Grothendieck's simultaneous resolution and the Springer correspondence: Addendum

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1 An interpretation in terms of perverse sheaves

1.1 Small and semismall maps

Let $f : X \to Y$ be a proper surjective map of varieties with X nonsingular. We say that a decomposition of $Y = \bigsqcup_k S_k$ into finitely many non-singular locally closed subvarieties S_k is a stratification relative to f if each restriction $f : f^{-1}(S_k) \to S_k$ is a topological locally trivial fibration.

Proposition 1.1. The following conditions are equivalent.

(a) $Rf_*\underline{\mathbb{C}}_X[\dim X]$ is a perverse sheaf on Y,

(b) $\dim X \underset{Y}{\times} X \leq \dim X$, or

(c) $\dim S_k + 2(\dim f^{-1}(S_k) - \dim S_k) \leq \dim X$ for all k.

We say that f is *semismall* if any of the conditions in Proposition 1.1 hold and *small* if the inequality in (c) is strict for all strata S_k which are not dense in Y. Call a stratum *relevant* if equality holds in (c).

Proposition 1.2. If f is small, then

$$Rf_*\underline{\mathbb{C}}_X[n] = IC(Y, \mathcal{L}),$$

where $\mathcal{L} = Rf_*\underline{\mathbb{C}}_X|_{Y_0}$ is the restriction of the (derived) push-forward to the open stratum.

For small and semismall maps, the Decomposition theorem takes on the following especially nice form.

Theorem 1.3 (Decomposition theorem)**.** If $f : X \to Y$ is semismall, then

$$Rf_*\underline{\mathbb{C}}_X[\dim X] = \bigoplus_k IC(\overline{S}_k, \mathcal{L}_k),$$

where the sum is over relevant strata S_k and local systems $\mathcal{L}_k$ on S_k.

1.2 Semismallness of the Springer resolution

Let us see how these results apply to our situation. Depending on the context, the map π will be either small or semismall.

Proposition 1.4. The Springer map $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ admits a stratification by G-orbits under the adjoint action. Relative to this stratification, π is semismall, and every orbit is a relevant stratum.

Proof. Let $\mathcal{O}$ be a nilpotent orbit in $\mathcal{N}$ and $\xi \in \mathcal{O}$ an element. It is clear that $\widetilde{\mathcal{O}} \to \mathcal{O}$ is a fibration with fiber $\mathcal{B}_\xi$, so nilpotent orbits give a valid stratification. Now, $\widetilde{\mathcal{N}}$ is smooth, and π is evidently proper and surjective. Some dimension estimates from our first talk together imply that

$$\dim \mathcal{O} + 2 \dim \mathcal{B}_x = \dim Z_{\mathcal{O}} = \dim \mathcal{N},$$

so we conclude that π is semi-small and that each stratum is relevant. □

Proposition 1.5. The simultaneous resolution $\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is small.

Proof. Again, π is proper surjective with $\widetilde{\mathfrak{g}}$ smooth, so it suffices to verify the dimension condition (c) for some stratification. Let $\mathfrak{g}_n \subset \mathfrak{g}$ be

$$\mathfrak{g}_n = \{x \in \mathfrak{g} \mid \dim \mathcal{B}_x = n\}.$$

Take a stratification of $\mathfrak{g}$ which is a possible refinement of

$$\mathfrak{g} = \mathfrak{g}_{\mathrm{sr}} \cup (\mathfrak{g}_0 - \mathfrak{g}_{\mathrm{sr}}) \cup \bigcup_{n \geq 1} \mathfrak{g}_n$$

which makes $\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ a fibration above each stratum. On $\mathfrak{g}_{\mathrm{sr}}$, π is a $|W|$-to-1 cover, so (c) follows trivially. For any other stratum $S \subset \mathfrak{g}_n$, we see that

$$\dim S + n = \dim \pi^{-1}(S) = \dim \mathcal{B} + \dim(S \cap \mathfrak{b}),$$

hence

$$\dim S + 2n = \dim \mathcal{B} + \dim(S \cap \mathfrak{b}) + n.$$

On the other hand, we see that

$$\dim(S \cap \mathfrak{b}) + n = \dim(S \cap \mathfrak{b}) \underset{\mathfrak{g}}{\times} \widetilde{\mathfrak{g}} < \dim \mathfrak{b} \underset{\mathfrak{g}}{\times} \widetilde{\mathfrak{g}},$$

where the strict inequality is because S is not the dense stratum. Thus, to show smallness, it suffices to check that

$$\dim \mathfrak{b} \underset{\mathfrak{g}}{\times} \widetilde{\mathfrak{g}} = \dim \mathfrak{b}.$$

For this, observe that

$$\mathfrak{b} \underset{\mathfrak{g}}{\times} \widetilde{\mathfrak{g}} = \{(x, \mathfrak{b}') \mid x \in \mathfrak{b}, x \in \mathfrak{b}'\},$$

so the second projection equips it with a map to $\mathcal{B}$. Its pullback over each Schubert cell $X(w) = BwB/B \subset \mathcal{B}$ is given by

$$\{(x, gB) \mid x \in \mathfrak{b} \cap \mathrm{ad}_g(\mathfrak{b}), gB \in BwB\},$$

so it suffices to check that $\dim \mathfrak{b} \cap \mathrm{ad}_g(\mathfrak{b}) + \dim X(w) = \dim \mathfrak{b}$, which is true because $\mathfrak{b} \cap \mathrm{ad}_g(\mathfrak{b})$ is the Lie algebra of the stabilizer of the B-action on $X(w)$. $\square$

1.3 Action on the derived pushforward

Because $\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ is small, we obtain that

$$R\pi_* \underline{\mathbb{C}}_{\widetilde{\mathfrak{g}}}[\dim \mathfrak{g}] = IC(\mathfrak{g}, \mathcal{L}),$$

where $\mathcal{L}$ is the local system

$$\mathcal{L} = R\pi_* \underline{\mathbb{C}}_{\widetilde{\mathfrak{g}}}[\dim \mathfrak{g}]\big|_{\mathfrak{g}_{\mathrm{sr}}} = R\pi_* \underline{\mathbb{C}}_{\widetilde{\mathfrak{g}}_{\mathrm{sr}}}[\dim \mathfrak{g}].$$

Recall that $\widetilde{\mathfrak{g}}_{\mathrm{sr}}$ is a $|W|$-fold covering of $\mathfrak{g}_{\mathrm{sr}}$ and therefore $\underline{\mathbb{C}}_{\widetilde{\mathfrak{g}}_{\mathrm{sr}}}$ comes equipped with a W-action. By functoriality, this yields a W-action on $\mathcal{L}$ and hence $R\pi_* \underline{\mathbb{C}}_{\widetilde{\mathfrak{g}}}[\dim \mathfrak{g}]$. We now degenerate this action to $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$. Because π is proper, by base change on the inclusions $i : \mathcal{N} \to \mathfrak{g}$ and $\widetilde{i} : \widetilde{\mathcal{N}} \to \widetilde{\mathfrak{g}}$, we have that

$$i^* R\pi_* \underline{\mathbb{C}}_{\widetilde{\mathfrak{g}}}[\dim \mathfrak{g}] = R\pi_* \widetilde{i}^* \underline{\mathbb{C}}_{\widetilde{\mathfrak{g}}}[\dim \mathfrak{g}] = R\pi_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \mathfrak{g}],$$

so again by functoriality and dimension shift, we obtain a W-action on $R\pi_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}]$.

Recall now that $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is semismall, so by the Decomposition theorem we have that

$$R\pi_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}] = \bigoplus_k IC(\overline{S}_k, \mathcal{L}_k) \tag{1}$$

for some local systems $\mathcal{L}_k$ on strata S_k for which $\dim S_k + 2(\dim \pi^{-1}(S_k) - \dim S_k) = \dim \mathcal{N}$. By Proposition 1.4, $\mathcal{N}$ admits a stratification by nilpotent orbits $\mathcal{O}_\xi$; further, each simple local system $\mathcal{L}_{\xi,\chi}$ on $\mathcal{O}_\xi$ corresponds to a monodromy representation χ of $\pi_1(\mathcal{O}_\xi)$. Letting $V_{\xi,\chi}$ be the multiplicity space of χ in the monodromy representation corresponding to $\mathcal{L}_\xi$, we obtain the decomposition

$$R\pi_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}] = \bigoplus_{\xi, \chi \in \mathrm{Irr}(\pi_1(\mathcal{O}_\xi))} IC(\overline{\mathcal{O}}_\xi, \mathcal{L}_{\xi,\chi}) \otimes V_{\xi,\chi}.$$

We may characterize the multiplicity spaces more explicitly by examining the fibers of $R\pi_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}]$.

Proposition 1.6. For any nilpotent orbit $\mathcal{O}_\xi$, we have

$$H^{BM}_{2d_\xi}(\mathcal{B}_\xi) = \bigoplus_\chi \mathbb{C}_\chi \otimes V_{\xi,\chi},$$

where $d_\xi = \dim \pi^{-1}(\mathcal{O}_\xi) - \dim \mathcal{O}_\xi$.

Proof. Choose $\xi \in \mathcal{O}_\xi$; taking the cohomology in degree $-\dim \mathcal{O}_\xi$ of the stalks at ξ of (1) yields

$$H^{-\dim \mathcal{O}_\xi}(\mathcal{B}_\xi, \underline{\mathbb{C}}_{\mathcal{B}_\xi}[\dim \mathcal{N}]) = H^{-\dim \mathcal{O}_\xi}(R\pi_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}]_\xi) = \bigoplus_\chi (IC(\overline{\mathcal{O}_\xi}, \mathcal{L}_{\xi,\chi}) \otimes V_{\xi,\chi})_\xi = \bigoplus_\chi \mathbb{C}_\chi \otimes V_{\xi,\chi},$$

where we note that $(\mathcal{L}_{\xi',\chi})_\xi = 0$ unless $\mathcal{O}_\xi \subset \overline{\mathcal{O}_{\xi'}}$ and that

$$H^{-\dim \mathcal{O}_\xi}(\mathcal{O}_\xi, IC(\overline{\mathcal{O}_{\xi'}}, \mathcal{L}_{\xi',\chi'})) = 0$$

for $\mathcal{O}_\xi \subset \overline{\mathcal{O}_{\xi'}}$ by the support conditions on IC sheaves.

Observe now that $i_\xi : B_\xi \to \widetilde{\mathcal{N}}$ is the inclusion of a fiber into the total space of the fibration $\mathcal{B}_\xi \to \mathcal{O}_\xi$, hence we see that $i_\xi^! = i_\xi^*[-2\dim \mathcal{O}_\xi]$. Thus, we find that

$$\begin{aligned} H^{BM}_{2d_\xi}(\mathcal{B}_\xi) &= H^{-2d_\xi}(\mathcal{B}_\xi, i_\xi^! \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[2\dim \widetilde{\mathcal{N}}]) \\ &= H^{-2d_\xi}(\mathcal{B}_\xi, \underline{\mathbb{C}}_{\mathcal{B}_\xi}[2\dim \widetilde{\mathcal{N}} - 2\dim \mathcal{O}_\xi]) = H^{-\dim \mathcal{O}_\xi}(\mathcal{B}_\xi, \underline{\mathbb{C}}_{\mathcal{B}_\xi}[\dim \mathcal{N}]). \quad \square \end{aligned}$$

Given this decomposition, the W-action on $R\pi_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}]$ becomes a map

$$\mathbb{C}[W] \to \mathrm{End}(R\pi_* \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \mathcal{N}]) = \bigoplus_{\xi,\chi} \mathrm{End}(V_{\xi,\chi}), \tag{2}$$

where the equality follows because

$$\mathrm{End}(IC(\overline{\mathcal{O}}_\xi, \mathcal{L}_{\xi,\chi}), IC(\overline{\mathcal{O}}_{\xi'}, \mathcal{L}_{\xi',\chi'})) = \begin{cases} 0 & (\xi,\chi) \neq (\xi',\chi') \\ \mathbb{C} & (\xi,\chi) = (\xi',\chi'). \end{cases}$$

Theorem 1.7. The map (2) is an isomorphism of algebras. In particular, the multiplicity spaces $V_{\xi,\chi}$ form the complete set of irreducible representations of W.

Remark. Together, Proposition 1.6 and Theorem 1.7 imply that every irreducible representation of W is a direct summand of $H^{BM}_{2d_\xi}(\mathcal{B}_\xi)$ for some $\xi \in \mathcal{N}$. Further, if $\mathcal{O}_\xi$ is simply connected (which is the case for $W = S_n$), then each $H^{BM}_{2d_\xi}(\mathcal{B}_\xi)$ is itself irreducible.

Proof of Theorem 1.7. The proof consists of two steps. First, we show that (2) is injective. For this, observe that taking stalks at $0 \in \mathcal{N}$ and applying proper base change yields a map

$$\mathbb{C}[W] \to \mathrm{End}((R\pi_* \underline{\mathbb{C}}_{\mathcal{N}}[\dim \mathcal{N}])_0) = \mathrm{End}(R\pi_* \underline{\mathbb{C}}_{\mathcal{B}}) \to \mathrm{End}(H^*(\mathcal{B}, \underline{\mathbb{C}}_{\mathcal{B}}))$$

where we note that $\mathcal{B} = \mathcal{B}_0 = \pi^{-1}(0)$. Recall now that $H^*(\mathcal{B}, \underline{\mathbb{C}}_{\mathcal{B}})$ is isomorphic to the regular representation of W. We claim without proof that this action coincides with the action given by the map above, which would show that the composition is injective.[1]

To finish, we note that $\pi_1(\mathcal{O}_\xi) \simeq C(\xi)$. Further, because $H^{BM}_{2d_\xi}(\mathcal{B}_\xi)$ is top dimensional, it is generated by the fundamental classes of the irreducible components of $\mathcal{B}_\xi$, and the orbits of the monodromy representation of $\pi_1(\mathcal{O}_\xi)$ are exactly the orbits of the $C(\xi)$ action on irreducible components. Recall by Proposition 1.6 that the irreducible components of $Z_{\mathcal{O}_\xi}$ are in bijection with $C(\xi)$-orbits on pairs of irreducible components of $\mathcal{B}_\xi$. By the orbit stabilizer theorem, the number of such pairs is

$$\frac{1}{|C(\xi)|}\sum_{c\in C(\xi)} \mathrm{Fix}_c(\mathrm{Irred}(\mathcal{B}_\xi))^2 = \frac{1}{|C(\xi)|}\sum_{c\in C(\xi)} \mathrm{Tr}|_{H^{BM}_{2d_\xi}(\mathcal{B}_\xi)}(c)^2 = ||\mathrm{ch}_{H^{BM}_{2d_\xi}(\mathcal{B}_\xi)}||^2 = \sum_\chi (\dim V_{\xi,\chi})^2.$$

Combining this with our classification of irreducible components of Z, we then have

$$|W| = \sum_{\xi,\chi} (\dim V_{\xi,\chi})^2,$$

which implies that $\mathbb{C}[W]$ and $\mathrm{End}(R\pi_* \underline{\mathbb{C}}_{\mathcal{N}}[\dim \mathcal{N}])$ have the same dimension, so (2) is an isomorphism. $\square$

Proposition 1.8. The isomorphism of (2) is compatible with the isomorphism $\mathbb{C}[W] \simeq H^{BM}_{\dim \widetilde{\mathcal{N}}}(Z)$.

Proof. We will give an isomorphism of vector spaces

$$H^{BM}_{\dim \widetilde{\mathcal{N}}}(Z) \simeq \mathrm{End}(R\pi_* \mathbb{C}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}])$$

but omit the proof that it is compatible with the algebra structure (which may be found in Chriss-Ginzburg Chapter 8). Consider the Cartesian square

$$\begin{array}{ccc} Z & \xrightarrow{\ i\ } & \widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi \times \pi} \\ \mathcal{N} & \xrightarrow{\ \Delta\ } & \mathcal{N} \times \mathcal{N} \end{array}$$

Applying base change on this square and using Verdier duality, we have the chain of isomorphisms

$$\begin{aligned} H^{BM}_{\dim \widetilde{\mathcal{N}}}(Z) &\simeq H^{-\dim \widetilde{\mathcal{N}}}(Z, \mathbb{D}(Z)) \\ &\simeq H^0(Z, i^!(\underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}] \boxtimes \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}])) \\ &\simeq H^0(Z, i^!(\mathbb{D}(\underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}]) \boxtimes \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}])) \\ &\simeq H^0(\mathcal{N}, R\pi_* i^!(\mathbb{D}(\underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}]) \boxtimes \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}])) \\ &\simeq H^0(\mathcal{N}, \Delta^! R(\pi \times \pi)_*(\mathbb{D}(\underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}]) \boxtimes \underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}[\dim \widetilde{\mathcal{N}}])) \\ &\simeq H^0(\mathcal{N}, \Delta^!(R\pi_*(\mathbb{D}(\underline{\mathbb{C}}_{\widetilde{\mathcal{N}}})) \boxtimes R\pi_*(\underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}))) \\ &\simeq H^0(\mathcal{N}, \mathbb{D}(R\pi_*(\underline{\mathbb{C}}_{\widetilde{\mathcal{N}}})) \otimes R\pi_*(\underline{\mathbb{C}}_{\widetilde{\mathcal{N}}})) \\ &\simeq \mathrm{End}(R\pi_*(\underline{\mathbb{C}}_{\widetilde{\mathcal{N}}}), R\pi_*(\underline{\mathbb{C}}_{\widetilde{\mathcal{N}}})). \end{aligned}$$

$\square$

[1] Strictly speaking, this is cheating!