Grothendieck's simultaneous resolution and the Springer correspondence

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1 The Springer resolution

1.1 Symplectic structure on the cotangent bundle

Let G be a semisimple algebraic group with Lie algebra $\mathfrak{g}$. Fix a Borel subgroup B of G, and let $\mathcal{B} = G/B$ be the corresponding flag variety. The cotangent bundle $T^*\mathcal{B}$ comes equipped with symplectic form $\omega = d\lambda$ defined as follows. For $x \in \mathcal{B}$ and $\alpha \in T_x^*\mathcal{B}$, let $\pi : T^*\mathcal{B} \to \mathcal{B}$ be the projection and $\pi_* : T_\alpha T^*\mathcal{B} \to T_x\mathcal{B}$ be its differential. Then, for $\xi \in T_\alpha T^*\mathcal{B}$, we set

$$\langle \lambda, \xi \rangle = \langle \alpha, \pi_*(\xi) \rangle.$$

If q_i, p_i are dual coordinates on $T^*\mathcal{B}$ (with q_i coordinates on $\mathcal{B}$), then $\lambda = \sum_i p_i dq_i$ and therefore

$$\omega = \sum_i dp_i \wedge dq_i.$$

1.2 Poisson structure on coadjoint orbits

Recall that a coadjoint orbit $\mathcal{O}$ of G on $\mathfrak{g}^*$ is equipped with the Kirillov-Kostant-Souriau symplectic structure given as follows. For $\alpha \in \mathcal{O}$, we have $\mathcal{O} \simeq G/G^\alpha$, where G^α is the stabilizer of α under G. This means that

$$T_\alpha \mathcal{O} = \mathfrak{g}/\mathfrak{g}^\alpha,$$

where $\mathfrak{g}^\alpha = \{x \in \mathfrak{g} \mid \alpha([x, -]) = 0\}$ is the Lie algebra of G^α. The Poisson bivector ω on $\mathfrak{g}^*$ is defined by

$$\omega_\alpha(x, y) = \alpha([x, y]).$$

Evidently, ω is skew-symmetric.

Exercise. The Poisson bivector ω is non-degenerate when restricted to each $\mathcal{O}$, and its dual on $\mathcal{O}$ is a symplectic form.

1.3 Moment map

We recall briefly the definition of a moment map. Let (M, ω) be a symplectic manifold, with a G-action preserving the symplectic form (meaning that $\omega(x, y) = \omega(gx, gy)$ for $x, y \in T_m M$). The action is *Hamiltonian* if there is a Lie algebra map $\mathfrak{g} \to \mathcal{O}(M)$ given by $x \mapsto H_x$ so that the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[d, "H"] \arrow[dr] & \\ \mathcal{O}(M) \arrow[r] & \text{symplectic vector fields on } M \end{tikzcd}$$

commutes and so that the *moment map* $\mu : M \to \mathfrak{g}^*$ is given by

$$\mu(m) = \Big(x \mapsto H_x(m)\Big)$$

is G-equivariant. Here, the diagonal arrow is given by the differential of the G-action and the horizontal map by sending f to ξ_f (which is defined by $\omega(-, \xi_f) = \langle -, df\rangle$).

Exercise. The pullback $\mu^* : \mathcal{O}(\mathfrak{g}^*) \to \mathcal{O}(M)$ is a map of Poisson algebras.

Exercise. If G is connected, then μ is automatically G-equivariant with respect to the adjoint action of G on $\mathfrak{g}^*$, so this assumption can be removed from the definition of a Hamiltonian action.

1.4 Resolution of the nilpotent cone

Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ via the Killing form so that $\mathfrak{g}$ inherits a Poisson structure from $\mathfrak{g}^*$. Recall that $x \in \mathfrak{g}$ is *nilpotent* if $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is nilpotent, and let $\mathcal{N} \subset \mathfrak{g}$ denote the cone of nilpotent elements. Now, define

$$\widetilde{\mathcal{N}} = \{(x, \mathfrak{b}) \mid x \in \mathcal{N}, \mathfrak{b} \ni x\} \subset \mathcal{N} \times \mathcal{B},$$

where we now view $\mathcal{B}$ as the variety of Borel subalgebras. The second projection $\widetilde{\mathcal{N}} \to \mathcal{B}$ is a vector bundle with fiber $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ above each $\mathfrak{b}$; on the other hand, we see that $T^*_{\mathfrak{b}}\mathcal{B} \simeq \mathfrak{n}$ for each $\mathfrak{b} \in \mathcal{B}$, which shows that $\widetilde{\mathcal{N}} \simeq T^*\mathcal{B}$. For $\xi \in \mathcal{N}$, write $\mathcal{B}_\xi = \pi^{-1}(\xi)$.

Proposition 1.1. The map $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is a resolution of singularities.

Proof. Recall that an element is regular if its centralizer has minimal dimension. Note that regular nilpotent elements form a Zariski dense set in $\mathcal{N}$; further, any regular nilpotent element is contained in a unique Borel subalgebra, hence π is birational on $\mathcal{N}_{\mathrm{reg}}$, as needed. □

Corollary 1.2. We have $\dim \mathcal{N} = 2 \dim \mathcal{B}$.

The map of Proposition 1.1 is known as Springer's resolution.

Exercise. Springer's resolution corresponds to the moment map $T^*\mathcal{B} \to \mathfrak{g}^*$ induced by the G-action on $T^*\mathcal{B}$.

Remark. The corresponding map

$$\mathcal{O}(\mathfrak{g}) \to \mathcal{O}(\mathcal{N}) \to \mathcal{O}(T^*\mathcal{B})$$

of Poisson algebras may be quantized to a map

$$U(\mathfrak{g}) \to U(\mathfrak{g})/Z(\mathfrak{g}) \to \mathcal{D}_{\mathcal{B}},$$

where $\mathcal{D}_{\mathcal{B}}$ denotes the global sections of the sheaf of differential operators on $\mathcal{B}$. This connection will be pursued further in later lectures.

2 Grothendieck's simultaneous resolution

2.1 The simultaneous resolution

We may generalize the resolution of Proposition 1.1 as follows. Define the subvariety

$$\widetilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \mid x \in \mathfrak{g}, \mathfrak{b} \ni x\} \subset \mathfrak{g} \times \mathcal{B}$$

and equip it with the maps $\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ and $\theta : \widetilde{\mathfrak{g}} \to \mathcal{B}$ given by projection on the first and second coordinates. Evidently, θ makes $\widetilde{\mathfrak{g}}$ a vector bundle over $\mathcal{B}$ with fiber $\mathfrak{b}$, so $\widetilde{\mathfrak{g}}$ is smooth. Equip $\widetilde{\mathfrak{g}}$ also with the map $\psi : \widetilde{\mathfrak{g}} \to \mathfrak{h}$

given by $\psi(x, \mathfrak{b}) = x \pmod{[\mathfrak{b}, \mathfrak{b}]}$ in $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \simeq \mathfrak{h}$. Consider now the diagram below.

$$\begin{tikzcd}
 & \widetilde{\mathfrak{g}} \arrow[r, "\psi"] \arrow[d, "\pi"'] & \mathfrak{h} \arrow[d] \\
\widetilde{\mathcal{N}} \arrow[ur] \arrow[d, "\pi"'] & \mathfrak{g} \arrow[r, "\phi"] & \mathfrak{h}//W \\
\mathcal{N} \arrow[ur] \arrow[r] & \{0\} \arrow[ur] &
\end{tikzcd} \tag{1}$$

Here, $W \simeq N_G(\mathfrak{h})/\mathfrak{h}$ is the Weyl group of G, and $\phi : \mathfrak{g} \to \mathfrak{h}//W$ denotes the Chevalley map, which originates from the map of algebras

$$\mathbb{C}[\mathfrak{g}] \leftarrow \mathbb{C}[\mathfrak{g}]^G \simeq \mathbb{C}[\mathfrak{h}]^W,$$

where the latter isomorphism is realized by restriction from $\mathfrak{g}$ to $\mathfrak{h}$ (viewed as a subspace of $\mathfrak{g}$ here). We summarize the properties of this diagram below.

Proposition 2.1. The following properties hold:

(a) diagram (1) commutes, and

(b) for each $x \in \mathfrak{h}$, the map $\pi : \psi^{-1}(x) \to \phi^{-1}(x)$ is a resolution of singularities.

Observe that Proposition 1.1 is a special case of Proposition 2.1(b) with $x = 0$.

2.2 The regular locus

Recall that an element $x \in \mathfrak{g}$ is called *regular* if its centralizer

$$\mathfrak{z}_{\mathfrak{g}}(x) = \{y \in \mathfrak{g} \mid [y, x] = 0\}$$

has minimal dimension $\mathrm{rank}\mathfrak{g} = \dim \mathfrak{h}$ and *semisimple* if $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ is semisimple as a linear map. Let $\mathfrak{g}_{\mathrm{reg}}$ denote the locus of regular elements and $\mathfrak{g}_{\mathrm{sr}}$ denote the locus of semisimple regular elements.

Exercise. Show that $\mathrm{codim}(\mathfrak{g} - \mathfrak{g}_{\mathrm{reg}}) \geq 3$.

Proposition 2.2. The restriction to the regular locus

$$\begin{tikzcd}
\widetilde{\mathfrak{g}}_{\mathrm{reg}} \arrow[r, "\psi"] \arrow[d, "\pi"'] & \mathfrak{h} \arrow[d] \\
\mathfrak{g}_{\mathrm{reg}} \arrow[r, "\phi"] & \mathfrak{h}//W
\end{tikzcd}$$

of Grothendieck's resolution is Cartesian.

Proof. We will prove this for $\mathfrak{g}_{\mathrm{sr}}$ instead of $\mathfrak{g}_{\mathrm{reg}}$. In this case, $\mathfrak{h}_{\mathrm{sr}} \to \mathfrak{h}_{\mathrm{sr}}//W$ is a ramified covering with group of deck transformations W. On the other hand, if $x \in \mathfrak{g}_{\mathrm{sr}}$, it lies in a unique Cartan subalgebra $\mathfrak{h} = Z_{\mathfrak{g}}(x)$, hence any Borel containing x must contain $\mathfrak{h}$, so the set of Borels containing $\mathfrak{h}$ also gives a W-covering. $\square$

In light of Proposition 2.2, we see that $\widetilde{\mathfrak{g}}_{\mathrm{sr}} \to \mathfrak{g}_{\mathrm{sr}}$ is a $|W|$-to-1 covering map. In particular, this means that $\widetilde{\mathfrak{g}}_{\mathrm{sr}}$ is equipped with a W-action. This action does not extend to all of $\widetilde{\mathfrak{g}}$, however.

2.3 The Steinberg variety

The Steinberg variety is defined as the fiber product

$$Z = \widetilde{\mathcal{N}} \underset{\mathcal{N}}{\times} \widetilde{\mathcal{N}} = \{(x, \mathfrak{b}, \mathfrak{b}') \mid x \in \mathfrak{b}, x \in \mathfrak{b}'\}.$$

Observe that Z is equipped with a natural map $i : Z \to \mathcal{B} \times \mathcal{B}$. Recalling the Schubert decomposition

$$\mathcal{B} \times \mathcal{B} = \bigsqcup_{w \in W} \mathcal{O}(w) = \bigsqcup_{w \in W} G \cdot (B, wB),$$

we define the variety $Z_w = i^{-1}(\mathcal{O}(w))$. We identify Z_w as a subvariety of $T^*\mathcal{B} \times T^*\mathcal{B}$. Now, consider the isomorphism

$$T^*\mathcal{B} \times T^*\mathcal{B} \to T^*(\mathcal{B} \times \mathcal{B})$$

given by

$$\Big((x_1, \mathfrak{b}_1), (x_2, \mathfrak{b}_2)\Big) \mapsto \Big((x_1, -x_2), (\mathfrak{b}_1, \mathfrak{b}_2)\Big).$$

Note that there is a sign change applied to x_2.

Proposition 2.3. Viewed as a subvariety of $T^*(\mathcal{B} \times \mathcal{B})$, Z_w is the conormal bundle to $\mathcal{O}(w)$.

Proof. The fiber of the conormal bundle of $\mathcal{O}(w)$ at a point $\alpha = (\mathfrak{b}, \mathfrak{b}') \in \mathcal{B} \times \mathcal{B}$ consists of $\Big((x_1, x_2), (\mathfrak{b}_1, \mathfrak{b}_2)\Big)$ with $x_1 \in [\mathfrak{b}_1, \mathfrak{b}_1], x_2 \in [\mathfrak{b}_2, \mathfrak{b}_2]$ such that (x_1, x_2) annihilates the tangent space $T_\alpha \mathcal{O}(w) \simeq \mathfrak{g}/\mathfrak{b} \times \mathfrak{g}/\mathfrak{b}'$. This condition is equivalent to $x_1 + x_2 = 0$, hence coincides with the fiber of Z_w under the identification above. □

Proposition 2.4. The irreducible components of Z are $\overline{Z}_w$ for $w \in W$.

Proof. The Z_w partition Z, hence it suffices to check that they are irreducible of the same dimension, which follows Proposition 2.3. □

On the other hand, for an orbit $\mathcal{O} \subset \mathcal{N}$, let $Z_{\mathcal{O}}$ denote its preimage under the map $Z \to \mathcal{N}$. Assuming the following technical lemma, we now undertake a more detailed analysis of the dimension of $Z_{\mathcal{O}}$.

Lemma 2.5 (Chriss-Ginzburg Theorem 3.3.7)**.** For any $\mathfrak{n} \subset \mathfrak{g}$, each irreducible component of $\mathcal{O} \cap \mathfrak{n}$ is a Lagrangian subvariety of $\mathcal{O}$, hence has dimension $\frac{1}{2} \dim \mathcal{O}$.

Lemma 2.6. Each $Z_{\mathcal{O}}$ has dimension $\dim Z_{\mathcal{O}} = \dim Z = \dim \mathcal{N}$.

Proof. Let $\widetilde{\mathcal{O}} = \pi^{-1}(\mathcal{O})$ be the preimage of $\mathcal{O}$ in $\widetilde{\mathcal{N}}$. Viewed as a subvariety of $\widetilde{\mathcal{N}}$, we see that

$$\widetilde{\mathcal{O}} = G \times^B (\mathcal{O} \cap \mathfrak{n}),$$

which is a fibration over $\mathcal{B}$ with fiber $\mathcal{O} \cap \mathfrak{n}$. By Lemma 2.5, this implies that $\widetilde{\mathcal{O}}$ has pure dimension

$$\dim \widetilde{\mathcal{O}} = \dim \mathcal{B} + \frac{1}{2} \dim \mathcal{O}.$$

Finally, because $\widetilde{\mathcal{O}} \to \mathcal{O}$ is a fiber bundle, we see that $Z_{\mathcal{O}} \simeq \widetilde{\mathcal{O}} \underset{\mathcal{O}}{\times} \widetilde{\mathcal{O}}$ has pure dimension

$$\dim Z_{\mathcal{O}} = 2 \dim \widetilde{\mathcal{O}} - \dim \mathcal{O} = 2 \dim \mathcal{B} = \dim \mathcal{N},$$

where the last equality follows from Corollary 1.2. □

Because each $Z_{\mathcal{O}}$ is top dimensional in Z, the irreducible components of Z divide into irreducible components of $Z_{\mathcal{O}}$ for some $\mathcal{O}$. To understand the irreducible components of $Z_{\mathcal{O}}$, for $\xi \in \mathcal{O}$ let $G(\xi)$ be the stabilizer of ξ and let $\mathcal{B}_\xi$ be the set of Borel subalgebras containing ξ. Then we have

$$Z_{\mathcal{O}} \simeq G \times^{G(\xi)} (\mathcal{B}_\xi \times \mathcal{B}_\xi).$$

The following description of the irreducible components of $Z_{\mathcal{O}}$ now follows formally.

Proposition 2.7. The irreducible components of $Z_{\mathcal{O}}$ are indexed by orbits of $C(\xi) = G(\xi)/G^0(\xi)$ on pairs of irreducible components of $\mathcal{B}_\xi$.

Corollary 2.8. All irreducible components of $\mathcal{B}_\xi$ have the same dimension, which is determined by $\dim Z_{\mathcal{O}} = \dim \mathcal{O} + 2\dim \mathcal{B}_\xi$.

Proof. Combine Proposition 2.7 and Lemma 2.6. □

3 Springer representations

3.1 Preliminaries on Borel-Moore homology

In this talk, we consider topological spaces X which satisfy the following technical conditions.

- X is locally compact,
- X has the homotopy type of a finite CW-complex, and
- X admits a closed embedding into a C^∞ manifold.

We will not concern ourselves too much with these assumptions but note only that any complex algebraic variety will satisfy them. We now define Borel-Moore homology, which will be our main geometric tool.

For a space X, we define the complex $C_*^{BM}(X)$ of infinite singular chains

$$\sum_{i=0}^{\infty} c_i \sigma_i,$$

where the σ_i are singular chains of the same dimension and where any compact set $D \subset X$ intersects the support of only finitely many of the σ_i with c_i non-zero. The *Borel-Moore homology* $H_\star^{BM}(X)$ of X is defined to be the homology of $C_*^{BM}(X)$ under the standard boundary map.

Remark. There are a number of equivalent definitions of Borel-Moore homology; we list a few useful ones below.

- If $\overline{X} \supset X$ is compact so that $(\overline{X}, \overline{X} \setminus X)$ is a CW-pair, then
$$H_*^{BM}(X) = H_*(\overline{X}, \overline{X} \setminus X).$$
- If M is a smooth oriented manifold with $\dim M = m$ in which X is a closed subset, then
$$H_*^{BM}(X) = H^{m-*}(M, M - X).$$
This is a version of Poincare duality for Borel-Moore homology.
- Recall that for $p : X \to \{\text{pt}\}$, the *dualizing sheaf* $\mathbb{D}_X$ on X is the complex $\mathbb{D}_X = p^!(\mathbb{C})$ of constructible sheaves on X.[1] Then we have
$$H_d^{BM}(X) = H^{-d}(X, \mathbb{D}_X).$$

Because taking preimages preserves compact sets for proper maps, they induce valid pushforwards in Borel-Moore homology. Now, suppose that $V \xrightarrow{j} X \xleftarrow{i} U$ is a decomposition of X into an open subset U and its closed complement $V = X - U$. Then we have a pullback map $i^* : H_*^{BM}(X) \to H_*^{BM}(U)$ and a pushforward map $j_* : H_*^{BM}(V) \to H_*^{BM}(X)$.

[1] Here, we work in $D_c^b(X)$, the bounded derived category of constructible sheaves on X. For a map $f : X \to Y$, the *exceptional inverse image* $f^!$ is a functor $D_c^b(Y) \to D_c^b(X)$ given by $f^! = f^* R\Gamma_X$, where Γ_X is the functor of sections supported on X. If f is smooth of relative dimension d, then $f^! = f^*[2d]$.

3.2 Fundamental classes

If X is a smooth oriented manifold with $\dim X = n$, then taking a (possibly infinite) CW-complex decomposition of X yields a non-trivial cycle in $H_n^{BM}(X)$, which is known as the *fundamental class* $[X]$ of X. Such a class only exists for compact manifolds in ordinary homology, but allowing infinite chains allows us to extend it in Borel-Moore homology.

If X is a complex algebraic variety of complex dimension n, this construction works without the the smoothness condition. For X singular, let $X^{\text{reg}} \subset X$ be the (open dense) non-singular locus with fundamental class $[X^{\text{reg}}] \in H_{2n}^{BM}(X^{\text{reg}})$. Then $X - X^{\text{reg}}$ has real codimension at least 2, hence the long exact sequence in relative homology and the second definition of Borel-Moore homology shows that the restriction $H_{2n}^{BM}(X) \to H_{2n}^{BM}(X^{\text{reg}})$ is an isomorphism. We take $[X]$ to be the preimage of $[X^{\text{reg}}]$ under this restriction.

This construction allows us a convenient geometric description of the top dimensional Borel-Moore homology of a complex algebraic variety.

Proposition 3.1. Let X be a complex algebraic variety of complex dimension n with top dimensional irreducible components $X_1, \dots, X_m$. Then the top dimensional Borel-Moore homology $H_{2n}^{BM}(X)$ of X has basis $[X_1], \dots, [X_m]$, where $[X_i]$ is the (pushforward of) the fundamental class of X_i.

3.3 Kunneth formula, smooth pullback, and intersection product

We discuss now a few operations on Borel-Moore homology which are necessary for our next construction. If M_1 and M_2 are two spaces, then there is a Kunneth isomorphism

$$\boxtimes : H_i^{BM}(M_1) \otimes H_j^{BM}(M_2) \to H_{i+j}^{BM}(M_1 \times M_2)$$

which is defined similarly to ordinary homology on chains. The fact that it is an isomorphism can be seen from the relative Kunneth formula in ordinary homology.

Using this isomorphism, for a trivial fibration $\pi : B \times F \to B$ with F smooth oriented of real dimension $\dim F = d$, we may define the *smooth pullback* $\pi^* : H_*^{BM}(B) \to H_{*+d}^{BM}(B)$ by $\pi^*(-) = - \boxtimes [F]$, where $[F]$ is the fundamental class of F. It is possible to define this map more generally on any locally trivial fibration with oriented fibers, but we mention only that it restricts to the map we have described over any open set where the fibration is trivial.

Finally, for M_1, M_2 closed subsets of a smooth oriented manifold M with $\dim M = m$, consider the relative cup product

$$H^{m-i}(M, M-M_1) \otimes H^{m-j}(M, M-M_2) \to H^{2m-i-j}(M, (M-M_1) \cup (M-M_2)) = H^{2m-i-j}(M, M-(M_1 \cap M_2)).$$

Under Poincare duality, this becomes the operation

$$\cap : H_i^{BM}(M_1) \otimes H_j^{BM}(M_2) \to H_{i+j-m}^{BM}(M_1 \cap M_2),$$

which is known as the *intersection product* in Borel-Moore homology. We note that if $[N_1]$ and $[N_2]$ are two fundamental classes which intersect transversely, then the intersection product satisfies

$$[N_1] \cap [N_2] = [N_1 \cap N_2].$$

3.4 Borel-Moore homology as a convolution algebra

If $f : X \to Y$ is a proper map of complex varieties with X non-singular, then letting $Z = X \underset{Y}{\times} X$, we give $H_*^{BM}(Z)$ an algebra structure as follows. Let $\pi_{ij} : X \underset{Y}{\times} X \underset{Y}{\times} X \to Z$ denote the projection in the i^{th} and j^{th} coordinates. Then for $\alpha \in H_i^{BM}(Z)$ and $\beta \in H_j^{BM}(Z)$, we define their *convolution product* to be

$$\alpha \star \beta = (\pi_{13})_*(\pi_{12}^*(\alpha) \cap \pi_{23}^*(\beta)) \in H_{i+j-\dim X}^{BM}(Z),$$

where π_{12}^* and π_{23}^* are the smooth pullbacks $H_*^{BM}(Z) \to H_*^{BM}(Z \times X)$ and $H_*^{BM}(Z) \to H_*^{BM}(X \times Z)$ and $\cap$ is the intersection product. If $\alpha = [M_1]$ and $\beta = [M_2]$ are cycles such that $\pi_{12}^{-1}(M_1)$ and $\pi_{23}^{-1}(M_2)$ intersect tranversely, then $\alpha \star \beta$ is the class of the set-theoretic convolution of M_1 and M_2.

We may check that this product is associative and that the fundamental class of the diagonal $\Delta \subset Z$ is an identity element. This endows $H_*^{BM}(Z)$ with its *convolution algebra* structure. If X has complex dimension n, we note that $H_{2n}^{BM}(Z)$ is a subalgebra of $H_*^{BM}(Z)$.

3.5 Convolution structure of the Steinberg variety

We now specialize to our specific situation. By the previous subsection, the Borel-Moore homology $H^{BM}_{\dim_{\mathbb{R}} \widetilde{\mathcal{N}}}(Z)$ of the Steinberg variety Z is endowed with a natural convolution algebra structure. Recall that Z, $\widetilde{\mathcal{N}}$, and $\mathcal{N}$ have the same dimension, hence by Proposition 3.1, $H^{BM}_{\dim_{\mathbb{R}} \widetilde{\mathcal{N}}}(Z)$ has a natural basis given by the fundamental classes of the irreducible components of Z. By Proposition 2.4, these irreducible components $\overline{Z}_w$ are labeled by elements $w \in W$, suggesting the following theorem.

Theorem 3.2. There is an isomorphism of algebras

$$\mathbb{C}[W] \xrightarrow{\sim} H^{BM}_{\dim_{\mathbb{R}} \widetilde{\mathcal{N}}}(Z).$$

Remark. We caution that the isomorphism of Proposition 3.2 is *not* given by $w \mapsto [\overline{Z}_w]$, where $\overline{Z}_w$ are the components of Proposition 2.4.

We now construct the map of Theorem 3.2. For $w \in W$, define $\Lambda^w \subset \widetilde{\mathfrak{g}} \underset{\mathfrak{g}}{\times} \widetilde{\mathfrak{g}}$ by

$$\Lambda^w = \{(x, \mathfrak{b}, \mathfrak{b}') \mid (\mathfrak{b}, \mathfrak{b}') \in \mathcal{O}(w), x \in \mathfrak{b} \cap \mathfrak{b}'\},$$

where $\mathcal{O}(w) = G \cdot (B, wB)$ is the Bruhat cell corresponding to W. For $(\mathfrak{b}, \mathfrak{b}') \in \mathcal{O}(w)$, $\mathfrak{b} \cap \mathfrak{b}'$ is the Lie algebra of the stabilizer of the G-action on $\mathcal{O}(w)$, meaning that Λ^w is a vector bundle over $\mathcal{O}(w)$ and in particular has dimension $\dim G$. We thus obtain a decomposition

$$\widetilde{\mathfrak{g}} \underset{\mathfrak{g}}{\times} \widetilde{\mathfrak{g}} = \bigsqcup_{w \in W} \Lambda^w$$

of $\widetilde{\mathfrak{g}} \underset{\mathfrak{g}}{\times} \widetilde{\mathfrak{g}}$ into irreducible components. For $h \in \mathfrak{h}$, let $\widetilde{\mathfrak{g}}_h$ be the fiber of $\widetilde{\mathfrak{g}}$ above $h \in \mathfrak{h}$ (recalling the map $\widetilde{\mathfrak{g}} \to \mathfrak{h}$), and let Λ^w_h be the preimage of Λ^w above $\widetilde{\mathfrak{g}}_h$ along the second projection.

If $h \in \mathfrak{h}_{\mathrm{sr}}$, then observe that Λ^w_h is the graph of the map $\widetilde{\mathfrak{g}}_h \to \widetilde{\mathfrak{g}}_{w(h)}$ induced by the W-action on $\widetilde{\mathfrak{g}}_{\mathrm{sr}}$. Therefore, in the Borel-Moore homology of $\widetilde{\mathfrak{g}} \underset{\mathfrak{g}}{\times} \widetilde{\mathfrak{g}}$, we have

$$[\Lambda^{yw}_h] = [\Lambda^y_{w(h)}] \star [\Lambda^w_h] \tag{2}$$

for every $h \in \mathfrak{h}_{\mathrm{sr}}$. To proceed, we will degenerate each class $[\Lambda^w_h]$ as $h \to 0$ to a class in $\Lambda^w_0 \simeq Z$. For this, we must discuss another construction in Borel-Moore homology.

3.6 Specialization in Borel-Moore homology

Fix a base space S which is a smooth manifold of real dimension d and a point $o \in S$; denote $S - \{o\}$ by S^*. For a map $\pi : Z \to S$, write $Z_o = \pi^{-1}(o)$ and $Z^* = \pi^{-1}(S^*)$ for the special fiber and its neighborhood. Suppose that $\pi : Z^* \to S^*$ is a locally trivial fibration with fiber F. We will construct a specialization map

$$H_*^{BM}(Z^*) \to H_{*-d}^{BM}(Z_o)$$

to the Borel-Moore homology of the special fiber. For this, we may assume that $(S, o) = (\mathbb{R}^d, 0)$. In this case, write S_+ for the positive half plane in the first coordinate, I_+ for the positive first coordinate axis, and $I_{\geq 0}$ for the non-negative first coordinate axis. The specialization is then given by the composition

$$\begin{aligned} H_*^{BM}(Z^*) &\to H_*^{BM}(\pi^{-1}(S_+)) \simeq H_{*-d}^{BM}(F) \otimes H_d^{BM}(S_+) \\ &\to H_{*-d}^{BM}(F) \otimes H_1^{BM}(I_+) \simeq H_{*-d+1}^{BM}(\pi^{-1}(I_+)) \to H_{*-d}^{BM}(Z_o), \end{aligned}$$

where the first map is given by restriction, the middle maps by the Kunneth theorem, and the last map by the exact sequence of the pair $(\pi^{-1}(I_{\geq 0}), \pi^{-1}(I_+))$. It is known that the specialization map is independent of the choices of coordinates made above; further, it is known that specialization is compatible with a convolution structure on Z.

For a fixed $w \in W$, choose a 2-dimensional real subspace $\mathfrak{l} \subset \mathfrak{h}$ whose non-zero elements lie in $\mathfrak{h}_{\mathrm{sr}}$; write $\mathfrak{l}^* = \mathfrak{l} - \{0\}$. Letting $\Lambda^w_{\mathfrak{l}}$ and $\Lambda^w_{\mathfrak{l}^*}$ be the preimages of Λ^w above $\mathfrak{l}$ and $\mathfrak{l}^*$, we may form the Cartesian diagram

$$\begin{tikzcd} \Lambda^w_0 \arrow[r] \arrow[d] & \Lambda^w_{\mathfrak{l}} \arrow[d] \\ \{0\} \arrow[r] & \mathfrak{l} \end{tikzcd}$$

where $\Lambda^w_{\mathfrak{l}^*} \to \mathfrak{l}^*$ is a locally trivial fibration because $\mathfrak{l}^* \subset \mathfrak{h}_{\mathrm{sr}}$. We may therefore apply the specialization construction above to obtain a map

$$H^{BM}_*(\Lambda^w_{\mathfrak{l}^*}) \to H^{BM}_{*-2}(\Lambda^w_0) = H^{BM}_{*-2}(Z).$$

We define $[\Lambda^w_0]$ to be the image of the fundamental class $[\Lambda^w_{\mathfrak{l}^*}]$ under this map, which may be checked to be independent of the choice of $\mathfrak{l}^*$. Define the map of Theorem 3.2 by

$$w \mapsto [\Lambda^w_0].$$

Proof of Theorem 3.2. We must check that $w \mapsto [\Lambda^w_0]$ is a map of algebras and that it induces an isomorphism of vector spaces. For the first property, if we take $\mathfrak{l}$ to be the complex span of some $h \in \mathfrak{h}_{\mathrm{sr}}$, then

$$\Lambda^w_{\mathfrak{l}^*} \simeq \Lambda^w_h \times \mathfrak{l}^*$$

is a trivial fibration and $[\Lambda^w_{\mathfrak{l}^*}] \simeq [\Lambda^w_h] \boxtimes [\mathfrak{l}^*]$. Combining this with (2), we see that

$$[\Lambda^{yw}_{\mathfrak{l}^*}] = [\Lambda^y_{w(\mathfrak{l}^*)}] \star [\Lambda^w_{\mathfrak{l}^*}],$$

from which we conclude by specialization that

$$[\Lambda^{yw}_0] = [\Lambda^y_0] \star [\Lambda^w_0].$$

It remains now to check that the map $\mathbb{C}[W] \to H^{BM}_{\dim_{\mathbb{R}} \mathcal{N}}(Z)$ is an isomoprhism of vector spaces. First, because the projection of each $[\Lambda^w_h]$ to $\mathcal{B} \times \mathcal{B}$ is supported in $\mathcal{O}(w)$, the projection of $[\Lambda^w_0]$ is supported at most on $\overline{\mathcal{O}(w)}$. Therefore, recalling that $\{[\overline{Z}_w]\}_{w\in W}$ form a basis by Proposition 2.4, we may write

$$[\Lambda^w_0] = \sum_{v \leq w} c_{v,w}[\overline{Z}_v]$$

for some $c_{v,w}$ and where $v \leq w$ is taken under the Bruhat ordering. We claim that $c_{w,w} = 1$ for all $w \in W$. For this, notice that the restriction of Λ^w_h to $\mathcal{O}(w)$ is given by

$$G \times^{B \cap wB} (h + \mathfrak{n} \cap w(\mathfrak{n})),$$

which is a flat family of affine bundles above $\mathcal{O}(w)$. Thus, as $h \to 0$, we see that $[\Lambda^w_h]|_{\mathcal{O}(w)}$ degenerates to the fundamental class of $G \times^{B \cap wB} (\mathfrak{n} \cap w(\mathfrak{n}))$, which is exactly $[Z_w] = [\overline{Z}_w]|_{\mathcal{O}(w)}$. This shows that $c_{w,w} = 1$, completing the proof. $\square$

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