THE TRIGONOMETRIC CASIMIR CONNECTION

YI SUN

ABSTRACT. Notes for a talk at the 2014 UT Austin Workshop on Yangians and Quantum Loop Algebras.

CONTENTS

1. UNIVERSAL TRIGONOMETRIC CONNECTIONS

1.1. **Connections valued in algebras.** Let A be an algebra and X a manifold. An A-valued connection ∇ on X consists of the data of a connection operator $M \in A \otimes T^*(X)$ so that

$$\nabla = d + M.$$

For any representation V of A, the image of M under the evaluation map $A \otimes T^*(X) \to \mathrm{End}(V) \otimes T^*(X)$ is a connection matrix for ∇_V, which is a connection in the traditional sense on the trivial vector bundle $X \times V \to X$. We now describe a general way to construct an A-valued connection.

1.2. **The geometric setup and definition.** Let $\mathfrak{g}$ be a simple Lie algebra with root system $\Phi \subset \mathfrak{h}^*$ and weight and root lattices $P, Q \subset \mathfrak{h}^*$. Fix a choice of positive roots $\Phi_+ \subset \Phi$. Let $H = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{C}^*)$ be the torus with Lie algebra $\mathfrak{h}$. For $\lambda \in P$, let e^λ be the evaluation map in $\mathbb{C}[H]$, and define the regular locus in H to be

$$H_{\mathrm{reg}} = H - \bigcup_{\alpha \in \Phi} \{e^\alpha = 1\}.$$

Let A be an algebra equipped with:

- a choice of elements $\{t_\alpha\}_{\alpha \in \Phi}$ so that $t_\alpha = t_{-\alpha}$, and
- a linear map $\tau : \mathfrak{h} \to A$.

Given the data of A, $\{t_\alpha\}$, and τ, the *universal trigonometric connection* on H_{reg} with values in A is defined by

$$\nabla = d - \sum_{\alpha \in \Phi_+} \frac{d\alpha}{e^\alpha - 1} t_\alpha - du_i \cdot \tau(u^i), \tag{1}$$

Date: May 8, 2014.

where u_i and u^i are dual bases in $\mathfrak{h}^*$ and $\mathfrak{h}$ and summation is implicit in the last term. The expression of (1) depends on the choice of $\Phi_+ \subset \Phi$; however, we may characterize it relative to any other choice of positive roots.

Proposition 1.1. For any set of positive roots $\Phi'_+ \subset \Phi$, the connection of (1) is also given by

$$\nabla = d - \sum_{\alpha \in \Phi'_+} \frac{d\alpha}{e^\alpha - 1} t_\alpha - du_i \tau'(u^i),$$

where

$$\tau'(v) = \tau(v) - \sum_{\alpha \in \Phi_+ \cap \Phi'_+} \alpha(v) t_\alpha.$$

If $\Phi'_+ = w\Phi_+$ for some $w \in W$, denote the resulting τ' by τ_w. Combining (1) and the expression given by Proposition 1.1 for $\Phi'_+ = \Phi_-$ yields the expression

$$\nabla = d - \frac{1}{2} \sum_{\alpha \in \Phi} \frac{d\alpha}{e^\alpha - 1} t_\alpha - du_i \delta(u^i), \tag{2}$$

where $\delta : \mathfrak{h} \to A$ is defined by

$$\delta(v) = \tau(v) - \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha(v) t_\alpha.$$

1.3. **Flatness criterion.** A subset $\Psi \subset \Phi$ is a *root subsystem* if it is closed under $\mathbb{Z}$-linear combination in Φ; a root subsystem is *complete* if it is closed under $\mathbb{R}$-linear combination in Φ. Evidently, a choice of positive roots for Φ restricts to a choice of positive roots for Ψ. We now give criteria for flatness and W-equivariance of ∇ which rely crucially on rank 2 root subsystems.

Theorem 1.2. The universal connection ∇ is flat if and only if $\{t_\alpha\}$ and τ satisfy the following relations:

- For any rank 2 root subsystem $\Psi \subset \Phi$ and $\alpha \in \Psi$, we have

$$\left[t_\alpha, \sum_{\beta \in \Psi_+} t_\beta \right] = 0; \tag{3}$$

- For any $u, v \in \mathfrak{h}$, we have

$$[\tau(u), \tau(v)] = 0; \tag{4}$$

- For any $\alpha \in \Phi_+$, $w \in W$ such that $w^{-1}\alpha$ is simple, and $u \in \mathfrak{h}$ such that $\alpha(u) = 0$, we have

$$[t_\alpha, \tau_w(u)] = 0. \tag{5}$$

Further, assuming (3), condition (5) is equivalent to

$$[t_\alpha, \delta(u)] = 0 \text{ for all } u \text{ with } \alpha(u) = 0. \tag{6}$$

Proof. We only sketch the proof of necessity of a few of the conditions. Write $\nabla = d - A$, where $dA = 0$, so that ∇ is flat if and only if $A \wedge A = 0$. We compute

$$A \wedge A = \frac{1}{2} \sum_{\alpha,\beta} \frac{d\alpha}{e^\alpha - 1} \wedge \frac{d\beta}{e^\beta - 1} [t_\alpha, t_\beta] + \sum_{\alpha,i} \frac{d\alpha}{e^\alpha - 1} \wedge du_i [t_\alpha, \tau(u^i)] + \frac{1}{2} \sum_{i,j} du_i \wedge du_j [\tau(u^i), \tau(u^j)].$$

View $T = \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{C}^*)$ as a complex torus which is a quotient of H. We may also view ∇ as a connection on T with singularities on $T_\alpha = \{e^\alpha = 1\} \subset T$. Define the coordinates $z_i = e^{-\alpha_i}$ on T so that the dual basis of $\{\lambda_i^\vee\}$ is $du^i = -dz_i/z_i$, meaning that

$$du_i \tau(u^i) = -\frac{dz_i}{z_i} \tau(\lambda_i^\vee).$$

Thus, for $\alpha = \sum_i m_i \alpha_i$, we have $e^\alpha = \prod_i z_i^{-m_i}$ so that

$$\frac{d\alpha}{e^\alpha - 1} = \frac{e^{-\alpha}}{1 - e^{-\alpha}} d\alpha = -\sum_i m_i z_i^{-1} \frac{\prod_j z_j^{m_j}}{1 - \prod_j z_j^{m_j}} dz_i.$$

The coordinates z_i give an embedding $T(\mathbb{C}^*)^n$, and let $\overline{T} \simeq \mathbb{C}^n$ be the compactification. Let $T_i = \{z_i = 0\}$ and $\iota_i : T_i \to \overline{T}$ be the inclusion so that

$$\mathrm{res}_{T_i}(A \wedge A) = \iota_i^* \sum_{\alpha>0} \frac{d\alpha}{e^\alpha - 1}[t_\alpha, \tau(\lambda_i^\vee)] + \iota_i^* \sum_{j\neq i} \frac{dz_j}{z_j}[\tau(\lambda_i^\vee), \tau(\lambda_j^\vee)].$$

This implies that

$$\mathrm{res}_{T_j \cap T_i}\mathrm{res}_{T_i}(A \wedge A) = [\tau(\lambda_j^\vee), \tau(\lambda_i^\vee)] = 0,$$

which is (4). $\square$

1.4. **W-equivariance criterion.** Suppose that A is equipped with an action of the Weyl group W.

Theorem 1.3. The universal connection ∇ is W-equivariant if and only if

$$s_i(t_\alpha) = t_{s_i\alpha} \tag{7}$$

for all $\alpha \in \Phi$ and $s_i \in W$, and

$$s_i(\tau(x)) - \tau(s_i x) = (\alpha_i, x)t_{\alpha_i} \tag{8}$$

for all $x \in \mathfrak{h}$ and $s_i \in W$.

Proof. We compute

$$\begin{aligned} s_i^*\nabla &= d - \sum_{\alpha>0} \frac{d\alpha}{e^{-\alpha} - 1} s_i(t_\alpha) - s_i(\tau(u^j))d(s_i u_j) \\ &= d - \sum_{\alpha>0} \frac{d\alpha}{e^{\alpha} - 1} s_i(t_\alpha) - d\alpha_i s_i(t_{\alpha_i}) - s_i(\tau(u^j))d(s_i u_j). \end{aligned}$$

Taking residues along $\{e^\alpha = 1\}$ in $s_i^*\nabla = \nabla$ yields (7). For (8), notice that

$$s_i(\tau(u^j))d(s_i u_j) = s_i(\tau(s_i(u^j)))du_j,$$

so we obtain that $\nabla = s_i^*\nabla$ if and only if

$$\tau(u^j)du_j = s_i(t_{\alpha_i})d\alpha_i + s_i\tau(s_i u^j)du_j.$$

Pairing this against u^k yields (8), showing the necessity of the two relations. Sufficiency follows by running this argument backwards. $\square$

2. The trigonometric Casimir connection

We will now realize a flat, W-equivariant connection with $A = Y(\mathfrak{g})^\mathfrak{h} \subset Y(\mathfrak{g})$, where we note that the W-action on $Y(\mathfrak{g})^\mathfrak{h}$ is induced by the fact that $Y(\mathfrak{g})$ is integrable under the adjoint action of $\mathfrak{g}$. Here the reflection s_i acts by

$$s_i(\pi) = \exp(\pi(f_i))\exp(-\pi(e_i))\exp(\pi(f_i)).$$

2.1. **Definition.** For each $\alpha \in \Phi_+$, let $\mathfrak{sl}_2^\alpha = \langle x_\alpha, x_{-\alpha}, h_\alpha\rangle$ be an $\mathfrak{sl}_2$-triple with $x_\alpha \in \mathfrak{g}_\alpha$ and $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ so that $(x_\alpha, x_{-\alpha}) = 1$. Let

$$\kappa_\alpha = x_\alpha x_{-\alpha} + x_{-\alpha}x_\alpha$$

be the (truncated) Casimir operator corresponding to α. Note that κ_α is independent of the choice of generators $x_\alpha, x_{-\alpha}$. We may now apply the construction of (1) with $t_\alpha = \kappa_\alpha$ and

$$\tau(u) = -2J(u) + \frac{\hbar}{2}\sum_{\beta>0}(u,\beta)\kappa_\beta = -2T(u)_1 + \hbar(u, t^j)t_j^2,$$

where Drinfeld's loop generators $T(-)_1 : \mathfrak{h} \to Y(\mathfrak{g})$ are given by

$$T(v)_1 = (v, t^i)T_{i,1}$$

for a basis $\{t^i\}$ of $\mathfrak{h}$ dual to the basis $\{t_i = d_i\alpha_i^\vee\}$ of $\mathfrak{h}^*$. We note that this corresponds to

$$\delta(u) = -2J(u)$$

in the form of (2).

The *trigonometric Casimir connection* of a simple Lie algebra $\mathfrak{g}$ is given by

$$\nabla = d - \hbar \sum_{\alpha \in \Phi_+} \frac{d\alpha}{e^\alpha - 1} \cdot \kappa_\alpha + du_i \Big(2T(u^i)_1 - (u^i, t^j) t_j^2 \Big) \tag{9}$$

$$= d - \hbar \sum_{\alpha \in \Phi_+} \frac{d\alpha}{e^\alpha - 1} \cdot \kappa_\alpha + du_i \Big(2J(u^i) - \frac{\hbar}{2} \sum_{\beta > 0} (u^i, \beta) \kappa_\beta \Big). \tag{10}$$

2.2. **Checking flatness and W-equivariance.**

Proposition 2.1. The trigonometric Casimir connection is flat and W-equivariant.

Proof. It suffices to check conditions (3), (4), (5) for flatness and conditions (7) and (8) for W-equivariance, where we have $t_\alpha = \hbar \kappa_\alpha$ and

$$\tau(u) = -2J(u) + \frac{\hbar}{2} \sum_{\beta > 0} (u, \beta) \kappa_\beta = -2T(u)_1 + (u, t^j) T_{j,0}^2.$$

For (3), we may either do an explicit check or note that

$$\hbar \sum_{\beta \in \Psi_+} \kappa_\beta \in U(\mathfrak{h}_\Psi) + \kappa_\Psi,$$

where κ_Ψ is the Casimir operator of the Lie subalgebra $\mathfrak{g}_\Psi \subset \mathfrak{g}$ and $\mathfrak{h}_\Psi$ is the Cartan subalgebra of $\mathfrak{g}_\Psi$. Thus κ_α commutes with $\sum_{\beta \in \Psi_+} \kappa_\beta$ because it commutes with both κ_Ψ and $U(\mathfrak{h}_\Psi)$. For (4), we note that

$$[\tau(u), \tau(v)] = \Big[(u, t^j)(2T_{j,1} - T_{j,0}^2), (v, t^l)(2T_{l,1} - T_{l,0}^2) \Big] = 0$$

because all $T_{r,s}$ commute. For (6), for α and u so that $\alpha(u) = 0$, we have

$$[t_\alpha, \delta(u)] = -2[\kappa_\alpha, J(u)] = -(\alpha, \alpha) \Big(J([e_\alpha, u]) f_\alpha + e_\alpha J([f_\alpha, u]) + J([f_\alpha, u]) e_\alpha + f_\alpha J([e_\alpha, u]) \Big) = 0.$$

Together, these checks establish flatness of ∇. For W-equivariance, we must check (7) and (8). First, (7) holds because the W-action of $w \in W$ on $\mathfrak{g}$ sends κ_α to $\kappa_{w(\alpha)}$. For (8), we note that

$$s_i(\tau(u)) - \tau(s_i u) = -\frac{\hbar}{2} \sum_{\beta > 0} (u, s_i^{-1}(\beta)) \kappa_\beta + \frac{\hbar}{2} \sum_{\beta > 0} (u, \beta) \kappa_{s_i \beta} = \hbar (u, \alpha_i) \kappa_{\alpha_i}.$$ □

Remark. The proof is homogeneous in t_α and $\tau(u)$, meaning it also applies to the family of connections

$$\nabla = d - \lambda^{-1} \left(\hbar \sum_{\alpha > 0} \frac{d\alpha}{e^\alpha - 1} \kappa_\alpha + du_i \Big(2T(u^i)_1 - (u^i, t^j) t_j^2 \Big) \right).$$

2.3. **The trigonometric Casimir connection of $\mathfrak{gl}_N$.** In the case of $\mathfrak{gl}_N$, we must modify the previous construction. Let H_{reg} be the set of regular diagonal elements of GL_N. The *trigonometric Casimir connection of* $\mathfrak{gl}_N$ is the $Y(\mathfrak{gl}_N)$-valued connection on H_{reg} given by

$$\nabla = d - \sum_{i<j} \frac{d(\theta_i - \theta_j)}{e^{\theta_i - \theta_j} - 1} \kappa_{\theta_i - \theta_j} - \sum_{i=1}^n d\theta_i D_i.$$

where elements D_i in $Y(\mathfrak{gl}_N)$ are given by

$$D_i = 2t_{ii}^{(2)} - \sum_{j<i} \kappa_{\theta_j - \theta_i} - E_{ii}^2.$$

Proposition 2.2. The trigonometric Casimir connection of $Y(\mathfrak{gl}_N)$ is flat and W-equivariant.

3. Commutation relations with rational q-KZ equations

3.1. **Recollections on the rational q-KZ equations.** Let $V_1, \ldots, V_n$ be $Y(\mathfrak{g})$-modules and $d_i \in GL(V_i)$ satisfying

$$d_i d_j R^{ij}(u) = R^{ij}(u) d_i d_j.$$

For $c \in \mathbb{C}^\times$ and $a_1, \ldots, a_n \in \mathbb{C}$ distinct, define the operators $A_i \in \mathrm{End}(V_1 \otimes \cdots \otimes V_n)$ by

$$A_i = R^{i-1,i}(a_{i-1} - a_i - c)^{-1} \cdots R^{1,i}(a_1 - a_i - c)^{-1} d_i R^{i,n}(a_i - a_n) \cdots R^{i,i+1}(a_i - a_{i+1}).$$

The *rational q-KZ* difference equations are given by

$$T_i f = A_i f$$

where $T_i f(a_1, \ldots, a_n) = f(a_1, \ldots, a_{i-1}, a_i + c, a_{i+1}, \ldots, a_n)$. It was shown in the previous talk that $\widetilde{T}_i = A_i^{-1} T_i$ are a commuting family of difference operators.

3.2. **Proof of compatibility.** Let $V_1, \ldots, V_n$ be representations of $Y(\mathfrak{g})$, integrable as $\mathfrak{g}$-modules. Denote by $V_i(a_i)$ the pullback of V_i under τ_{a_i} and by $\Delta_{a_1,a_2} = (\tau_{a_1} \otimes \tau_{a_2}) \circ \Delta$. Consider the trigonometric Casimir connection ∇' valued in the $Y(\mathfrak{g})$-module

$$V_1(-a_1) \otimes \cdots \otimes V_n(-a_n)$$

and scaled by $\frac{1}{2c}$, meaning that

$$\nabla' = d - \frac{1}{2c} \frac{\hbar}{2} \sum_{\alpha > 0} \left(\frac{e^\alpha + 1}{e^\alpha - 1} d\alpha \kappa_\alpha - 2 du_i J(u^i) \right).$$

Fix $d_i \in GL(V_i)$ to be the function on H so that

$$d_i(e^u) = (e^{-u})^{(i)}.$$

Theorem 3.1. The q-KZ operators $\widetilde{T}_i$ commute with the (rescaled) trigonometric Casimir connection ∇'.

Proof. We break the computation into three steps. First, computing with the Yang-Baxter relation yields

$$\widetilde{T}_1 \cdots \widetilde{T}_i = \widetilde{A}_i^{-1} T_1 \cdots T_i, \tag{11}$$

where

$$\widetilde{A}_i = d_1 \cdots d_i R^{1,n}(a_1 - a_n) \cdots R^{i,n}(a_1 - a_n) \cdots R^{1,i+1}(a_1 - a_{i+1}) \cdots R^{i,i+1}(a_i - a_{i+1}).$$

It now suffices to check that

$$[\nabla', \widetilde{T}_1 \cdots \widetilde{T}_i](T_1 \cdots T_i)^{-1} = [\nabla', \widetilde{A}_i^{-1} T_1 \cdots T_i](T_1 \cdots T_i)^{-1} = d\widetilde{A}_i^{-1} - \frac{1}{2c} \widetilde{A}_i^{-1} (1 - \mathrm{Ad}_{T_1 \cdots T_i})(B) - \frac{1}{2c}[B, \widetilde{A}_i^{-1}] = 0, \tag{12}$$

where $B = B_1 + B_2$ with

$$B_1 = \frac{\hbar}{2} \sum_{\alpha > 0} \frac{e^\alpha + 1}{e^\alpha - 1} d\alpha \kappa_\alpha \text{ and } B_2 = -2 du_i J(u^i).$$

We will show that the first two terms in (12) cancel and that the last term is zero. For the former, it suffices to check that

$$d\widetilde{A}_i \widetilde{A}_i^{-1} = -\frac{1}{2c}(1 - \mathrm{Ad}_{T_1 \cdots T_i})(B).$$

Indeed, notice that

$$d\widetilde{A}_i \widetilde{A}_i^{-1} = -\sum_{j=1}^{i} du_a (u^a)^{(j)}.$$

On the other hand, we have $(1-\mathrm{Ad}_{T_1\cdots T_i})(B_1) = 0$ because B_1 has coefficients in $U(\mathfrak{g})$, hence is independent of shifts. Further, we note that

$$\begin{aligned}(1-\mathrm{Ad}_{T_1\cdots T_i})(B_2) &= -2du_a(1-\mathrm{Ad}_{T_1\cdots T_i})J(u^a)\\ &= -2du_a\left((1-\mathrm{Ad}_{T_1\cdots T_i})\Big(\sum_i J(u^a)^{(i)} + \frac{\hbar}{2}\sum_{1\le i<j\le n}[(u^a)^{(i)}, t^{ij}]\Big)\right)\\ &= -2du_a\left(-c\sum_{j=1}^{i}(u^a)^{(j)}\right)\\ &= -2c\, d\widetilde{A}_i\widetilde{A}_i^{-1}.\end{aligned}$$

For the final check, notice that $[B, \widetilde{A}_i^{-1}] = 0$ if and only if $[B, \widetilde{A}_i] = 0$. For the latter, observe that

$$\widetilde{A}_i = \Delta^{(i)}(d_1)(\Delta^{(i)}_{a_1-a_i,\dots,a_{i-1}-a_i,0}\otimes\Delta^{(n-i)}_{a_{i+1}-a_n,\dots,a_{n-1}-a_n,0})(R(a_i-a_n)),$$

where we use that

$$\begin{aligned}(\Delta\otimes 1)(R(u)) &= R^{13}(u)R^{23}(u)\\ (1\otimes\Delta)(R(u)) &= R^{13}(u)R^{12}(u)\\ \tau_{v,w}R(u) &= R(u+v-w).\end{aligned}$$

It therefore suffices to check the statement for $n = 2$ and $i = 1$. In this case, we see that

$$\begin{aligned}d_1^{-1}[d_1R(a_1-a_2), \Delta_{a_1,a_2}(B)]R(a_1-a_2)^{-1} &= (1-\mathrm{Ad}(d_1^{-1}))(\Delta_{a_1,a_2}(B)) + (1-\mathrm{Ad}_{R(a_1-a_2)})(\Delta(B))\\ &= (1-\mathrm{Ad}(d_1^{-1}))(\Delta_{a_1,a_2}(B)) + \Delta_{a_1,a_2}(B) - P\Delta_{a_1,a_2}(B)\\ &= (1-\mathrm{Ad}(d_1^{-1}))(\Delta_{a_1,a_2}(B)) + 2[u^i\otimes 1, t]du_i,\end{aligned}$$

where we used that

$$\Delta^{21}(x) = R(u)\Delta(x)R(u)^{-1}.$$

Observe now that

$$(1-\mathrm{Ad}(d_1^{-1}))\Delta_{a_1,a_2}(B_2) = -\hbar(1-\mathrm{Ad}(d_1^{-1}))[u^i\otimes 1, t]du_i$$

and that

$$\begin{aligned}(1-\mathrm{Ad}(d_1^{-1}))\Delta_{a_1,a_2}(B_1) &= \hbar\sum_{\alpha>0} d\alpha\frac{e^\alpha+1}{e^\alpha-1}(1-\mathrm{Ad}(d_1^{-1}))(\overline{\kappa}_\alpha)\\ &= \hbar\sum_{\alpha>0} d\alpha\frac{e^\alpha+1}{e^\alpha-1}(1-e^\alpha)(x_\alpha\otimes x_{-\alpha} - e^{-\alpha}x_{-\alpha}\otimes x_\alpha)\\ &= -\hbar\sum_{\alpha>0} d\alpha\Big((e^\alpha+1)x_\alpha\otimes x_{-\alpha} - (e^{-\alpha}+1)x_{-\alpha}\otimes x_\alpha\Big)\\ &= -\hbar\sum_{\alpha>0} du_i[u^i\otimes 1, (1+\mathrm{Ad}(d_1^{-1}))\overline{\kappa}_\alpha]\\ &= -\hbar du_i[u^i\otimes 1, (1+\mathrm{Ad}(d_1^{-1}))t],\end{aligned}$$

where $\overline{\kappa}_\alpha = x_\alpha\otimes x_{-\alpha} + x_{-\alpha}\otimes x_\alpha$. Combining these three computations gives the result. □

References

[1] V. Toledano Laredo, The trigonometric Casimir connection of a simple Lie algebra, J. Algebra **329** (2011), 286-327.

E-mail address: yisun@math.mit.edu