# Grothendieck's simultaneous resolution and the Springer correspondence: Part 2

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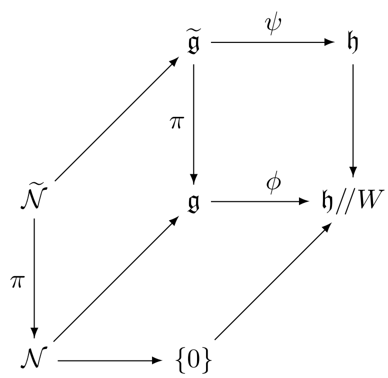
## 1 Recap of last time

We first give a brief summary of where we left off in the last talk. We defined the Springer resolution $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$, which fit into the commutative diagram




(1)

where $\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is Grothendieck's simultaneous resolution. Recalling that the Steinberg variety was defined as $Z = \widetilde{\mathcal{N}} \underset{\mathcal{N}}{\times} \widetilde{\mathcal{N}}$ and using the fact that $\pi$ is a $W$-covering over the semisimple regular locus $\mathfrak{g}_{\text{sr}} \subset \mathfrak{g}$, we constructed a map

$$\mathbb{C}[W] \to H^{BM}_{\dim_{\mathbb{R}} \widetilde{\mathcal{N}}}(Z)$$

which sends $w \in W$ to a class $[\Lambda_0^w]$ given as a certain specialization. The main result from last time was the following.

**Theorem 1.1.** The map

$$\mathbb{C}[W] \overset{\sim}{\to} H^{BM}_{\dim_{\mathbb{R}} \widetilde{\mathcal{N}}}(Z)$$

is an isomorphism of algebras.

## 2 Conclusion of the Springer correspondence

### 2.1 Realizing irreducible representations of $W$

Using Theorem 1.1, we now find a parametrization of all irreducible representations of $W$. Recall that for $\xi \in \mathcal{N}$, the *Springer fiber* $\mathcal{B}_\xi$ is defined to be the fiber $\pi^{-1}(\xi) \subset \widetilde{\mathcal{N}}$ above $\xi$. Let $G(\xi)$ be the stabilizer of $\xi$ and $C(\xi) = G(\xi)/G(\xi)^0$ the component group of $G(\xi)$. The main result is then the following theorem.

**Theorem 2.1.** The spaces $H^{BM}_{2d_\xi}(\mathcal{B}_\xi)^\chi$ for $\chi \in \text{Irred}(C(\xi))$ are all the irreducible representations of $W$.

We now discuss how to obtain this theorem from Theorem 1.1. Partially order the nilpotent orbits of $\mathcal{N}$ by closure, and for such an orbit $\mathcal{O}$, let $Z_{<\mathcal{O}}$, $Z_{\mathcal{O}}$, and $Z_{\leq \mathcal{O}}$ be the corresponding preimages in $Z$. Note that

$H^{BM}_{\dim_{\mathbb{R}}\mathcal{N}}(Z_{<\mathcal{O}})$ and $H^{BM}_{\dim_{\mathbb{R}}\mathcal{N}}(Z_{\leq\mathcal{O}})$ are both two-sided ideals in $H^{BM}_{\dim_{\mathbb{R}}\mathcal{N}}(Z)$. On the other hand, we know that $H^{BM}_{\dim_{\mathbb{R}}\mathcal{N}}(Z)$ is semisimple because it is isomorphic to $\mathbb{C}[W]$, so we obtain an isomorphism

$$H^{BM}_{\dim_{\mathbb{R}}\mathcal{N}}(Z) \simeq \bigoplus_{\mathcal{O}} H^{BM}_{\dim_{\mathbb{R}}\mathcal{N}}(Z_{\leq\mathcal{O}})/H^{BM}_{\dim_{\mathbb{R}}\mathcal{N}}(Z_{<\mathcal{O}}) =: \bigoplus_{\mathcal{O}} H_{\mathcal{O}}.$$

Observe that $H_{\mathcal{O}} := H^{BM}_{\dim_{\mathbb{R}}\mathcal{N}}(Z_{\leq\mathcal{O}})/H^{BM}_{\dim_{\mathbb{R}}\mathcal{N}}(Z_{<\mathcal{O}})$ itself inherits a convolution algebra structure. Now, because $H^{BM}_{\dim_{\mathbb{R}}\mathcal{N}}(Z_{\leq\mathcal{O}})$ and $H^{BM}_{\dim_{\mathbb{R}}\mathcal{N}}(Z_{<\mathcal{O}})$ each have bases given by fundamental classes of the irreducible components of their respective spaces, $H_{\mathcal{O}}$ has a basis given by the fundamental classes of the irreducible components of $Z_{\mathcal{O}}$.

Recall that $Z_{\mathcal{O}}$ is a $G$-equivariant fiber bundle over $\mathcal{O}$ with fiber $\mathcal{B}_\xi \times \mathcal{B}_\xi$ over $\xi \in \mathcal{O}$; in addition, its irreducible components are the $G$-orbits of the orbits of $C(\xi) = G(\xi)/G(\xi)^0$ on pairs of irreducible components of $\mathcal{B}_\xi$.

**Proposition 2.2.** We have an algebra isomorphism

$$H_{\mathcal{O}} \simeq \mathrm{End}_{C(\xi)}(H^{BM}_{2d_\xi}(\mathcal{B}_\xi)),$$

where $d_\xi = \dim \pi^{-1}(\mathcal{O}_\xi) - \dim \mathcal{O}_\xi$.

*Proof.* The convolution structure of $H_{\mathcal{O}}$ acts fiberwise, so the characterization of the irreducible components of $Z_{\mathcal{O}}$ implies that

$$H_{\mathcal{O}} \simeq H^{BM}_{4d_\xi}(\mathcal{B}_\xi \times B_\xi)^{C(\xi)}.$$

Now, the Kunneth isomorphism and the fact that $H^{BM}_{2d_\xi}(\mathcal{B}_\xi)_L \simeq H^{BM}_{2d_\xi}(\mathcal{B}_\xi)^\vee_R$ as $H_{\mathcal{O}}$-modules (where the $L$ and $R$ denote the left and right action) implies that

$$H^{BM}_{4d_\xi}(\mathcal{B}_\xi \times B_\xi)^{C(\xi)} \simeq (H_{2d_\xi}(\mathcal{B}_\xi)_L \otimes H^{BM}_{2d_\xi}(\mathcal{B}_\xi)^\vee_L)^{C(\xi)} \simeq \mathrm{End}_{C(\xi)}(H^{BM}_{2d_\xi}(\mathcal{B}_\xi)_L)$$

where we note that the first identification is on the level of $H_{\mathcal{O}}$-bimodules. □

We conclude formally from Proposition 2.2 and our previous analysis the following characterization of all irreducible representations of $W$.

*Proof of Theorem 2.1.* We have the chain of isomorphisms

$$\mathbb{C}[W] \simeq H^{BM}_{\dim_{\mathbb{R}}\mathcal{N}}(Z) \simeq \bigoplus_{\mathcal{O}} H_{\mathcal{O}} \simeq \bigoplus_{\mathcal{O}} \mathrm{End}_{C(\xi)}(H^{BM}_{2d_\xi}(\mathcal{B}_\xi)_L) = \bigoplus_{\mathcal{O},\chi} \mathrm{End}_{\mathbb{C}}(H^{BM}_{2d_\xi}(\mathcal{B}_\xi)^\chi),$$

where $H^{BM}_{2d_\xi}(\mathcal{B}_\xi)^\chi$ is the $\chi$-isotypic subspace of $H^{BM}_{2d_\xi}(\mathcal{B}_\xi)$. □

**Remark.** For $G = GL_n$, it turns out that $C(\xi)$ is trivial, which shows that the irreducible representations of $W = S_{n-1}$ correspond to nilpotent orbits. Such orbits are parametrized by the structure of the Jordan blocks of their orbits, which correspond to partitions of $n-1$. Thus we recover the classical classification of representations of the symmetric group.

Let us see $H^{BM}_{2d_\xi}(\mathcal{B}_\xi)$ explicitly in some cases. Assume that $G = GL_n$, so that $C(\xi)$ is always trivial.

- If $\xi$ is regular nilpotent, then $\mathcal{B}_\xi$ is a point, hence $H^{BM}_{2d_\xi}(\mathcal{B}_\xi)$ corresponds to the trivial representation.
- If $\xi = 0$, then $\mathcal{B}_\xi$ is the entire flag variety, which is a single irreducible component, hence $H^{BM}_{2d_\xi}(\mathcal{B}_\xi)$ is one-dimensional. The action of $W$ is then the sign representation.
- If $\xi$ has Jordan type $(n-1, 1)$, then $\mathcal{B}_\xi$ consists of $(n-1)$ copies of $\mathbb{P}^1$ connected sequentially, corresponding to the Dynkin diagram of type $A_{n-1}$. The action of $W$ yields the $(n-1)$-dimensional irreducible subrepresentation of the permutation representation of $S_n$, where each reflection acts by exchanging the corresponding $\mathbb{P}^1$'s.

## References

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