies of Schubert varieties suggested by Theorem 3.13. This will allow us to finally obtain a formula for the multiplicities of $L_{-w\rho-\rho}$ in the Verma modules $M_{-w\rho-\rho}$. Before the computation proper, we first introduce the Hecke algebra of a Weyl group and its Kazhdan-Lusztig polynomials, which are necessary to state the answer, and the Bott-Samelson resolution of the Schubert varieties, which is necessary to perform the computation. We then prove the main technical result, Theorem 4.14, using an approach due originally to R. MacPherson. Finally, we gather everything together to give a proof of Theorem 4.4. This section draws from a large number of sources. We mainly follow the exposition of [Ric10a], [Ric10b], and [Spr81] for the specific approach we take. We follow [Soe97] for the proof of Proposition 4.1. Finally, we are generally indebted to [dCM09] and [HTT08].

4.1. **Hecke algebras and Kazhdan-Lusztig polynomials.** Let W be a Weyl group (or more generally a Coxeter group) with a generating set of reflections $s_1, \ldots, s_n$. The *Hecke algebra of* W is the $\mathbb{Z}[q, q^{-1}]$-algebra $\mathcal{H}(W)$ with $\mathbb{Z}[q, q^{-1}]$-basis T_w for $w \in W$ and multiplication given by $T_e = 1$ and the relations

$$T_s T_w = \begin{cases} T_{sw} & \ell(sw) > \ell(w), \\ (q^2 - 1)T_w + q^2 T_{sw} & \ell(sw) < \ell(w) \end{cases} \tag{6}$$

for s a reflection and $w \in W$. Since any $w \in W$ is the product of simple reflections, (6) uniquely defines an algebra structure on $\mathcal{H}(W)$.

Remark. The specializations $q = 1$ and $q = -1$ of $\mathcal{H}(W)$ simply give the group algebra $\mathbb{Z}[W]$ of W.

Applying the second relation of (6) for $w = s$ shows that $T_s^2 = (q^2 - 1)T_s + q^2$, so T_s is invertible with

$$T_s^{-1} = T_s q^{-2} - 1 + q^{-2}. \tag{7}$$

Combining this with the first relation shows that each T_w is invertible. We may therefore define a canonical involution of $\mathbb{Z}$-algebras $i : \mathcal{H}(W) \to \mathcal{H}(W)$ by

$$i(q) = q^{-1} \qquad \text{and} \qquad i(T_w) = T_{w^{-1}}^{-1}.$$

It is easy to check that i respects the relations of $\mathcal{H}(W)$. We would like now to consider some special elements $C_w \in \mathcal{H}(W)$ which are invariant under i and whose coefficients when expressed in the T_w-basis form an upper triangular matrix under the Bruhat-Chevalley order. More precisely, we have the following.

Proposition 4.1. *There exist unique elements $C_w \in \mathcal{H}(W)$ and polynomials $P_{v,w}(q) \in \mathbb{Z}[q]$ such that*

(i) $i(C_w) = C_w$;
(ii) $C_w = q^{-\ell(w)} \sum_{v \leq w} P_{v,w}(q) T_v$;
(iii) $P_{w,w} = 1$, *and* $P_{v,w}(q)$ *is a polynomial of degree at most* $\ell(w) - \ell(v) - 1$ *for* $v < w$.

Moreover, the odd degree coefficients of $P_{v,w}(q)$ vanish.

Proof. We first prove the existence by induction on the Bruhat-Chevalley order. For the base case $e \in W$, we may take $C_e = 1$. Further, for a simple reflection $s \in W$, take $C_s = q^{-1}(T_s + 1)$, so that by (7) we have

$$i(C_s) = q^{-1}(T_s q^{-2} - 1 + q^{-2} + 1) = C_s.$$

This establishes existence for $\ell(w) \leq 1$. Suppose now that there exist C_v and polynomials $P_{v',v}(q)$ satisfying Properties (i), (ii), and (iii) for $v', v < w$. Choose a reflection s such that $\ell(sw) < \ell(w)$. By our inductive hypothesis, we have

$$C_{sw} = q^{-\ell(sw)} \sum_{v \leq sw} P_{v,sw}(q) T_v.$$

Multiplying on the left by $C_s = q^{-1}(T_s + 1)$ and noting that

$$(T_s + 1)T_v = (T_{sv} + T_v) \cdot \begin{cases} 1 & \ell(sv) > \ell(v) \\ q^2 & \ell(sv) < \ell(v), \end{cases}$$

we obtain

$$C_s \cdot C_{sw} = q^{-\ell(w)} \sum_{v \leq sw} P_{v,sw}(q)(T_s + 1)T_v = q^{-\ell(w)} \left[T_w + \sum_{v < w} R_{v,w}(q) T_v \right]$$

for some polynomials $R_{v,w}(q)$ with

$$\begin{aligned}\deg R_{v,w}(q) &\leq \max\{\deg P_{v,sw}(q), \deg P_{sv,sw}(q) + 2\} \\ &\leq \max\{\ell(w) - \ell(v) - 2, \ell(w) - 1 - \ell(v) - 1 + 2\} = \ell(w) - \ell(v),\end{aligned}$$

where we've applied the induction hypothesis. Let $r_{v,w}$ be the coefficient of $q^{\ell(w)-\ell(v)}$ in $R_{v,w}$. Now, define

$$C_w = C_s \cdot C_{sw} - \sum_{v<w} r_{v,w} C_v. \tag{8}$$

Note that $i(C_w) = C_w$ by the inductive hypothesis and the fact that $R_{v,w}(0) \in \mathbb{Z}$. Then, we see that

$$\begin{aligned}C_w &= q^{-\ell(w)} \left[T_w + \sum_{v<w} R_{v,w}(q) T_v - \sum_{v<w} r_{v,w} C_v\right] \\ &= q^{-\ell(w)} \left[T_w + \sum_{v<w} R_{v,w}(q) T_v - \sum_{v<w} \sum_{u \leq v} q^{\ell(w)-\ell(v)} r_{v,w} P_{u,v}(q) T_u\right] \\ &= q^{-\ell(w)} \left[T_w + \sum_{v<w} \Big(R_{v,w}(q) - \sum_{v \leq u < w} q^{\ell(w)-\ell(u)} r_{u,w} P_{v,u}(q)\Big) T_v\right]\end{aligned}$$

Notice that

$$\deg \Big(R_{v,w}(q) - \sum_{v \leq u < w} q^{\ell(w)-\ell(u)} r_{u,w} P_{v,u}(q)\Big) \leq \ell(w) - \ell(v) - 1$$

because this expression is the difference of two polynomials of degree at most $\ell(w) - \ell(v)$ whose $q^{\ell(w)-\ell(v)}$ coefficients are the same. Further, all terms T_v which appear in C_w satisfy $v \leq w$ and the coefficient of T_w is 1. Therefore, we conclude that C_w takes the form

$$C_w = q^{-\ell(w)} \sum_{v \leq w} P_{v,w}(q) T_v$$

with $P_{w,w}(q) = 1$ and $\deg P_{v,w} \leq \ell(w) - \ell(v) - 1$ for $v < w$. This implies that C_w and $P_{v,w}(q)$ satisfy properties (i), (ii), and (iii), completing the induction.

Let us now check the uniqueness. Suppose that we had two sets of elements C_w and C'_w satisfying the conditions for two sets of polynomials $P_{v,w}(q)$ and $P'_{v,w}(q)$. Set $D_w = C_w - C'_w$ and $Q_{v,w}(q) = P_{v,w}(q) - P'_{v,w}(q)$, so that $i(D_w) = D_w$, $\deg Q_{v,w}(q) \leq \ell(w) - \ell(v) - 1$, and

$$D_w = q^{-\ell(w)} \sum_{v<w} Q_{v,w}(q) T_v.$$

Choose v maximal so that $Q_{v,w}(q)$ is non-zero. Observe now that the form of (7) implies that $T_{v^{-1}}^{-1}$, as an element of the Hecke algebra, takes the form

$$T_{v^{-1}}^{-1} = q^{-2\ell(v)} T_v + \sum_{u<v} G_{u,v}(q) T_u$$

for some Laurent polynomials $G_{u,v}(q)$. Further, for $u < w$, the only element $i(T_u)$ with non-zero coefficient for T_v is $i(T_v)$. Therefore, the coefficient of T_v in $i(D_w) = D_w$ is given by

$$q^{\ell(w)} Q_{v,w}(q^{-1}) q^{-2\ell(v)} = q^{-\ell(w)} Q_{v,w}(q),$$

implying that

$$q^{2\ell(w)-2\ell(v)} Q_{v,w}(q^{-1}) = Q_{v,w}(q),$$

which is impossible because the left hand side has degree at least $\ell(w) - \ell(v) + 1$ and the right had side has degree at most $\ell(w) - \ell(v) - 1$. Thus, we see that $D_w = 0$ for all w, so the elements C_w are unique.

It remains only to verify that $P_{v,w}(q)$ has non-zero coefficients only in even degrees. This follows by noting that in construction (8), $r_{v,w} = 0$ if $\ell(w)$ and $\ell(v)$ do not have the same parity. $\square$

The polynomials $P_{v,w}$ are known as the *Kazhdan-Lusztig polynomials*. Their values will be the multiplicities of the simple $\mathfrak{g}$-modules in the composition series of a Verma module. Let us see some examples of $P_{v,w}(q)$.

Example 4.2. We first consider some small general cases.

- If $w = 1$, then $P_{w,w}(q) = 1$.
- If $w = s$, then $C_s = q^{-1}(T_s + 1)$, so $P_{1,s}(q) = P_{s,s}(q) = 1$.
- If $w = s_1 s_2$, then
$$C_{s_1} \cdot C_{s_2} = q^{-2}(T_{s_1 s_2} + T_{s_1} + T_{s_2} + 1),$$
so $R_{v,s_2}(q) = 0$ for all v, hence $C_w = q^{-2}(T_{s_1 s_2} + T_{s_1} + T_{s_2} + 1)$ and $P_{v,w}(q) = 1$ for $v \leq w$.

Example 4.3. Having seen a number of instances where $P_{v,w}(q)$ is trivial, let us consider type A_3, one of the simplest cases where these polynomials can be nontrivial. In this case, the Weyl group S_4 is generated by s_1, s_2, s_3 with the relations $s_1 s_2 s_1 = s_2 s_1 s_2$, $s_2 s_3 s_2 = s_3 s_2 s_3$, and $s_1 s_3 = s_3 s_1$. In particular, the element $s_2 s_1 s_3 s_2$ is a reduced decomposition. By our previous computation, we have
$$C_{s_2 s_1} = q^{-2}(T_{s_2 s_1} + T_{s_1} + T_{s_2} + 1)$$
and
$$C_{s_3 s_2} = q^{-2}(T_{s_3 s_2} + T_{s_3} + T_{s_2} + 1).$$
Multiplying and considering the degrees of the coefficients of the T_w, we find that $C_{s_2 s_1 s_3 s_2} = C_{s_2} C_{s_1} C_{s_3} C_{s_2}$ and that the terms
$$(q^2 + 1)T_{s_2} \text{ and } (q^2 + 1)$$
have coefficient not equal to 1. This means that
$$P_{s_2, s_2 s_1 s_3 s_2}(q) = P_{1, s_2 s_1 s_3 s_2}(q) = q^2 + 1.$$

Remark. While the proof of Proposition 4.1 gives in principle an algorithm for explicitly computing $P_{v,w}(q)$ for specific choices of W, v, and w, there is no explicit general formula. As can be seen from the last example, computation by this method is quite inefficient.

4.2. **Statement of the conjecture.** We are now ready to state the main result of this paper, which gives an explicit description of the simple modules occurring in the composition series of the Verma modules. This result was first conjectured by D. Kazhdan and G. Lusztig in [KL79] and is known as the *Kazhdan-Lusztig conjecture.*

Theorem 4.4 (Kazhdan-Lusztig conjecture)**.** *In the Grothendieck group of the category $\mathcal{O}_{-\rho}$ of $\mathfrak{g}$-modules, we have*
$$[L_{-w\rho-\rho}] = \sum_{v \leq w} (-1)^{\ell(v)-\ell(w)} P_{v,w}(1)[M_{-v\rho-\rho}].$$

The proof of Theorem 4.4 uses the correspondences presented in the previous two sections to reduce the problem to a computation of intersection cohomology on Schubert varieties. This final computation was first done by A. Beilinson and J. Bernstein using characteristic p methods. We will present a purely geometric proof using the Bott-Samelson resolution of Schubert varieties due to R. MacPherson and presented by T. A. Springer in [Spr81]. We defer the proof to first see some consequences. First, the following corollary, which may be obtained from Theorem 4.4 by formal computations using Kazhdan-Lusztig polynomials, shows that Theorem 4.4 indeed provides the promised multiplicities of the simple objects of $\mathcal{O}_{-\rho}$ in the Verma modules.

Corollary 4.5 ([KL79, Theorem 3.1])**.** *In the Grothendieck group of the category $\mathcal{O}_{-\rho}$ of $\mathfrak{g}$-representations, we have*
$$[M_{-w\rho-\rho}] = \sum_{v \leq w} P_{w_0 w, w_0 v}(1)[L_{-v\rho-\rho}].$$

Example 4.6. We consider some explicit examples for small types.

- **Type A_1:** We have $\mathfrak{g} = \mathfrak{sl}_2$, so the root lattice of $\mathfrak{g}$ is $2\mathbb{Z}$, $\rho = 1$, and $W = S_2$ acts by reflection about the origin. Further, $\mathcal{O}_{-\rho}$ contains two simple objects, L_{-2} and L_0, and two Verma modules, M_{-2} and M_0. By Example 4.2, we see that all non-zero Kazhdan-Lusztig polynomials are equal to 1 in this case, so Theorem 4.4 implies in this case that
$$[L_{-2}] = [M_{-2}] \text{ and } [L_0] = [M_0] - [M_{-2}].$$
We see that M_{-2} is irreducible and that $[M_0] = [L_0] + [L_2]$, with the latter equality coming from the exact sequence
$$0 \to M_2 \simeq L_2 \to M_2 \to L_2 \to 0.$$

- **Type A_2:** We have $\mathfrak{g} = \mathfrak{sl}_3$. The root lattice of $\mathfrak{g}$ is generated by two simple roots α and β so that $\{\alpha, \beta, \alpha+\beta, -\alpha, -\beta, -\alpha-\beta\}$ form the root system. Then $\rho = \alpha+\beta$, and $W = S_3$ acts via the two reflections exchanging $\alpha+\beta$ and one of α or β. Further, $\mathcal{O}_{-\rho}$ contains 6 simple objects and 6 Verma modules with highest weights at $0, -\alpha, -\beta, -2\alpha-\beta, -2\beta-\alpha, -2\beta-2\alpha$. They correspond to the set $-W\rho-\rho$ and fit into the Bruhat-Chevalley order as follows.

$$\begin{tikzcd}
& 0 \arrow[dl] \arrow[dr] & \\
-\alpha \arrow[d] \arrow[drr] & & -\beta \arrow[d] \arrow[dll] \\
-2\beta-\alpha \arrow[dr] & & -2\alpha-\beta \arrow[dl] \\
& -2\beta-2\alpha &
\end{tikzcd}$$

Thus, noting that we may extend the calculations of Example 4.2 slightly to check that all non-zero Kazhdan-Lusztig polynomials are equal to 1 in this case, we may summarize the conclusion of Theorem 4.4 by the following matrix

$$\begin{bmatrix} [L_0] \\ [L_{-\alpha}] \\ [L_{-\beta}] \\ [L_{-2\beta-\alpha}] \\ [L_{-2\alpha-\beta}] \\ [L_{-2\beta-2\alpha}] \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & -1 \\ 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} [M_0] \\ [M_{-\alpha}] \\ [M_{-\beta}] \\ [M_{-2\beta-\alpha}] \\ [M_{-2\alpha-\beta}] \\ [M_{-2\beta-2\alpha}] \end{bmatrix}.$$

Inverting this matrix gives the multiplicities of the simple modules in the Verma modules

$$\begin{bmatrix} [M_0] \\ [M_{-\alpha}] \\ [M_{-\beta}] \\ [M_{-2\beta-\alpha}] \\ [M_{-2\alpha-\beta}] \\ [M_{-2\beta-2\alpha}] \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} [L_0] \\ [L_{-\alpha}] \\ [L_{-\beta}] \\ [L_{-2\beta-\alpha}] \\ [L_{-2\alpha-\beta}] \\ [L_{-2\beta-2\alpha}] \end{bmatrix}.$$

- **Type A_3:** We have $\mathfrak{g} = \mathfrak{sl}_4$ and $W = S_4$. In this case, $\mathcal{O}_{-\rho}$ will have $|W| = 24$ simple objects, so the full multiplicity matrix will be a bit unwieldy to calculate. However, we note that for types A_1 and A_2, each multiplicity we have encountered has been either 0 or 1. For type A_3, this will no longer be true. Indeed, notice that the longest element $w_0 \in S_4$ is given by $w_0 = s_1 s_2 s_3 s_1 s_2 s_1$, hence our computations of

$$P_{s_2, s_2 s_1 s_3 s_2}(q) = P_{1, s_2 s_1 s_3 s_2}(q) = q^2 + 1$$

in Example 4.3 show that

$$[M_{-s_1 s_2 s_3 s_2 s_1 \rho - \rho} : L_{-s_3 s_1 \rho - \rho}] = [M_{-s_1 s_2 s_3 s_1 s_2 s_1 \rho - \rho} : L_{-s_3 s_1 \rho - \rho}] = 2.$$

We note that the computations for this example were done in Sage.

4.3. **A modification of the Schubert varieties.** Instead of calculating directly on the Schubert varieties, we instead consider a modification which will later allow us to define a convolution product on it. Define $\mathcal{X} = X \times X$, and consider the diagonal G-action on $\mathcal{X}$. By the Bruhat decomposition, each G-orbit on $\mathcal{X}$ has a unique element of the form $(B/B, wB/B)$ for $w \in W$, hence we obtain a stratification

$$\mathcal{X} = \bigsqcup_{w \in W} G \cdot (B/B, wB/B). \tag{9}$$

Take $\mathcal{X}_w^o = G \cdot (B/B, wB/B)$ to be the cells of this decomposition, and let $\mathcal{X}_w = \overline{G \cdot (B/B, wB/B)}$ be the closure of $\mathcal{X}_w^o$. This is called the *G-Schubert variety.* We may characterize $\mathcal{X}_w^o$ and $\mathcal{X}_w$ in the following manner, which establishes a strong analogy to the case of Schubert varieties and shows that the $\mathcal{X}_w^o$ provide a stratification of $X \times X$, each of whose strata is a fiber bundle over X.

Proposition 4.7. *The following properties hold:*

(i) We have the decomposition

$$\mathcal{X}_w = \bigsqcup_{v \leq w} G \cdot (B/B, vB/B);$$

(ii) the first projection $\pi : \mathcal{X}_w^o \to X$ realizes $G \cdot (B/B, wB/B)$ as a fiber bundle over X with fiber X_w^o;
(iii) the first projection $\pi : \mathcal{X}_w \to X$ realizes $\mathcal{X}_w$ as a fiber bundle over X with fiber X_w.

Proof. Each of these properties follows easily from the decomposition of Proposition 2.2(iv). □

Example 4.8. We see that $\mathcal{X}_1 = G \cdot (B/B, B/B) \simeq \Delta(X)$ is simply the diagonal embedding of X in $\mathcal{X}$. Further, for a simple reflection $s \in W$, $\mathcal{X}_s$ is a $X_s \simeq \mathbb{P}^1$-bundle over X.

Proposition 4.7 allows us to relate intersection cohomologies on $\mathcal{X}_w$ and X_w.

Proposition 4.9. *Let $v \leq w$. Then we have the following:*

(i) for $v \leq w$, $H^i(IC(\mathcal{X}_w))$ is constant along $\mathcal{X}_v^o$, and
(ii) $H^i(IC(\mathcal{X}_w))_{(B/B,vB/B)} = H^{i+\dim X}(IC(X_w))_{vB/B}$.

Proof. For (i), notice that $IC(\mathcal{X}_w)$ is a G-equivariant element of $\mathrm{Perv}(\mathcal{X})$, hence $H^i(IC(\mathcal{X}_w))$ is a G-equivariant sheaf on $\mathcal{X}$, meaning that its stalks are constant on the G-orbits $\mathcal{X}_v^o$.

For (ii), choose an affine open neighborhood U of eB/B which is isomorphic to $\mathbb{A}^{\dim X}$, and let $U' = \pi^{-1}(U)$ so that $U' \simeq X_w \times U$. Denote by $j : U \to X$ and $j' : U' \to \mathcal{X}_w$ the inclusions and by $p : U' \to X_w$ a choice of projection compatible with stratification. Notice that $(B/B, vB/B) \in U'$, so we have

$$H^i(IC(\mathcal{X}_w))_{(B/B,vB/B)} = H^i((j')^* IC(\mathcal{X}_w))_{(B/B,vB/B)} = H^i(IC(U'))_{(B/B,vB/B)}.$$

By the Kunneth theorem for intersection cohomology (see [CGJ92, Proposition 2]), we see that

$$IC(U') = \pi^* IC(U') \otimes p^* IC(X_w) = \pi^* \underline{\mathbb{C}}_{U'}[\dim X] \otimes p^* IC(X_w) = p^* IC(X_w)[\dim X],$$

which shows that

$$\begin{aligned} H^i(IC(U'))_{(B/B,vB/B)} &= H^i(p^* IC(X_w)[\dim X])_{(B/B,vB/B)} \\ &= H^{i+\dim X}(IC(X_w))_{p((B/B,vB/B))} \\ &= H^{i+\dim X}(IC(X_w))_{vB/B}, \end{aligned}$$

implying the desired

$$H^i(IC(\mathcal{X}_w))_{(B/B,vB/B)} = H^{i+\dim X}(IC(X_w))_{vB/B}.$$

□

Remark. In Proposition 4.9(ii) and in the remainder of this paper, it is sensible for us to consider $H^i(IC(\mathcal{X}_w))_{(B/B,vB/B)}$ and $H^{i+\dim X}(IC(X_w))_{vB/B}$ because $H^i(IC(\mathcal{X}_w))$ and $H^{i+\dim X}(IC(X_w))$ are constant along the open strata $\mathcal{X}_v^o$ and X_v^o.

4.4. **The Bott-Samelson resolution.** In this subsection we construct and briefly summarize the main properties of the Bott-Samelson resolution of the G-Schubert variety. This resolution is again defined in close analogy to the usual Bott-Samelson resolution of the Schubert variety. Let $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$ be a reduced decomposition of some $w \in W$, and define the *G-Bott-Samelson variety* to be the closed subvariety

$$\mathcal{Z}_{\mathfrak{w}} := \left\{ (x_0, \dots, x_{n+1}) \in X^{n+1} \mid (x_k, x_{k+1}) \in \mathcal{X}_{s_{i_k}} \right\}$$

of X^{n+1}.[4] In this way, $\mathcal{Z}_{\mathfrak{w}}$ is equipped with a map $\pi_{\mathfrak{w}} : \mathcal{Z}_{\mathfrak{w}} \to X \times X$ given by projection on the first and last coordinates. It is evident that the image of $\pi_{\mathfrak{w}}$ lies in $\mathcal{X}_w$, and in fact the following proposition shows that $\pi_{\mathfrak{w}}$ is a resolution of $\mathcal{X}_w$.

Proposition 4.10. *We have the following:*

(i) the Bott-Samelson variety $\mathcal{Z}_{\mathfrak{w}}$ is smooth,
(ii) the map $\pi_{\mathfrak{w}} : \mathcal{Z}_{\mathfrak{w}} \to \mathcal{X}_w$ is projective, and
(iii) $\pi_{\mathfrak{w}}$ is an isomorphism on $G \cdot (B/B, wB/B)$, making it a resolution of singularities of $\mathcal{X}_w$.

Proof. For (i), defining $\mathfrak{w}[k] = (s_{i_1}, \dots, s_{i_k})$ to consist of the first k reflections in the word $\mathfrak{w}$ (so that $\mathfrak{w} = \mathfrak{w}[n]$), we see that

$$\mathcal{Z}_{\mathfrak{w}} = \mathcal{Z}_{\mathfrak{w}[n]} \to \mathcal{Z}_{\mathfrak{w}[n-1]} \to \cdots \to \mathcal{Z}_{\mathfrak{w}[1]} = X_{s_{i_1}} \simeq \mathbb{P}^1$$

realizes $\mathcal{Z}_{\mathfrak{w}}$ as a sequence of $\mathbb{P}^1$ bundles, since by Proposition 4.7(iii) each map $\mathcal{Z}_{\mathfrak{w}[k]} \to \mathcal{Z}_{\mathfrak{w}[k-1]}$ is a $\mathbb{P}^1$-bundle.

For (ii), we note simply that X is projective and that $\pi_{\mathfrak{w}}$ factors through $\mathcal{Z}_{\mathfrak{w}} \to X^{n+1} \to \mathcal{X}_w$, where the last map is a projection.

For (iii), notice that the points of $\mathcal{Z}_{\mathfrak{w}}$ take the form $z = (gB/B, gr_1B/B, gr_1r_2B/B, \dots, gr_1r_2\cdots r_nB/B)$, where r_k is either s_{i_k} or 1. In particular, $\pi_{\mathfrak{w}}(z) \in G \cdot (B/B, wB/B)$ if and only if $r_k = s_{i_k}$ for all k. On the domain where this occurs, $\pi_{\mathfrak{w}}$ is evidently an isomorphism onto $G \cdot (B/B, wB/B)$. □

Finally, we note the following alternate formulation of $\mathcal{Z}_{\mathfrak{w}}$ which will be more convenient for defining the convolution product.

Proposition 4.11. *We may realize $\mathcal{Z}_{\mathfrak{w}}$ as*

$$\mathcal{Z}_{\mathfrak{w}} = \mathcal{X}_{s_{i_1}} \underset{X}{\times} \mathcal{X}_{s_{i_2}} \underset{X}{\times} \cdots \underset{X}{\times} \mathcal{X}_{s_{i_n}}.$$

Proof. This is clear from looking at the points of $\mathcal{Z}_{\mathfrak{w}}$ and the fiber product on the right. □

4.5. **The convolution product.** We are now ready to introduce the key technical tool of our computation. Let $D^b_c(\mathcal{X})$ be the bounded derived category of sheaves on $\mathcal{X}$ which are constructible with respect to the stratification in (9). Let $\pi_i : \mathcal{X} \underset{X}{\times} \mathcal{X} \to \mathcal{X}$ be the projection from the i^{th} coordinate, and let $\pi' : \mathcal{X} \underset{X}{\times} \mathcal{X} \to \mathcal{X}$ be the projection

$$\mathcal{X} \underset{X}{\times} \mathcal{X} \to (X \times X) \underset{X}{\times} (X \times X) \to X \times X$$

from the first and last copy of X. Define the *convolution product* on $D^b_c(\mathcal{X})$ by the map

$$\circ : D^b_c(\mathcal{X}) \times D^b_c(\mathcal{X}) \to D^b_c(\mathcal{X})$$

given by

$$\mathcal{F}_1 \circ \mathcal{F}_2 = \pi'_* \left(\pi_1^* \mathcal{F}_1 \underset{\mathbb{C}}{\otimes} \pi_2^* \mathcal{F}_2 \right).$$

We provide a reformulation which will be more convenient. Consider the projections

$$\pi_i : \underbrace{\mathcal{X} \underset{X}{\times} \mathcal{X} \underset{X}{\times} \cdots \underset{X}{\times} \mathcal{X}}_{n} \to \mathcal{X}$$

from the i^{th} coordinate, and let π' again be the projection from the first and last copy of X. Then we may characterize the iterated convolution product as follows.

[4]We will omit the "G-" to simplify our terminology.

Proposition 4.12. *For $\mathcal{F}_i \in D^b_c(\mathcal{X})$, we have*

$$\mathcal{F}_1 \circ \cdots \circ \mathcal{F}_n = \pi'_* \left(\pi_1^* \mathcal{F}_1 \underset{\mathbb{C}}{\otimes} \cdots \underset{\mathbb{C}}{\otimes} \pi_n^* \mathcal{F}_n \right)$$

for any choice of parenthetical order on the left side. In particular, $\circ$ is an associative product.

Proof. We first check the case $n = 3$, which will give associativity. Consider the Cartesian square

$$\begin{tikzcd} \mathcal{X} \underset{X}{\times} \mathcal{X} \underset{X}{\times} \mathcal{X} \arrow[r, "\rho_{12}"] \arrow[d, "\rho'_{12} \times \mathrm{id}_3"'] & \mathcal{X} \underset{X}{\times} \mathcal{X} \arrow[d, "\rho'"] \\ \mathcal{X} \underset{X}{\times} \mathcal{X} \arrow[r, "\rho_1"] & \mathcal{X}, \end{tikzcd}$$

where ρ' is the projection from the first and last copy of X and $\rho'_{12} \times \mathrm{id}_3$ is the base change of ρ' with $\mathcal{X}$ over X.[5] Let $\rho_2 : \mathcal{X} \underset{X}{\times} \mathcal{X} \to \mathcal{X}$ be the second projection. Applying base change to the diagram and then the projection formula, we see that

$$\begin{aligned} (\mathcal{F}_1 \circ \mathcal{F}_2) \circ \mathcal{F}_3 &= \rho'_* \Big(\rho_1^* \rho'_* (\rho_1^* \mathcal{F}_1 \otimes \rho_2^* \mathcal{F}_2) \otimes \rho_2^* \mathcal{F}_3 \Big) \\ &= \rho'_* \Big((\rho'_{12} \times \mathrm{id}_3)_* (\pi_1^* \mathcal{F}_1 \otimes \pi_2^* \mathcal{F}_2) \otimes \rho_2^* \mathcal{F}_3 \Big) \\ &= \rho'_* \Big((\rho'_{12} \times \mathrm{id}_3)_* \Big(\pi_1^* \mathcal{F}_1 \otimes \pi_2^* \mathcal{F}_2 \otimes (\rho'_{12} \times \mathrm{id}_3)^* \rho_2^* \mathcal{F}_3 \Big) \Big) \\ &= \pi'_* (\pi_1^* \mathcal{F}_1 \otimes \pi_2^* \mathcal{F}_2 \otimes \pi_3^* \mathcal{F}_3). \end{aligned}$$

An obvious analogue of this computation shows that $\mathcal{F}_1 \circ (\mathcal{F}_2 \circ \mathcal{F}_3)$ is equal to this expression, implying that the product is associative.

For the general case, we induct on n. If the claim holds for some $n - 1$, then note

$$\begin{aligned} (\mathcal{F}_1 \circ \cdots \circ \mathcal{F}_{n-1}) \circ \mathcal{F}_n &= \rho'_* \left(\rho_1^* \pi'_* (\pi_1^* \mathcal{F}_1 \underset{\mathbb{C}}{\otimes} \cdots \underset{\mathbb{C}}{\otimes} \pi_{n-1}^* \mathcal{F}_{n-1}) \otimes \rho_2^* \mathcal{F}_n \right) \\ &= \pi'_* \left(\pi_1^* \mathcal{F}_1 \underset{\mathbb{C}}{\otimes} \cdots \underset{\mathbb{C}}{\otimes} \pi_n^* \mathcal{F}_n \right) \end{aligned}$$

by the same base change and projection argument as in the case $n = 3$. □

4.6. **Intersection cohomology of Schubert varieties.** We are now ready to compute the intersection cohomology of the Schubert varieties. We will mainly work with the G-Schubert varieties and translate the result over. Our general strategy will be to consider the pushforwards $(\pi_{\mathfrak{w}})_* \underline{\mathbb{C}}_{\mathcal{Z}_{\mathfrak{w}}}$ of IC-sheaves on the Bott-Samelson varieties (which are simply constant because $\mathcal{Z}_{\mathfrak{w}}$ is smooth). We then decompose these pushforwards into IC-sheaves (which are now non-trivial) on the singular G-Schubert varieties. Examining the decomposition carefully then gives the result.

Let us now proceed to the computation proper. For $\mathcal{F} \in D^b_c(\mathcal{X})$, define

$$h^i(\mathcal{F}_{(B/B, wB/B)}) := \dim H^i(\mathcal{F})_{(B/B, wB/B)}$$

to be the dimension of the stalk of $H^i(\mathcal{F})_{(B/B,wB/B)}$ at $(B/B, wB/B) \in \mathcal{X}$. Because $\mathcal{F}$ is constructible with respect to the stratification of (9), this describes $H^i(\mathcal{F})$ completely on the open stratum $\mathcal{X}^o_w$. Define the element $h(\mathcal{F}) \in \mathcal{H}(W)$ of the Hecke algebra by

$$h(\mathcal{F}) = \sum_{w \in W} \sum_{i \in \mathbb{Z}} h^i(\mathcal{F}_{(B/B, wB/B)}) q^i \cdot T_w.$$

The key step of the computation will be the following lemma, which describes the behavior of cohomology under the convolution product.

[5] We denote all projections in this diagram by ρ to avoid confusion with π, which will denote the projections from the n-fold fiber product.

Lemma 4.13. *Let $\mathcal{F} \in D_c^b(\mathcal{X})$ have non-zero cohomology in only a single parity class of degrees. Then for any simple reflection s, the same holds for $\underline{\mathbb{C}}_{\mathcal{X}_s} \circ \mathcal{F}$ and further we have*

$$h(\underline{\mathbb{C}}_{\mathcal{X}_s} \circ \mathcal{F}) = (T_s + 1) \cdot h(\mathcal{F}).$$

Proof. Let us first translate this statement into more concrete terms. By definition, we see that

$$\begin{aligned}(T_s+1)\cdot h(\mathcal{F}) &= \sum_{w\in W}\sum_{i\in\mathbb{Z}} h^i(\mathcal{F}_{(B/B,wB/B)})q^i \begin{cases} T_{sw}+T_w & \ell(sw)>\ell(w) \\ q^2(T_{sw}+T_w) & \ell(sw)<\ell(w). \end{cases} \\ &= \sum_{w\in W}\sum_{i\in\mathbb{Z}} T_w \begin{cases} h^i(\mathcal{F}_{(B/B,wB/B)})q^i + h^i(\mathcal{F}_{(B/B,swB/B)})q^{i+2} & \ell(sw)>\ell(w) \\ h^i(\mathcal{F}_{(B/B,wB/B)})q^{i+2} + h^i(\mathcal{F}_{(B/B,swB/B)})q^i & \ell(sw)<\ell(w) \end{cases} \\ &= \sum_{w\in W}\sum_{i\in\mathbb{Z}} q^i T_w \begin{cases} h^i(\mathcal{F}_{(B/B,wB/B)}) + h^{i-2}(\mathcal{F}_{(B/B,swB/B)}) & \ell(sw)>\ell(w) \\ h^{i-2}(\mathcal{F}_{(B/B,wB/B)}) + h^i(\mathcal{F}_{(B/B,swB/B)}) & \ell(sw)<\ell(w). \end{cases}\end{aligned}$$

Thus, it suffices for us to show that

$$h^i((\underline{\mathbb{C}}_{\mathcal{X}_S}\circ\mathcal{F})_{(B/B,wB/B)}) = \begin{cases} h^i(\mathcal{F}_{(B/B,wB/B)}) + h^{i-2}(\mathcal{F}_{(B/B,swB/B)}) & \ell(sw)>\ell(w) \\ h^{i-2}(\mathcal{F}_{(B/B,wB/B)}) + h^i(\mathcal{F}_{(B/B,swB/B)}) & \ell(sw)<\ell(w). \end{cases} \tag{10}$$

Indeed, combining (10) and the parity condition on cohomology for $\mathcal{F}$ yields the parity condition for $\underline{\mathbb{C}}_{\mathcal{X}_s} \circ \mathcal{F}$.

Now, let $\rho : (\pi')^{-1}(B/B, wB/B) \to \mathcal{X} \underset{X}{\times} \mathcal{X}$ be the inclusion. By proper base change, we see that

$$H^i\left(\underline{\mathbb{C}}_{\mathcal{X}_s}\circ\mathcal{F}\right)_{(B/B,wB/B)} = H^i\left(\pi'_*(\pi_1^*\underline{\mathbb{C}}_{\mathcal{X}_s} \underset{\mathbb{C}}{\otimes} \pi_2^*\mathcal{F})\right)_{(B/B,wB/B)} = H^i\left((\pi')^{-1}(B/B,wB/B), \rho^*(\pi_1^*\underline{\mathbb{C}}_{\mathcal{X}_s} \underset{\mathbb{C}}{\otimes} \pi_2^*\mathcal{F})\right).$$

On the other hand, as a sheaf on $X \times X \times X$, $\pi_1^*\underline{\mathbb{C}}_{\mathcal{X}_s} \underset{\mathbb{C}}{\otimes} \pi_2^*\mathcal{F}$ is supported on $\mathcal{X}_s \times X$. Thus for

$$\mathcal{Y}_{s,w} := (\pi')^{-1}(B/B, wB/B) \cap (\mathcal{X}_s \times X) = \{B/B\} \times P_s \times \{wB/B\} \simeq \mathbb{P}^1$$

and the inclusion $\tau : \mathcal{Y}_{s,w} \to \mathcal{X} \underset{X}{\times} \mathcal{X}$, we see that

$$H^i\left(\underline{\mathbb{C}}_{\mathcal{X}_s}\circ\mathcal{F}\right)_{(B/B,wB/B)} = H^i\left((\pi')^*(B/B,wB/B), \rho^*(\pi_1^*\underline{\mathbb{C}}_{\mathcal{X}_s} \underset{\mathbb{C}}{\otimes} \pi_2^*\mathcal{F})\right) = H^i\left(\mathcal{Y}_{s,w}, \tau^*(\pi_1^*\underline{\mathbb{C}}_{\mathcal{X}_s} \underset{\mathbb{C}}{\otimes} \pi_2^*\mathcal{F})\right).$$

To complete the proof, we recall that $\underline{\mathbb{C}}_{\mathcal{X}_s}$ and $\mathcal{F}$ are constructible with respect to the stratification of (9). Further, this stratification induces a stratification

$$\mathcal{X} \underset{X}{\times} \mathcal{X} = \bigsqcup_{v,w\in W} G\cdot(B/B, wB/B) \underset{X}{\times} G\cdot(B/B, vB/B)$$

with respect to which $\pi_1^*\underline{\mathbb{C}}_{\mathcal{X}_s} \otimes_{\mathbb{C}} \pi_2^*\mathcal{F}$ is constructible. Note that the closure of each cell of this stratification is given by

$$\overline{G\cdot(B/B, wB/B) \underset{X}{\times} G\cdot(B/B, vB/B)} = \bigsqcup_{w'\le w, v'\le v} G\cdot(B/B, w'B/B) \underset{X}{\times} G\cdot(B/B, v'B/B).$$

We will restrict this stratification to $\mathcal{Y}_{s,w}$. We split into two cases depending on the the value of $\ell(sw)$.

First, suppose that $\ell(sw) > \ell(w)$. Then, we see that $(B/B, bsB/B, wB/B)$ lies in the stratum

$$G\cdot(B/B, vB/B) \underset{X}{\times} G\cdot(B/B, v'B/B)$$

if and only if $sb^{-1}w \in Bv'B$ and $bs \in BvB$, which occurs if and only if $v = s$ and $v' = sw$ by Proposition 2.2(iii). On the other hand, $(B/B, B/B, wB/B)$ evidently lies in the stratum

$$G\cdot(B/B, B/B) \underset{X}{\times} G\cdot(B/B, wB/B),$$

so $\mathcal{Y}_{s,w}$ lies in two strata

$$\mathcal{Y}_{s,w} = (B/B, B/B, wB/B) \cup (B/B, BsB/B, wB/B) \simeq \{\mathrm{pt}\} \cup \mathbb{A}^1,$$

both of which are contractible. Let $i : (B/B, B/B, wB/B) \hookrightarrow \mathcal{Y}_{s,w}$ and $j : (B/B, BsB/B, wB/B) \hookrightarrow \mathcal{Y}_{s,w}$ be the inclusions, which give rise to a distinguished triangle

$$j_! j^* \tau^* (\pi_1^* \underline{\mathbb{C}}_{\mathcal{X}_s} \underset{\mathbb{C}}{\otimes} \pi_2^* \mathcal{F}) \to \tau^* (\pi_1^* \underline{\mathbb{C}}_{\mathcal{X}_s} \underset{\mathbb{C}}{\otimes} \pi_2^* \mathcal{F}) \to i_* i^* \tau^* (\pi_1^* \underline{\mathbb{C}}_{\mathcal{X}_s} \underset{\mathbb{C}}{\otimes} \pi_2^* \mathcal{F}) \overset{+1}{\to} . \tag{11}$$

Notice that

$$i^* \tau^* (\pi_1^* \underline{\mathbb{C}}_{\mathcal{X}_s} \underset{\mathbb{C}}{\otimes} \pi_2^* \mathcal{F}) = (\underline{\mathbb{C}}_{\mathcal{X}_s})_{(B/B, B/B)} \underset{\mathbb{C}}{\otimes} \mathcal{F}_{(B/B, wB/B)} = \mathcal{F}_{(B/B, wB/B)}.$$

Now, set $\mathcal{G} = j^* \tau^* (\pi_1^* \underline{\mathbb{C}}_{\mathcal{X}_s} \underset{\mathbb{C}}{\otimes} \pi_2^* \mathcal{F})$ so that the cohomologies of $\mathcal{G}$ are constant sheaves (as $\mathcal{G}$ is constructible with respect to the trivial stratification on $\mathbb{A}^1$). By Poincaré duality, we have

$$H_c^i(\mathbb{A}^1, \mathcal{G}) = H^{-i}(\mathbb{A}^1, \mathbb{D}\mathcal{G}) = H^{-i}(\mathbb{A}^1, \mathcal{RH}om(\mathcal{G}, \underline{\mathbb{C}}[2])) = H^{-i+2}(\mathbb{A}^1, \mathcal{RH}om(\mathcal{G}, \underline{\mathbb{C}})),$$

where the cohomologies of $\mathcal{RH}om(\mathcal{G}, \underline{\mathbb{C}})$ are constant sheaves. Therefore, we see that for $p = (B/B, swB/B) \in \mathbb{A}^1$, we have

$$H^{-i+2}(\mathbb{A}^1, \mathcal{RH}om(\mathcal{G}, \underline{\mathbb{C}})) = H^{-i+2}(\mathcal{RH}om(\mathcal{G}, \underline{\mathbb{C}}))_p = H^{-i+2}(\mathrm{RHom}(\mathcal{G}_p, \underline{\mathbb{C}}_p)) = H^{-i+2}(\mathcal{G}_p^\vee) = H^{i-2}(\mathcal{G}_p),$$

where for a complex of vector spaces $V^\bullet$, we set $(V^\vee)^i = V^{-i}$. Computing, we see that

$$\mathcal{G}_p = (\underline{\mathbb{C}}_{\mathcal{X}_s})_{(B/B, sB/B)} \underset{\mathbb{C}}{\otimes} \mathcal{F}_{(B/B, swB/B)} = \mathcal{F}_{(B/B, swB/B)},$$

which shows that

$$H_c^i(\mathbb{A}^1, j^* \tau^* (\pi_1^* \underline{\mathbb{C}}_{\mathcal{X}_s} \underset{\mathbb{C}}{\otimes} \pi_2^* \mathcal{F})) = H^{i-2}(\mathcal{F}_{(B/B, swB/B)})).$$

The long exact sequence in compactly supported cohomology associated to (11) is therefore

$$\cdots \to H^{i-2}(\mathcal{F}_{(B/B, swB/B)})) \to H_c^i(\mathcal{Y}_{s,w}, \tau^* (\pi_1^* \underline{\mathbb{C}}_{\mathcal{X}_s} \underset{\mathbb{C}}{\otimes} \pi_2^* \mathcal{F})) \to H^i(\mathcal{F}_{(B/B, wB/B)}) \\ \to H^{i-1}(\mathcal{F}_{(B/B, swB/B)})) \to \cdots .$$

By our assumption that the cohomology of $\mathcal{F}$ is non-zero in only a single parity class, this long exact sequence breaks up into the short exact sequences

$$0 \to H^{i-2}(\mathcal{F}_{(B/B, swB/B)}) \to H_c^i(\mathcal{Y}_{s,w}, \tau^* (\pi_1^* \underline{\mathbb{C}}_{\mathcal{X}_s} \underset{\mathbb{C}}{\otimes} \pi_2^* \mathcal{F})) = H^i((\underline{\mathbb{C}}_{\mathcal{X}_s} \circ \mathcal{F})_{(B/B, wB/B)}) \to H^i(\mathcal{F}_{(B/B, wB/B)}) \to 0,$$

which gives (10) in this case. Here we recall that $\mathcal{Y}_{s,w} \simeq \mathbb{P}^1$, hence

$$H_c^i(\mathcal{Y}_{s,w}, \tau^* (\pi_1^* \underline{\mathbb{C}}_{\mathcal{X}_s} \underset{\mathbb{C}}{\otimes} \pi_2^* \mathcal{F})) = H^i(\mathcal{Y}_{s,w}, \tau^* (\pi_1^* \underline{\mathbb{C}}_{\mathcal{X}_s} \underset{\mathbb{C}}{\otimes} \pi_2^* \mathcal{F})).$$

It remains to consider the case $\ell(sw) < \ell(w)$. By a similar analysis, we see that $\mathcal{Y}_{s,w}$ lies in two strata, with $(B/B, sB/B, wB/B)$ the unique point of $\mathcal{Y}_{s,w}$ lying in

$$G \cdot (B/B, sB/B) \underset{X}{\times} G \cdot (B/B, swB/B)$$

and all other points of $\mathcal{Y}_{s,w}$ lying in

$$G \cdot (B/B, sB/B) \underset{X}{\times} G \cdot (B/B, wB/B).$$

Taking i to be the inclusion of $(B/B, sB/B, wB/B)$ and j the inclusion of its complement, we thus obtain the distinguished triangle (11) for these differently defined i and j. Again applying compactly supported cohomology gives, after application of our parity condition on cohomology, the short exact sequences

$$0 \to H^{i-2}(\mathcal{F}_{(B/B, wB/B)}) \to H_c^i(\mathcal{Y}_{s,w}, \tau^* (\pi_1^* \underline{\mathbb{C}}_{\mathcal{X}_s} \underset{\mathbb{C}}{\otimes} \pi_2^* \mathcal{F})) = H^i((\underline{\mathbb{C}}_{\mathcal{X}_s} \circ \mathcal{F})_{(B/B, wB/B)}) \to H^i(\mathcal{F}_{(B/B, swB/B)}) \to 0,$$

which imply (10) in this case. This completes the proof. $\square$

We are now ready to complete the computation. We will perform our main work on the G-Schubert varieties $\mathcal{X}_w$ and then use Proposition 4.9 to transfer it to the Schubert varieties X_w.

Theorem 4.14. *For $w \in W$, we have that*

$$h(IC(\mathcal{X}_w)) = q^{-\dim X} \cdot C_w,$$

where $C_w \in \mathcal{H}(W)$ are the elements of Proposition 4.1.

Proof. Let $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$ be a reduced decomposition of w, where $n = \ell(w)$. We first note that the map $\pi_{\mathfrak{w}} : \mathcal{Z}_{\mathfrak{w}} \to \mathcal{X}_w$ of Proposition 4.10 may easily be seen to be the restriction of the map $\pi' : \mathcal{X} \underset{X}{\times} \cdots \underset{X}{\times} \mathcal{X} \to \mathcal{X}$ when viewing $\mathcal{Z}_{\mathfrak{w}}$ in the form of Proposition 4.11. Our proof will center on the study of $\pi'_* IC(\mathcal{Z}_{\mathfrak{w}})$. Because $\mathcal{Z}_{\mathfrak{w}}$ is smooth, we see that $IC(\mathcal{Z}_{\mathfrak{w}}) = \underline{\mathbb{C}}_{\mathcal{Z}_{\mathfrak{w}}}[n + \dim X]$, where

$$\underline{\mathbb{C}}_{\mathcal{Z}_{\mathfrak{w}}} = \pi_1^* \underline{\mathbb{C}}_{\mathcal{X}_{s_{i_1}}} \underset{\mathbb{C}}{\otimes} \cdots \underset{\mathbb{C}}{\otimes} \pi_n^* \underline{\mathbb{C}}_{\mathcal{X}_{s_{i_n}}}.$$

By Proposition 4.12, this implies that

$$\pi'_* IC(\mathcal{Z}_{\mathfrak{w}}) = (\underline{\mathbb{C}}_{\mathcal{X}_{s_{i_1}}} \circ \cdots \circ \underline{\mathbb{C}}_{\mathcal{X}_{s_{i_n}}})[n + \dim X].$$

Because each $\underline{\mathbb{C}}_{\mathcal{X}_{s_{i_k}}}$ has cohomology only in degree 0, Lemma 4.13 gives

$$h(\pi'_* IC(\mathcal{Z}_{\mathfrak{w}})) = q^{-n-\dim X}(T_{s_{i_1}} + 1) \cdot (T_{s_{i_2}} + 1) \cdots (T_{s_{i_n}} + 1). \tag{12}$$

On the other hand, the decomposition theorem shows that

$$\pi'_* IC(\mathcal{Z}_{\mathfrak{w}}) = \bigoplus_{v \le w} \bigoplus_k IC(\mathcal{X}_v)^{\oplus m_{v,w,k}}[k] \tag{13}$$

for some multiplicities $m_{v,w,k}$, which implies that

$$h(\pi'_* IC(\mathcal{Z}_{\mathfrak{w}})) = \sum_{v \le w} \sum_k q^{-k} m_{v,w,k} \cdot h(IC(\mathcal{X}_v)). \tag{14}$$

Since we know $h(\pi'_* IC(\mathcal{Z}_{\mathfrak{w}}))$ explicitly by (12), it remains only to formally invert (14). We will do this by making three observations about the $m_{v,w,k}$.
Observation 1: By Proposition 4.10, π'_* is an isomorphism over $G \cdot (B/B, wB/B)$. In particular, the right hand side of (13) must then have rank 1 over $G \cdot (B/B, wB/B)$, so we must have

$$m_{w,w,k} = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Observation 2: Because π' is proper by Proposition 4.10, it commutes with duality, meaning that

$$\pi'_* IC(\mathcal{Z}_{\mathfrak{w}}) = \mathbb{D}(\pi'_* IC(\mathcal{Z}_{\mathfrak{w}})) = \bigoplus_{v \le w} \bigoplus_k IC(\mathcal{X}_v)^{\oplus m_{v,w,k}}[-k],$$

which shows that $m_{v,w,k} = m_{v,w,-k}$.
Observation 3: We show that $i(q^{\dim X} h(IC(\mathcal{X}_w))) = q^{\dim X} h(IC(\mathcal{X}_w))$ by induction on w. Suppose that the statement holds for all $v < w$. Combining Observations 1 and 2, the formula (14) becomes

$$h(\pi'_* IC(\mathcal{Z}_{\mathfrak{w}})) = h(IC(\mathcal{X}_w)) + \sum_{v<w} h(IC(\mathcal{X}_v)) \sum_k q^{-k} m_{v,w,k}$$

Rearranging and applying (12), we find that

$$q^{\dim X} h(IC(\mathcal{X}_w)) = q^{-\ell(w)}(T_{s_{i_1}} + 1) \cdots (T_{s_{i_n}} + 1) - \sum_{v<w} q^{\dim X} h(IC(\mathcal{X}_v)) \sum_k q^{-k} m_{v,w,k}.$$

By (7) we have $i(T_s + 1) = \frac{1}{q^2}(T_s + 1)$, and combining this with Observation 2 yields

$$\begin{aligned} i(q^{\dim X} h(IC(\mathcal{X}_w))) &= i\left(q^{-\ell(w)}(T_{s_{i_1}} + 1) \cdots (T_{s_{i_n}} + 1) - \sum_{v<w} q^{\dim X} h(IC(\mathcal{X}_v)) \sum_k q^{-k} m_{v,w,k}\right) \\ &= q^{-\ell(w)}(T_{s_{i_1}} + 1) \cdots (T_{s_{i_n}} + 1) - \sum_{v<w} q^{\dim X} h(IC(\mathcal{X}_v)) \sum_k q^{k} m_{v,w,k} \\ &= q^{\dim X} h(IC(\mathcal{X}_w)). \end{aligned}$$

Finishing the proof: We are now ready to show that $q^{\dim X} h(IC(\mathcal{X}_w))$ satisfies the same properties as C_w, which will yield the desired conclusion by Proposition 4.1. Again, we will induct on w. Suppose that

$q^{\dim X} h(IC(\mathfrak{X}_v)) = C_v$ for all $v < w$. Then, notice that

$$q^{\dim X} h(IC(\mathfrak{X}_w)) = q^{-\ell(w)} \left[(T_{s_{i_1}} + 1) \cdots (T_{s_{i_n}} + 1) - \sum_{v<w} C_v \sum_k q^{k+\ell(w)} m_{v,w,k} \right] \tag{15}$$

$$= q^{-\ell(w)} \sum_{v \leq w} Q_{v,w}(q) T_v \tag{16}$$

for some polynomials $Q_{v,w}(q)$ with $Q_{w,w}(q) = 1$, where we applied the multiplication rule (6) and property (ii) of C_v given by Proposition 4.1.

On the other hand, recall that $IC(\mathfrak{X}_w)$ was defined to satisfy

$$\dim \operatorname{supp} H^{-j}(IC(\mathfrak{X}_w)) < j,$$

meaning that

$$H^{-j}(IC(\mathfrak{X}_w)_{(B/B, vB/B)}) = 0 \text{ for } j \leq \dim X + \ell(v) = \dim \mathfrak{X}_v.$$

This implies that

$$q^{\dim X} h(IC(\mathfrak{X}_w)) = \sum_{w \in W} \sum_{i < -\dim X - \ell(v)} h^i(IC(X_w)_{(B/B, vB/B)}) \cdot q^{i + \dim X} \cdot T_w = q^{-\ell(w)} \sum_{v \leq w} Q_{v,w}(q) T_w,$$

which shows that $Q_{v,w}(q) = 0$ for $v > w$, $Q_{w,w}(q) = 1$, and

$$Q_{v,w}(q) = \sum_{i < -\dim X - \ell(v)} h^i(IC(\mathfrak{X}_w)_{(B/B, vB/B)}) \cdot q^{i + \dim X + \ell(w)}$$

has degree at most $\ell(w) - \ell(v) - 1$. Combining this with Observation 3 and (16) allows us to apply the uniqueness portion of Proposition 4.1 to see that

$$q^{\dim X} h(IC(\mathfrak{X}_w)) = C_w \text{ and } P_{v,w}(q) = Q_{v,w}(q),$$

completing the proof. □

Theorem 4.14 gives us complete knowledge of the intersection cohomology of $\mathfrak{X}_w$. We summarize the consequences as follows.

- If $i - \dim X$ is even, the final line of Proposition 4.1 shows that $P_{v,w}(q)$ has only odd non-zero coefficients, meaning that
$$h^{i - \dim X}(IC(\mathfrak{X}_w)) = 0.$$
- If $i - \dim X$ is odd, we find that $h^{i - \dim X}(IC(\mathfrak{X}_w)_{B/B, vB/B})$ is the coefficient of $q^{i + \ell(w)}$ in $P_{v,w}(q)$.

It is now easy for us to compute the intersection cohomology of X_w. For $\mathcal{F} \in D^b_c(X)$, define an element $h(\mathcal{F}) \in \mathcal{H}(W)$ by

$$h(\mathcal{F}) = \sum_{w \in W} \sum_{i \in \mathbb{Z}} h^i(\mathcal{F}_{wB/B}) q^i \cdot T_w.$$

Note that we are abusing notation here by defining $h(\mathcal{F})$ for $\mathcal{F}$ in both $D^b_c(\mathfrak{X})$ and $D^b_c(X)$.

Corollary 4.15. *For $w \in W$, we have*

$$h(IC(X_w)) = C_w.$$

Proof. This follows immediately from Theorem 4.14 and Proposition 4.9. □

4.7. **Putting it all together.** We now combine our work so far to give a proof of Theorem 4.4.

Proof of Theorem 4.4. By Theorems 3.3 and 3.8, it suffices to write $[\mathcal{M}_w]$ in terms of $[\mathcal{L}_w]$ in the Grothendieck group of $\mathrm{Perv}(X)^B$. For this, consider the character map

$$\chi : K(\mathrm{Perv}(X)^B) \to \mathbb{Z}[W]$$

defined by

$$\chi([\mathcal{M}]) = \sum_{w \in W} \sum_{i \in \mathbb{Z}} (-1)^i h^i(\mathcal{M}_{wB/B}) \cdot w.$$

Notice that

$$\sum_{i \in \mathbb{Z}} (-1)^i h^i(\mathcal{M}_{wB/B})$$

is the Euler characteristic of $\mathcal{M}_{wB/B}$, hence χ is a well-defined map out of $K(\mathrm{Perv}(X)^B)$. Let us now consider the images of $[\mathcal{M}_w]$ and $[\mathcal{L}_w]$ under this map.

Recall that $\mathcal{M}_w = \underline{\mathbb{C}}_{X_w^\circ}[\dim X_w]$, meaning that

$$(\mathcal{M}_w)_{vB/B} = \begin{cases} \mathbb{C}[\ell(w)] & v = w \\ 0 & \text{otherwise,} \end{cases}$$

which implies that

$$\chi([\mathcal{M}_w]) = (-1)^{\ell(w)} w.$$

Therefore, χ sends a basis of $K(\mathrm{Perv}(X)^B)$ to a basis of $\mathbb{Z}[W]$, hence is an isomorphism.

On the other hand, recalling that $\mathcal{L}_w = IC(X_w)$, we see that

$$\begin{aligned} \chi([\mathcal{L}_w]) &= \sum_{v\in W} \sum_{i\in\mathbb{Z}} (-1)^i h^i(IC(X_w)_{vB/B}) \cdot v \\ &= \sum_{v\in W} (-1)^{-\ell(w)} P_{v,w}(-1) \cdot v \\ &= \sum_{v\in W} (-1)^{\ell(v)-\ell(w)} P_{v,w}(-1)\chi([\mathcal{M}_w]), \end{aligned}$$

which shows that

$$[\mathcal{L}_w] = \sum_{v\in W} (-1)^{\ell(v)-\ell(w)} P_{v,w}(-1)[\mathcal{M}_w]$$

in $K(\mathrm{Perv}(X)^B)$ because χ is an isomorphism. Translating back to the Grothendieck group of $\mathcal{O}_{-\rho}$, noting that $P_{v,w}(-1) = P_{v,w}(1)$ because $P_{v,w}(q)$ has no non-zero coefficients in odd degrees, and recalling $P_{v,w} = 0$ for $v > w$, we obtain the desired

$$[L_{-w\rho-\rho}] = \sum_{v\le w} (-1)^{\ell(v)-\ell(w)} P_{v,w}(1) \cdot [M_{-w\rho-\rho}]. \tag{17}$$ □

References

[BB81] Alexander Beĭlinson and Joseph Bernstein, *Localisation de g-modules*, C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 1, 15–18. MR 610137 (82k:14015)

[Ber83] Joseph Bernstein, *Algebraic Theory of D-Modules*, `http://www.math.uchicago.edu/~mitya/langlands/Bernstein/Bernstein-dmod.ps`, 1983.

[BGK+87] A. Borel, P.-P. Grivel, B. Kaup, A. Haefliger, B. Malgrange, and F. Ehlers, *Algebraic D-modules*, Perspectives in Mathematics, vol. 2, Academic Press Inc., Boston, MA, 1987. MR 882000 (89g:32014)

[CGJ92] Daniel C. Cohen, Mark Goresky, and Lizhen Ji, *On the Künneth formula for intersection cohomology*, Trans. Amer. Math. Soc. **333** (1992), no. 1, 63–69. MR 1052904 (92k:55009)

[dCM09] Mark Andrea A. de Cataldo and Luca Migliorini, *The decomposition theorem, perverse sheaves and the topology of algebraic maps*, Bull. Amer. Math. Soc. (N.S.) **46** (2009), no. 4, 535–633. MR 2525735 (2011a:14012)

[Dix77] Jacques Dixmier, *Enveloping algebras*, North-Holland Publishing Co., Amsterdam, 1977, North-Holland Mathematical Library, Vol. 14, Translated from the French. MR 0498740 (58 #16803b)

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, *D-modules, perverse sheaves, and representation theory*, Progress in Mathematics, vol. 236, Birkhäuser Boston Inc., Boston, MA, 2008, Translated from the 1995 Japanese edition by Takeuchi. MR 2357361 (2008k:32022)

[KL79] David Kazhdan and George Lusztig, *Representations of Coxeter groups and Hecke algebras*, Inventiones Mathematicae **53** (1979), 165–184.

[Ric10a] Simon Riche, *D-modules, faisceaux pervers et conjecture de Kazhdan-Lusztig*, `http://math.univ-bpclermont.fr/~riche/exposeKL-v2.pdf`, 2010.

[Ric10b] ———, *Perverse sheaves on flag manifolds and Kazhdan-Lusztig polynomials*, `http://math.univ-bpclermont.fr/~riche/exposeKL2.pdf`, 2010.

[Soe97] Wolfgang Soergel, *Kazhdan-Lusztig polynomials and a combinatoric[s] for tilting modules*, Represent. Theory **1** (1997), 83–114 (electronic). MR 1444322 (98d:17026)

[Spr81] Thomas A. Springer, *Quelques applications de la cohomologie d'intersection*, Séminar N. Bourbaki **589** (1981), 249–273.

[Spr98] ———, *Linear algebraic groups*, second ed., Progress in Mathematics, vol. 9, Birkhäuser Boston Inc., Boston, MA, 1998. MR 1642713 (99h:20075)