Tricks in Trigonometry

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1 Warmup

Problem 1 (HMMT 2008). Compute $\arctan(\tan 65° - 2 \tan 40°)$.

2 Some Trigonometry Techniques

- **Direct Manipulation:** Many problems directly involve trigonometric expressions that must be manipulated. In this case, the main strategy is to combine ordinary algebraic manipulation with trigonometric identities. Some things to try are:

  - telescoping using sum-to-product and product-to-sum identities
  - using Fact 3 to transform trigonometric expressions into algebraic ones

- **Trigonometric Substitution:** If there are awkward constraints or strange, inhomogeneous conditions (especially in constrained inequalities), you should consider substituting trigonometric expressions. Some things to keep in mind are:

  - make sure the substituted functions have the same range as the variables for which you are substituting
  - attempt to mimic some existing trigonometric identities to simplify the result

3 Useful Facts

You should be familiar with all of the facts listed below. In particular, this means that you should be able to prove them if necessary.

Fact 1 (Double Angle Formulas). We have

$$\sin 2A = 2 \sin A \cos A \qquad \cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A \qquad \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

Fact 2 (Half Angle Formulas). We have

$$\sin \frac{A}{2} = \pm \sqrt{\frac{1 - \cos A}{2}} \qquad \cos \frac{A}{2} = \pm \sqrt{\frac{1 + \cos A}{2}} \qquad \tan \frac{A}{2} = \frac{\sin A}{1 + \cos A} = \frac{1 - \cos A}{\sin A}.$$

Fact 3. Let $t = \tan \frac{x}{2}$. Then, we have

$$\sin x = \frac{2t}{1 + t^2}, \cos x = \frac{1 - t^2}{1 + t^2}, \text{ and } \tan x = \frac{2t}{1 - t^2}.$$

Fact 4 (Addition Formulas). We have

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \ \text{ and } \ \cos(A+B) = \cos A \cos B - \sin A \sin B,$$

which together give

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

Notice here that

$$\tan(45° + A) = \frac{1 + \tan A}{1 - \tan A} \ \text{ and } \ \tan(45° - A) = \frac{1 - \tan A}{1 + \tan A}.$$

Fact 5 (Sum to Product). We have

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) \ \text{ and } \ \cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right).$$

Fact 6 (Product to Sum). We have

$$\cos A \cos B = \frac{1}{2}\left[\cos\left(A+B\right) + \cos\left(A-B\right)\right] \ \text{ and } \ \sin A \sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right].$$

Fact 7. In a triangle ABC, we have:

- $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = \dfrac{1}{2R}$

- $a^2 = b^2 + c^2 - 2bc \cos A$

- $\dfrac{a-b}{a+b} = \dfrac{\tan\frac{1}{2}(A-B)}{\tan\frac{1}{2}(A+B)}$

- $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

- $\cos A + \cos B + \cos C = 1 + \dfrac{r}{R}$

- $\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{A}{2}\tan\frac{C}{2} = 1$

- $\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2} = \cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

- $\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C = 1$

4   Problems

Problem 2. Evaluate the expression

$$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}.$$

Problem 3 (Melanie 2005). Prove

$$\sum_{k=1}^{n} \cot^{-1}(2k^2) = \cot^{-1}\left(1 + \frac{1}{n}\right).$$

Problem 4 (Melanie 2005)**.** Evaluate (for $a \notin \pi \mathbb{Z}$)

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{a}{2^n}.$$

Problem 5 (Melanie 2005)**.** Let $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$. Prove that

$$\sum_{k=1}^{180} F_k \sin k° = \frac{F_{180} \cos 1° + \frac{1}{2}(F_{179} + F_{181}) + 1}{2 \sin 1° + \frac{1}{2} \csc 1°}.$$

Problem 6 (HMMT 2008)**.** Let $\mathcal{C}$ be the hyperbola $y^2 - x^1 = 1$. Given a point P_0 on the x-axis, we construct a sequence of points $\{P_n\}$ on the x-axis in the following manner: let ℓ_n be the line with slope 1 passing through P_n. Then, P_{n+1} is the orthogonal projection of the point of intersection of ℓ_n and $\mathcal{C}$ onto the x-axis. (If $P_n = 0$, then the sequence simply terminates.)
Let N be the number of starting positions P_0 on the x-axis such that $P_0 = P_{2008}$. Determine the remainder of N when divided by 2008.

Problem 7 (USAMO 1996)**.** Let ABC be a triangle, and M an interior point such that $\angle MAB = 10°$, $\angle MBA = 20°$, $\angle MAC = 40°$, and $\angle MCA = 30°$. Prove that triangle ABC is isosceles.

Problem 8 (USAMO 1995)**.** A calculator is broken so that the only keys that still work are the sin, cos, tan, arcsin, arccos, and arctan buttons. The display initially shows 0. Given any positive rational number q, show that pressing some finite sequence of buttons will yield q. Assume that the calculator does real number calculations with infinite precisions. All functions are in terms of radians.

Problem 9 (USAMO 1996)**.** Prove that the average of the numbers $n \sin n°$ for $n = 2, 4, 6, \ldots, 180$ is $\cot 1°$.

Problem 10 (USAMO 2001)**.** Let $a, b, c \geq 0$ be real numbers satisfying

$$a^2 + b^2 + c^2 + abc = 4.$$

Show that

$$ab + bc + ca - abc \leq 2.$$

Problem 11 (USAMO 2002)**.** Let ABC be a triangle such that

$$\left(\cot \frac{A}{2} \right)^2 + \left(2 \cot \frac{B}{2} \right)^2 + \left(3 \cot \frac{C}{2} \right)^2 = \left(\frac{6s}{7r} \right)^2,$$

where s and r denote its semiperimeter and inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisors and determine these integers.

Problem 12 (TST 2009)**.** Find all triples (x, y, z) of real numbers that satisfy the system of equations

$$\begin{cases} x^3 &= 3x - 12y + 50, \\ y^3 &= 12y + 3z - 2, \\ z^3 &= 27z + 27x. \end{cases}$$

Problem 13 (USAMO 2003). A convex polygon P in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon P are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

Problem 14 (USAMO 2000). Let S be the set of all triangles ABC for which

$$5\left(\frac{1}{AP} + \frac{1}{BQ} + \frac{1}{CR}\right) - \frac{3}{\min\{AP, BQ, CR\}} = \frac{6}{r},$$

where r is the inradius and P, Q, R are the points of tangency of the incircle with sides AB, BC, CA, respectively. Prove that all triangles in S are isosceles and similar to one another.

Problem 15 (China TST 2004). Let a, b, c be the sides of a triangle whose perimeter does not exceed 2π. Prove that $\sin a, \sin b, \sin c$ are the sides of a triangle.

Problem 16 (USAMO 1998). Let $a_0, a_1, \ldots, a_n$ be numbers from the interval $\left(0, \frac{\pi}{2}\right)$ such that

$$\tan\left(a_0 - \frac{\pi}{4}\right) + \tan\left(a_1 - \frac{\pi}{4}\right) + \cdots + \tan\left(a_n - \frac{\pi}{4}\right) \geq n - 1.$$

Prove that

$$\tan a_0 \tan a_1 \cdots \tan a_n \geq n^{n+1}.$$

Problem 17 (TST 2002). Let A, B, C be the angles of a triangle. Prove that

$$\sin\frac{3A}{2} + \sin\frac{3B}{2} + \sin\frac{3C}{2} \leq \cos\frac{A-B}{2} + \cos\frac{B-C}{2} + \cos\frac{C-A}{2}.$$

Problem 18 (TST 2003). Let A, B, C be real numbers in the interval $\left(0, \frac{\pi}{2}\right)$. Prove that

$$\frac{\sin A \sin(A-B)\sin(A-C)}{\sin(B+C)} + \frac{\sin B \sin(B-C)\sin(B-A)}{\sin(C+A)} + \frac{\sin C \sin(C-A)\sin(C-B)}{\sin(A+B)} \geq 0.$$

Problem 19 (China 2005). For $i = 1, 2, 3, 4$, take $\theta_i \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Prove that there exists $x \in \mathbb{R}$ such that

$$\cos^2\theta_1 \cos^2\theta_2 - (\sin\theta_1 \sin\theta_2 - x)^2 \geq 0 \text{ and } \cos^2\theta_3 \cos^2\theta_4 - (\sin\theta_3 \sin\theta_4 - x)^2 \geq 0$$

if and only if

$$\sum_{i=1}^{4} \sin^2\theta_i \leq 2\left(1 + \prod_{i=1}^{4}\sin\theta_i + \prod_{i=1}^{4}\cos\theta_i\right).$$

Problem 20 (TST 2007). Let θ be an angle in the interval $(0, \pi/2)$. Given that $\cos\theta$ is irrational and that both $\cos k\theta$ and $\cos[(k+1)\theta]$ are rational for some positive integer k, show that $\theta = \pi/6$.

Problem 21 (China 2006). Triangle ABC contains a convex polygon P. Prove that there exists a triangle congruent to ABC that also contains P and has a side parallel to a side of P.