

# Polynomials

Yi Sun

MOP 2010

## 1 Warmup

**Problem 1** (USAMO 1974). Let  $a, b$ , and  $c$  be three distinct integers, and let  $P(x)$  be a polynomial with integer coefficients. Show that we cannot have  $P(a) = b, P(b) = c$ , and  $P(c) = a$ .

## 2 Common Techniques

Here are some common ideas that occur in problems involving polynomials:

- **Roots:** Every polynomial  $P(x)$  can be placed in the form  $P(x) = a(x - r_1) \cdots (x - r_n)$ , where  $r_1, \dots, r_n$  are the (complex) roots of  $P(x)$ . Vieta's formulas allow us to relate these roots to the coefficients.
- **Interpolation:** A polynomial of degree  $n$  is determined uniquely by a choice of  $n + 1$  points, and we may compute the coefficients of this polynomial explicitly using Fact 7.
- **Divisibility:** The most important idea for integer polynomials is Fact 8. It is sometimes used in an iterated fashion – for  $P(x)$  a polynomial with integer coefficients, we have

$$a - b \mid P(a) - P(b) \mid P(P(a)) - P(P(b)) \mid \cdots .$$

- **Intermediate Values:** If  $P$  is a polynomial (or any continuous function) such that  $P(a)$  and  $P(b)$  have different sign for  $a < b$ , then  $P$  has a root on the interval  $(a, b)$ .
- **Construct a different polynomial:** Sometimes translating the given information into facts about a different but related polynomial will lead to a simpler interpretation in terms of standard facts about polynomials.

## 3 Useful Facts

Here are some facts that may be useful when dealing with polynomials.

**Fact 1.** The sum of coefficients of a polynomial  $P(x)$  is  $P(1)$ , and the constant coefficient is  $P(0)$ .

**Fact 2.** For a polynomial  $P(x)$ , if  $P(a) = 0$ , then  $(x - a)$  divides  $P(x)$ .

**Fact 3.** Let  $P(x)$  and  $Q(x)$  be polynomials. Then, we may write

$$P(x) = T(x)Q(x) + R(x)$$

for some polynomials  $T(x)$  and  $R(x)$  with  $\deg R(x) < \deg Q(x)$ .

**Fact 4.** Let  $P(x)$  be a polynomial. Then, for any  $a$ , we may write

$$P(x) = b_0 + b_1(x - a) + b_2(x - a)^2 + \cdots + b_n(x - a)^n$$

for a unique choice of  $b_0, \dots, b_n$ .

**Fact 5.** If  $a_{-k} = a_k$  for  $0 \leq k \leq n$  in the rational function

$$R(x) = a_{-n}x^{-n} + a_{-n+1}x^{-n+1} + \cdots + a_{n-1}x^{n-1} + a_nx^n,$$

then we may write  $R(x) = P\left(x + \frac{1}{x}\right)$  for some polynomial  $P(x)$  of degree  $n$ .

**Fact 6** (Vieta's Formulas). If  $P(x) = a_0 + a_1x + \cdots + a_nx^n$  is a polynomial with roots  $r_1, \dots, r_n$ , then for  $0 \leq k \leq n$  we have

$$\frac{a_k}{a_n} = (-1)^{n-k} \sum_{i_1 < \cdots < i_{n-k}} \prod_{j=0}^{n-k} r_{i_j},$$

where the right hand side is the  $k^{\text{th}}$  elementary symmetric function in  $r_1, \dots, r_n$ .

**Fact 7** (Lagrange Interpolation). Let  $P(x)$  be a polynomial of degree  $n$  passing through the points  $(x_0, y_0), (x_1, y_1), \dots, (x_n, y_n)$ . Then, we may write

$$P(x) = \sum_{i=0}^n y_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}.$$

**Fact 8.** For integers  $a, b$  and a polynomial with integer coefficients  $P(x)$ , we have  $a - b \mid P(a) - P(b)$ .

**Fact 9** (Gauss' Lemma). Let  $P(x)$  be a monic polynomial with integer coefficients. Then if  $a$  is a rational root of  $P$ , then  $a$  is an integer.

**Fact 10.** Let  $P(x)$  be a polynomial that takes integer values on the integers. For any  $k$ , define

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}$$

as a polynomial in  $x$ . Then, we may write

$$P(x) = b_0 \binom{x}{0} + b_1 \binom{x}{1} + \cdots + b_n \binom{x}{n}$$

for a unique choice of integers  $b_0, \dots, b_n$ .

## 4 Problems

### 4.1 Polynomials

**Problem 2** (HMMT 2007). The complex numbers  $\alpha_1, \alpha_2, \alpha_3$ , and  $\alpha_4$  are the four distinct roots of the equation  $x^4 + 2x^3 + 2 = 0$ . Determine the unordered set

$$\{\alpha_1\alpha_2 + \alpha_3\alpha_4, \alpha_1\alpha_3 + \alpha_2\alpha_4, \alpha_1\alpha_4 + \alpha_2\alpha_3\}.$$

**Problem 3** (Putnam 2008). Let  $n \geq 3$  be an integer. Let  $f(x)$  and  $g(x)$  be polynomials with real coefficients such that the points  $(f(1), g(1)), (f(2), g(2)), \dots, (f(n), g(n))$  in  $\mathbb{R}^2$  are the vertices of a regular  $n$ -gon in counterclockwise order. Prove that at least one of  $f(x)$  and  $g(x)$  has degree greater than or equal to  $n - 1$ .

**Problem 4** (USAMO 2002). Prove that any monic polynomial of degree  $n$  with real coefficients is the average of two monic polynomials of degree  $n$  with  $n$  real roots.

**Problem 5** (TST 2005). Consider the polynomials

$$f(x) = \sum_{k=1}^n a_k x^k \text{ and } g(x) = \sum_{k=1}^n \frac{a_k}{2^k - 1} x^k,$$

where  $a_1, a_2, \dots, a_n$  are real numbers and  $n$  is a positive integer. Show that if 1 and  $2^{n+1}$  are zeros of  $g$ , then  $f$  has a positive zero less than  $2^n$ .

**Problem 6** (MOP 1997). Let  $S = \{s_1, s_2, \dots, s_n\}$  be a set of  $n$  distinct complex numbers, for some  $n \geq 9$ , exactly  $n - 3$  of which are real. Prove that there are at most two quadratic polynomials  $f(z)$  with complex coefficients such that  $f(S) = S$  (that is,  $f$  permutes the elements of  $S$ ).

**Problem 7** (China 1995). Alice and Bob play a game with a polynomial of degree at least 4:

$$x^{2n} + \square x^{2n-1} + \square x^{2n-2} + \dots + \square x + 1.$$

They fill in the empty boxes with real numbers in turn. If the resulting polynomial has no real root, Alice wins; otherwise, Bob wins. If Alice goes first, who has a winning strategy?

**Problem 8** (WOP 2004). A number of points are given on a unit circle so that the product of the distances from any point on the circle to the given points does not exceed 2. Prove that the given points are the vertices of a regular polygon.

## 4.2 Integer polynomials

**Problem 9** (BAMO 2004). Find (with proof) all monic polynomials  $f(x)$  with integer coefficients that satisfy the following two conditions.

1.  $f(0) = 2004$ .
2. If  $x$  is irrational, then  $f(x)$  is also irrational.

**Problem 10** (TST 2010). Let  $P(x)$  be a polynomial with integer coefficients such that  $P(0) = 0$  and

$$\gcd(P(x), P(1), P(2), \dots) = 1.$$

Prove that there are infinitely many positive integers  $n$  such that

$$\gcd(P(n) - P(0), P(n+1) - P(1), P(n+2) - P(2), \dots) = n.$$

**Problem 11** (China 2006). Let  $k \geq 3$  be an odd integer. Prove that there exists a  $k$ -th degree integer valued polynomial with non-integer coefficients that has the following properties: (1)  $f(0) = 0, f(1) = 1$ . (2) There exist infinitely many positive integers  $n$  so that if the equation:

$$n = f(x_1) + \dots + f(x_s)$$

has integer solutions  $x_1, \dots, x_s$ , then  $s \geq 2^k - 1$ .

**Problem 12** (Putnam 2000). Let  $f(x)$  be a polynomial with integer coefficients. Define a sequence  $a_0, a_1, \dots$  of integers such that  $a_0 = 0$  and  $a_{n+1} = f(a_n)$  for all  $n \geq 0$ . Prove that if there exists a positive integer  $M$  for which  $a_m = 0$  then either  $a_1 = 0$  or  $a_2 = 0$ .

**Problem 13** (IMO 2006). Let  $P(x)$  be a polynomial of degree  $n > 1$  with integer coefficients and let  $k$  be a positive integer. Consider the polynomial  $Q(x) = P(P(\dots P(P(x)) \dots))$ , where  $P$  occurs  $k$  times. Prove that there are at most  $n$  integers  $t$  such that  $Q(t) = t$ .

**Problem 14** (USAMO 1995). Suppose  $q_0, q_1, q_2, \dots$  is an infinite sequence of integers satisfying the following two conditions:

- (i)  $m - n$  divides  $q_m - q_n$  for  $m > n \geq 0$ ,
- (ii) there is a polynomial  $P$  such that  $|q_n| < P(n)$  for all  $n$

Prove that there is a polynomial  $Q$  such that  $q_n = Q(n)$  for all  $n$ .

**Problem 15** (TST 2002). Let  $p > 5$  be a prime number. For any integer  $x$ , define

$$f_p(x) = \sum_{k=1}^{p-1} \frac{1}{(px+k)^2}$$

Prove that for any pair of positive integers  $x, y$ , the numerator of  $f_p(x) - f_p(y)$ , when written as a fraction in lowest terms, is divisible by  $p^3$ .

**Problem 16** (Putnam 2008). Let  $p$  be a prime number. Let  $h(x)$  be a polynomial with integer coefficients such that  $h(0), h(1), \dots, h(p^2-1)$  are distinct modulo  $p^2$ . Show that  $h(0), h(1), \dots, h(p^3-1)$  are distinct modulo  $p^3$ .

**Problem 17** (USAMO 2006). For integral  $m$ , let  $p(m)$  be the greatest prime divisor of  $m$ . By convention, we set  $p(\pm 1) = 1$  and  $p(0) = \infty$ . Find all polynomials  $f$  with integer coefficients such that the sequence  $\{p(f(n^2)) - 2n\}_{n \geq 0}$  is bounded above. (In particular, this requires  $f(n^2) \neq 0$  for  $n \geq 0$ .)

**Problem 18** (TST 2008). Let  $n$  be a positive integer. Given an integer coefficient polynomial  $f(x)$ , define its signature modulo  $n$  to be the (ordered) sequence  $f(1), \dots, f(n)$  modulo  $n$ . Of the  $n^n$  such  $n$ -term sequences of integers modulo  $n$ , how many are the signature of some polynomial  $f(x)$  if

- (a)  $n$  is a positive integer not divisible by the square of a prime.
- (b)  $n$  is a positive integer not divisible by the cube of a prime.

**Problem 19** (TST 2009). For each positive integer  $n$ , let  $c(n)$  be the largest real number such that

$$c(n) \leq \left| \frac{f(a) - f(b)}{a - b} \right|$$

for all triples  $(f, a, b)$  such that

- $f$  is a polynomial of degree  $n$  taking integers to integers, and
- $a, b$  are integers with  $f(a) \neq f(b)$ .

Find  $c(n)$ .

**Problem 20** (TST 2007). For a polynomial  $P(x)$  with integer coefficients,  $r(2i - 1)$  (for  $i = 1, 2, 3, \dots, 512$ ) is the remainder obtained when  $P(2i - 1)$  is divided by 1024. The sequence

$$(r(1), r(3), \dots, r(1023))$$

is called the remainder sequence of  $P(x)$ . A remainder sequence is called complete if it is a permutation of  $(1, 3, 5, \dots, 1023)$ . Prove that there are no more than  $2^{35}$  different complete remainder sequences.

**Problem 21** (TST 2005). We choose a random monic polynomial with degree  $n$  and coefficients in the set  $1, 2, \dots, n!$ . Prove that the probability that this polynomial to be special is between 0.71 and 0.75, where a polynomial  $g$  is called special if for every  $k > 1$  in the sequence  $f(1), f(2), f(3), \dots$  there are infinitely many numbers relatively prime with  $k$ .

**Problem 22** (USAMO 1997). Prove that for any integer  $n$ , there exists a unique polynomial  $Q$  with coefficients in  $\{0, 1, \dots, 9\}$  such that  $Q(-2) = Q(-5) = n$ .

**Problem 23** (Czech-Slovak 1998). A polynomial  $P(x)$  of degree  $n \geq 5$  with integer coefficients and  $n$  distinct integer roots is given. Find all integer roots of  $P(P(x))$  given that 0 is a root of  $P(x)$ .