# QUANTUM HAMILTONIAN REDUCTION: PART 2

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## 1. Quantization of $\mathbb{C}[x_1, y_1, \ldots, x_n, y_n]^{S_n}$

1.1. **A $\mathfrak{gl}_n$-action on differential operators.** Recall the final result of the previous talk.

**Corollary 1.1.** Let $I_0 \subset A_0$ be the ideal generated by $\mu_0(\mathfrak{g})$. Let $x_1, \ldots, x_k$ be a basis for $\mathfrak{g}$. If $\mu_0(x_1), \ldots, \mu_0(x_k)$ form a regular sequence in $A_0$, then for any $\lambda$, the quantum reduction $R(\mathfrak{g}, A, \lambda)$ is a filtered quantization of the classical reduction $R(G, M, 0)$.

Fix $G = \mathrm{GL}_n$, $\mathfrak{g} = \mathfrak{gl}_n$, and $\mathfrak{h} \subset \mathfrak{g}$ a fixed choice of Cartan. Let $\mu : \mathfrak{g} \to D(\mathfrak{g} \times \mathbb{C}^n)$ be the quantum moment map given by differentiating the diagonal action of $G$ on $\mathfrak{g} \times \mathbb{C}^n$ given by $g \cdot (x, i) = (gxg^{-1}, g \cdot i)$. It extends to a quantum moment map $U(\mathfrak{g}) \to D(\mathfrak{g} \times \mathbb{C}^n)$ which has corresponding classical co-moment map $\mu_0 : \mathfrak{g} \to \mathbb{C}[T^*(\mathfrak{g} \times \mathbb{C}^n)]$. Recall from Barbara's talk that at $\lambda = 0$ the classical reduction along $\mu_0$ is

$$R(G, T^*(\mathfrak{g} \oplus \mathbb{C}^n)) := \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^{S_n}.$$

Recall further that Barbara showed the following fact.

**Proposition 1.2.** The space $\mathcal{M} = \{(A, B, i, j) \mid [A, B] + ij = 0\}$ is a complete intersection.

The defining ideal of $\mathcal{M}$ in $\mathbb{C}[T^*(\mathfrak{g} \times \mathbb{C}^n)]$ is generated by $\mu_0(\mathfrak{g})$, so in this setting, Corollary 1.1 provides a family of filtered quantizations $R(\mathfrak{g}, D(\mathfrak{g} \times \mathbb{C}^n), \lambda)$ of $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^{S_n}$ indexed by $\lambda \in \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \simeq \mathbb{C}$.

**Remark.** Recall the $(\mathbb{C}^*)^2$-action on $T^*(\mathfrak{g} \times \mathbb{C}^n)$ given by $(t_1, t_2) \cdot (A, B, i, j) = (t_1^{-1} A, t_2^{-1} B, t_1^{-1} i, t_2^{-1} j)$, which descends via reduction to the natural $(\mathbb{C}^*)^2$-action on $\mathrm{Sym}^n(\mathbb{C}^2)$ which scales each coordinate. In our context, it corresponds to the $(\mathbb{C}^*)^2$-action on $D_\hbar(\mathfrak{g} \times \mathbb{C}^n)$ given by $(t_1, t_2) \cdot (A, \partial_B, i, \partial_j) = (t_1^{-1} A, t_2^{-1} \partial_B, t_1^{-1} i, t_2^{-1} \partial_j)$.

1.2. **Identifying the quantum reduction.** In the remainder of the talk, we determine the quantum reduction $R(\mathfrak{g}, D(\mathfrak{g} \times \mathbb{C}^n), 0)$ at $\lambda = 0$ and identify it with $D(\mathfrak{h})^{S_n}$. Observe that equipping $D(\mathfrak{h})^{S_n}$ with the order filtration already makes it a filtered quantization of $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^{S_n}$; the following theorem identifies the two quantizations.

**Theorem 1.3.** The quantum reduction $R(\mathfrak{g}, D(\mathfrak{g} \oplus \mathbb{C}^n), 0)$ is isomorphic to $D(\mathfrak{h})^{S_n}$.

We will prove Theorem 1.3 in two stages. First, we reduce to a setting which is more convenient for explicit computation. Observe that $\mathfrak{z}$ acts non-trivially only on the second variable in $\mathfrak{g} \times \mathbb{C}^n$. This implies that

$$D(\mathfrak{g} \times \mathbb{C}^n)^{\mathfrak{z}} = D(\mathfrak{g}) \otimes \mathbb{C}[x_i \partial_j].$$

Define the quotient map $\widetilde{\pi} : D(\mathfrak{g} \times \mathbb{C}^n)^{\mathfrak{z}} \to D(\mathfrak{g})$.

**Lemma 1.4.** The map $\widetilde{\pi}$ factors through $R(\mathfrak{z}, D(\mathfrak{g} \times \mathbb{C}^n), 0)$.

*Proof.* This holds because $\mu(\mathfrak{z}) = \mathrm{span}(\sum_i x_i \partial_i) \subset \mathbb{C}[x_i \partial_j]$. □

Lemma 1.4 gives a map $R(\mathfrak{z}, D(\mathfrak{g} \times \mathbb{C}^n), 0) \to D(\mathfrak{g})$ of $\mathfrak{g}/\mathfrak{z}$-modules which induces a map

$$\pi : R(\mathfrak{g}, D(\mathfrak{g} \times \mathbb{C}^n), 0) \simeq R(\mathfrak{g}, R(\mathfrak{z}, D(\mathfrak{g} \times \mathbb{C}^n), 0), 0) \to R(\mathfrak{g}, D(\mathfrak{g}), 0)$$

between their $\mathfrak{g}$-reductions. We will now construct a map $\Phi : D(\mathfrak{g})^{\mathfrak{g}} \to D(\mathfrak{h}^{\mathrm{reg}})^W$, for which we recall the following classical facts.

**Lemma 1.5.** The restriction map $\phi : \mathbb{C}[\mathfrak{g}] \to \mathbb{C}[\mathfrak{h}]$ induces an isomorphism $\mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W$.

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We obtain a map $\widetilde{\Phi} : D(\mathfrak{g})^{\mathfrak{g}} \to D(\mathfrak{h}^{\text{reg}})^W$ as follows. By Lemma 1.5, we obtain an action of $D(\mathfrak{g})^{\mathfrak{g}}$ on $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \simeq \mathbb{C}[\mathfrak{h}]^W$. The map $\widetilde{\Phi}$ is defined as the composition of this action with the map $\text{Der}(\mathbb{C}[\mathfrak{h}]^W, \mathbb{C}[\mathfrak{h}]^W) \to D(\mathfrak{h}^{\text{reg}})^W$. For $f \in \mathbb{C}[\mathfrak{h}^{\text{reg}}]^W$, a lift $\widetilde{f} \in \phi^{-1}(\mathbb{C}[\mathfrak{h}^{\text{reg}}]^W)$ so that $\phi(\widetilde{f}) = f$, and $D \in D(\mathfrak{g} \times \mathbb{C}^n)^{\mathfrak{g}}$, this action satisfies

$$\widetilde{\Phi}(D)(f) = \phi(D(\widetilde{f})). \tag{1}$$

Now, define the map

$$\Phi = \delta(x) \circ \widetilde{\Phi} \circ \delta(x)^{-1}$$

to be the conjugation of $\widetilde{\Phi}$ by $\delta(x) = \prod_{\alpha>0}(\alpha, x)$.

**Proposition 1.6.** The map $\Phi$ factors through a map $\Psi : R(\mathfrak{g}, D(\mathfrak{g}), 0) \to D(\mathfrak{h}^{\text{reg}})^W$.

*Proof.* Because $\Phi$ is the composition of $\widetilde{\Phi}$ and conjugation by $\delta(x)$, it suffices to check that $\widetilde{\Phi}$ kills $(D(\mathfrak{g})\mu(x))^{\mathfrak{g}}$. This follows by (1) and the fact that $(\mu(x) \cdot f)(y) = \partial_{[x,y]} f(y) = 0$ for any $f \in D(\mathfrak{g})^{\mathfrak{g}}$. □

We now claim that the image of $\Psi \circ \pi$ lies in $D(\mathfrak{h})^W$. The proof relies on two technical lemmas, whose proofs we postpone until the end of the notes.

**Lemma 1.7.** The image of the Laplacian $\Delta_{\mathfrak{g}} = \sum_i \partial_{z_i}^2$ for $\{z_i\}$ an orthonormal basis of $\mathfrak{g}$ is given by $\Phi(\Delta_{\mathfrak{g}}) = \Delta_{\mathfrak{h}}$.

**Lemma 1.8.** The Poisson algebra $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W$ is generated by $\mathbb{C}[\mathfrak{h}]^W$ and $p_2 = \sum_i y_i^2 \in \mathbb{C}[\mathfrak{h}^*]^W$.

**Proposition 1.9.** The image of $\Psi \circ \pi$ lies in $D(\mathfrak{h})^W \subset D(\mathfrak{h}^{\text{reg}})^W$.

*Proof.* The proposition follows from the following three steps. First, the restriction of $\Psi \circ \pi$ to $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ coincides with the identification of the Chevalley isomorphism, so the proposition holds on $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$. Second, by Lemma 1.7 the proposition holds for the Laplacian $\Delta_{\mathfrak{g}} \otimes 1$. Finally, the top degree terms of $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ and $\Delta_{\mathfrak{g}} \otimes 1$ generate $\text{gr}\, R(\mathfrak{g}, D(\mathfrak{g} \times \mathbb{C}^n), 0) = \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W$ as a Poisson algebra by Lemma 1.8, so $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$ and $\Delta_{\mathfrak{g}} \otimes 1$ generate $R(\mathfrak{g}, D(\mathfrak{g} \times \mathbb{C}^n), 0)$, yielding the conclusion. □

*Proof of Theorem 1.3.* We must check that the composition

$$\Psi \circ \pi : R(\mathfrak{g}, D(\mathfrak{g} \times \mathbb{C}^n), 0) \to R(\mathfrak{g}, D(\mathfrak{g}), 0) \to D(\mathfrak{h})^W$$

is an isomorphism. By construction it is compatible with the order filtration on both sides, so it suffices to check that the Poisson map $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W \to \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W$ given by the associated graded is the identity. On $\text{gr}(\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}) \simeq \mathbb{C}[\mathfrak{h}]^W$, this follows from the fact that $\Phi$ is simply the Chevalley isomorphism on $\mathbb{C}[\mathfrak{g}]^{\mathfrak{g}}$. For $p_2 = \sum_i y_i^2$, this follows because $\Psi(\pi(\Delta_{\mathfrak{g}} \otimes 1)) = \Psi(\Delta_{\mathfrak{g}}) = \Delta_{\mathfrak{h}}$ by Lemma 1.7. The conclusion follows by Lemma 1.8. □

## 1.3. **A few technical proofs of lemmas.**

*Proof of Lemma 1.8.* Define the mixed power sum $p_{a,b} := \sum_i x_i^a y_i^b$, and let $V_n := \text{span}\{p_{a,b} \mid a + b = n\}$. For $p_2 = p_{0,2}$ and $q_2 = p_{2,0}$, notice that

$$\{p_2, p_{a,b}\} = a p_{a-1,b+1} \text{ and } \{q_2, p_{a,b}\} = b p_{a+1,b-1},$$

which shows that $p_2$, $q_2$, and $h_2(p_{a,b}) = (a - b)p_{a,b}$ form an irreducible representation of $\mathfrak{sl}_2$ on $V_n$. For each $n$, by the given we see that $p_{n,0}$ lies in the desired Poisson span, so we conclude that all of $V_n$ does. In particular, each $p_{a,b}$ lies in the span.

We now claim that $p_{a,b}$ generate $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W = \text{Sym}^n(\mathbb{C}[x, y])$ as an associative algebra. This follows from Lemma 1.10 below applied to $A = \mathbb{C}[x, y]$. □

**Lemma 1.10.** For any $\mathbb{C}$-algebra $A$, elements of the form

$$s(a) = \sum_i 1^{\otimes(i-1)} \otimes a \otimes 1^{\otimes(n-i)}$$

generate $\text{Sym}^n(A)$.

*Proof.* As a vector space, $\text{Sym}^n(A)$ is spanned by elements of the form $a^{\otimes n}$ for $a \in A$. Therefore, it suffices to check the conclusion for $A = \mathbb{C}[x]$, where it reduces to the statement that the ring of symmetric polynomials in $x_1, \ldots, x_n$ is generated by the power sums in $x_1, \ldots, x_n$. □

*Proof of Lemma 1.7.* The proof is by explicit computation. First, notice that

$$\Delta_{\mathfrak{g}} = \left( \sum_i \partial_{x_i}^2 + 2 \sum_{\alpha > 0} \partial_{f_\alpha} \partial_{e_\alpha} \right)$$

for $(e_\alpha, f_\alpha) = 1$. For $\widetilde{f} \in \mathbb{C}[\mathfrak{g} \times \mathbb{C}^n]^{\mathfrak{g}}$ so that $\phi(\widetilde{f}) = f$, we see that

$$\phi(\Delta_{\mathfrak{g}}(\widetilde{f})) = \sum_i \partial_{x_i}^2 f + 2 \sum_{\alpha > 0} \phi\left(\partial_{e_\alpha} \partial_{f_\alpha} \widetilde{f}\right).$$

We may compute

$$\begin{aligned} \partial_{e_\alpha} \partial_{f_\alpha} \widetilde{f}(x) &= \partial_t \partial_s|_{t=s=0} \widetilde{f}(x + t f_\alpha + s e_\alpha) \\ &= \partial_{ts}|_{t=s=0} \widetilde{f}\Big(\mathrm{Ad}_{e^{s\alpha(x)^{-1} e_\alpha}}(x + t f_\alpha + s e_\alpha)\Big) \\ &= \partial_{ts}|_{t=s=0} \widetilde{f}\Big(x + t f_\alpha + ts\alpha(x)^{-1} h_\alpha + o(t^2, s^2, ts)\Big) \\ &= \alpha(x)^{-1} \partial_{h_\alpha} f(x) \end{aligned}$$

for $h_\alpha = [e_\alpha, f_\alpha]$. Putting everything together yields

$$\widetilde{\Phi}(\Delta_{\mathfrak{g}})(f) = \phi(\Delta_{\mathfrak{g}}(\widetilde{f})) = \Delta_{\mathfrak{h}} f + 2 \sum_{\alpha > 0} \alpha(x)^{-1} \partial_{h_\alpha} f.$$

Observe also that

$$\begin{aligned} \delta(x)^{-1} \Delta_{\mathfrak{h}}(\delta(x) f) &= \Delta_{\mathfrak{h}}(f) + \delta(x)^{-1} \Delta_{\mathfrak{h}}(\delta(x)) \cdot f + \sum_i \delta(x)^{-1} \partial_{x_i}(\delta(x)) \partial_{x_i}(f) \\ &= \Delta_{\mathfrak{h}} f + \sum_i \partial_{x_i} f \sum_{j \neq i} \frac{(-1)^{1_{i<j}}}{x_i - x_j} \\ &= \Delta_{\mathfrak{h}} f + 2 \sum_{\alpha > 0} \alpha(x)^{-1} \partial_{h_\alpha} f, \end{aligned}$$

where $\Delta_{\mathfrak{h}} \delta(x) = 0$ because it is a $W$-antisymmetric polynomial of smaller degree than $\delta(x)$. Conjugating this by $\delta(x)$ shows that $\Phi(\Delta_{\mathfrak{g}}) = \Delta_{\mathfrak{h}}$. □

## References


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