LAGUERRE AND JACOBI ANALOGUES OF THE WARREN PROCESS

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ABSTRACT. We define Laguerre and Jacobi analogues of the Warren process. That is, we construct local dynamics on a triangular array of particles so that the projections to each level recover the Laguerre and Jacobi eigenvalue processes of König-O'Connell and Doumerc and the fixed time distributions recover the joint distribution of eigenvalues in multilevel Laguerre and Jacobi random matrix ensembles. Our techniques extend and generalize the framework of intertwining diffusions developed by Pal-Shkolnikov. One consequence is the construction of particle systems with local interactions whose fixed time distribution recovers the hard edge of random matrix theory.

CONTENTS

1. INTRODUCTION

The purpose of the present work is to construct and characterize Laguerre and Jacobi analogues of the Warren process. These are stochastic dynamics on a triangular array of particles such that each particle evolves independently outside of interactions implemented by reflections of particles on level n off particles on level $n-1$. We show that when started from Gibbs initial conditions, their projections to each level are Markovian and coincide in law with Laguerre and Jacobi analogues of Dyson Brownian motion introduced in [KO01] and [Dou05], and their fixed time distributions recover the joint distribution of eigenvalues in multilevel Laguerre and Jacobi random matrix ensembles. When projected to the left edge, our results yield

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a construction of an interacting particle system with local interactions whose fixed time distribution recovers the hard edge of random matrix theory.

Our motivation comes from the original construction of Warren in [War07] providing a coupling of Dyson Brownian motions via reflected Brownian motions. In the context of interacting particle systems, multilevel intertwinings of Brownian particles have appeared in the works [War07, WW09, MOW09, GS15a], while [O'C12, BC14, GS15b] study other intertwined processes whose single level projections are similar to Dyson Brownian motion but with more complicated multilevel interactions. The Laguerre and Jacobi Warren processes we consider in this work differ from the previously mentioned processes in incorporating both non-Brownian diffusion terms and reflection of particles from one level on particles from lower levels.

Our method of proof proceeds via the framework of intertwining diffusions introduced by Pal-Shkolnikov in [PS15]. We extend their results by giving a general partial existence criterion for intertwining diffusions with non-Brownian diffusion term; our criterion involves two new conditions which are not present in the Brownian case. The main results then follow by applying our new criterion to the Laguerre and Jacobi eigenvalues processes in conjunction with some general results on strong existence for stochastic differential equations with reflection on time-dependent barrier in one dimension.

In the remainder of this introduction, we state our results more precisely and provide additional motivation and background. For the reader's convenience, all notations will be redefined in later sections.

Remark. After this work was completed and during the final preparation of this article, the author was made aware of the recent preprint [AOW16], which obtains similar results using a different approach.

1.1. **Dyson Brownian motion and the Warren process.** For each $n \geq 1$, let $X_n(t)$ be a standard Brownian motion in the space of $n \times n$ Hermitian matrices. It was shown by Dyson in [Dys62] that the evolution of the ordered eigenvalues $\lambda_1^n(t) \leq \cdots \leq \lambda_n^n(t)$ of $X_n(t)$ is Markovian and solves the SDE

$$d\lambda_i^n(t) = dB_i^n(t) + \sum_{j \neq i} \frac{1}{\lambda_i^n(t) - \lambda_j^n(t)} dt, \qquad 1 \leq i \leq n,$$

where $B_i^n(t)$ are n independent standard real Brownian motions. The resulting process is known as *Dyson Brownian motion*. In [War07], Warren introduced the following system of stochastic differential equations with reflection valued in the Gelfand-Tsetlin cone

$$\mathbb{GT}_n := \{\mu_i^k \mid \mu_{i-1}^{k-1} \leq \mu_i^k \leq \mu_i^{k-1}\}_{1 \leq i \leq k, 1 \leq k \leq n}$$

and given by

$$d\mu_i^k(t) = dB_i^k(t) + \frac{1}{2} dL_i^{k,+}(t) - \frac{1}{2} dL_i^{k,-}(t), \qquad 1 \leq i \leq k, 1 \leq k \leq n, \tag{1.1}$$

where $B_i^k(t)$ are standard real Brownian motions, $L_i^{k,-}(t)$ is 0 if $i = k$ and the local time of $\mu_i^k(t) - \mu_i^{k-1}(t)$ at 0 otherwise, and $L_i^{k,+}(t)$ is 0 if $i = 1$ and the local time of $\mu_i^k(t) - \mu_{i-1}^{k-1}(t)$ at 0 otherwise. He proved the following theorem, which shows that the unique weak solution to (1.1), known as the *Warren process*, gives a coupling of Dyson Brownian motions with different numbers of particles. Let $\Delta(\mu) := \prod_{i<j}(\mu_i - \mu_j)$ denote the Vandermonde determinant.

Theorem 1.1 ([War07, Section 4])**.** The SDE (1.1) satisfies the following properties.

(a) It admits a unique weak solution $\{\mu_i^k(t)\}$ when started at 0 with entrance law

$$(2\pi)^{-n} t^{-n^2/2} \Delta(\mu^n) \prod_{i=1}^n e^{-(\mu_i^n)^2/2t} \prod_{k=1}^n \prod_{i=1}^k d\mu_i^k.$$

(b) For each k, the projection of $\{\mu_i^k(t)\}$ to level k is Markovian and coincides in law with Dyson Brownian motion started at 0 with entrance law

$$(2\pi)^{-k/2} t^{-k^2/2} \Delta(\mu^k)^2 \prod_{i=1}^k e^{-(\mu_i^k)^2/2t} d\mu_i^k.$$

1.2. **Statement of the main results.** The purpose of the present work is to provide a generalization of the Warren process of Theorem 1.1 for Laguerre and Jacobi analogues of Dyson Brownian motion. For the Laguerre case, it was shown in [KO01] that the largest $\min\{n,p\}$ singular values

$$0 \le \lambda_1^{(n)}(t) \le \cdots \le \lambda_{\min\{n,p\}}^{(n)}(t)$$

of a $n \times p$ rectangular matrix of complex Brownian motions satisfy the stochastic differential equation

$$d\lambda_i^{(n)}(t) = 2\sqrt{\lambda_i^{(n)}(t)}dB_i^{(n)}(t) + 2\Big(|n-p|+1\Big)dt + \sum_{j\neq i} \frac{4\lambda_i^{(n)}(t)}{\lambda_i^{(n)}(t) - \lambda_j^{(n)}(t)}dt,$$

where $B_i^{(n)}(t)$ are standard Brownian motions, and may be started at 0 with entrance law proportional to

$$\Delta(\lambda^{(n)})^2 \prod_{i=1}^{\min\{n,p\}} (\lambda_i^{(n)})^{|p-n|} e^{-\frac{\lambda_i^{(n)}}{2t}} d\lambda_i^{(n)}.$$

This process is known as the Laguerre eigenvalues process of rank p and level n. Consider the system of SDE's with reflection

$$(1.2)\quad dl_i^n(t) = 2\sqrt{l_i^n(t)}dB_i^n(t) + 2\Big(|p-n|+1\Big)dt + \frac{1}{2}dL_i^{n,+}(t) - \frac{1}{2}dL_i^{n,-}(t), 1 \le i \le \min\{n,p\}, 1 \le n \le m,$$

where $B_i^n(t)$ are standard real Brownian motions, $L_i^{n,+}(t)$ is 0 if $i=1$ and the local time at 0 of $l_i^n(t) - l_{i-1}^{n-1}(t)$ otherwise, and $L_i^{n,-}(t)$ is 0 if $i=n$ and the local time at 0 of $l_i^{n-1}(t) - l_i^n(t)$ otherwise. We say that an initial condition $l_i^n(0)$ for (1.2) is Gibbs if for any Borel B, we have

$$(1.3)\qquad \mathbb{P}_\nu(\{l_i^n(0)\} \in B \mid l^m(0) = \lambda\} = (\min\{m,p\}-1)!\frac{\mathrm{vol}(B)}{\Delta(\lambda)}.$$

Theorem 1.2, our first main result, states that a solution to (1.2) from a Gibbs initial condition provides a simultaneous coupling of Laguerre eigenvalues processes of different levels. It follows from Theorems 3.1 and 3.2 and Corollaries 3.3 and 3.4 in Section 3.1.

Theorem 1.2. The SDER (1.2) admits a unique strong solution, known as the Laguerre Warren process, for any Gibbs initial condition. This solution satisfies the following properties.

(a) Its projection to level n is Markovian and coincides in law with the Laguerre eigenvalues process of rank p and level n.
(b) Its fixed time distribution at any $t > 0$ is Gibbs.
(c) It may be started from $l_i^n(0) = 0$ with entrance law proportional to

$$\Delta(l^m) \prod_{i=1}^{\min\{m,p\}} (l_i^m)^{|p-m|} e^{-\frac{l_i^m}{2t}} \prod_{n=1}^{m} \prod_{i=1}^{\min\{n,p\}} dl_i^n.$$

Remark. The fixed-time marginals of Theorem 1.2(c) correspond to the joint distribution of eigenvalues at different levels of the Laguerre ensemble from random matrix theory. More precisely, let $X(t)$ be a matrix of complex Brownian motions with p columns and an infinite number of rows. Letting $X_n(t)$ consist of its top n rows, the distribution of Theorem 1.2(c) is the joint distribution of the largest $\min\{n,p\}$ singular values of $X_n(t)$ for $1 \le n \le m$.

For the Jacobi case, fix parameters (p,q) and $n \le p,q$, and let $N = p+q$. In [Dou05], it was shown that the singular values

$$0 \le \mu_1^{(n)}(t) \le \cdots \le \mu_n^{(n)}(t) \le 1$$

of the top left $n \times p$ submatrix of a Brownian motion on the space of unitary $N \times N$ matrices satisfy the stochastic differential equation

$$d\mu_i^{(n)}(t) = 2\sqrt{\mu_i^{(n)}(t)(1-\mu_i^{(n)}(t))}dB_i^{(n)}(t) + 2\Big((p-n+1)+(p+q-2n+2)\mu_i^{(n)}(t)\Big)dt + \sum_{j\neq i}\frac{4\mu_i^{(n)}(1-\mu_i^{(n)}(t))}{\mu_i^{(n)}(t)-\mu_j^{(n)}(t)}dt$$

for standard real Brownian motions $B_i^{(n)}(t)$ and may be started with invariant measure proportional to

$$\Delta(\mu^{(n)})^2 \prod_{i=1}^n (\mu_i^{(n)})^{p-n}(1-\mu_i^{(n)})^{q-n} d\mu_i^{(n)}.$$

This process is known as the Jacobi eigenvalues process with parameters (p,q) and level n. Consider the system of SDE's with reflection

$$dj_i^n(t) = 2\sqrt{j_i^n(t)(1-j_i^n(t))}dB_i^n(t) + 2\Big((p-n+1)+(p+q-2n+2)j_i^n(t)\Big)dt + \frac{1}{2}dL_i^{n,+}(t) - \frac{1}{2}dL_i^{n,-}(t), 1 \leq i \leq n, 1 \leq n \leq \min\{p,q\}, \tag{1.4}$$

where $B_i^n(t)$ are standard real Brownian motions, $L_i^{n,+}(t)$ is 0 if $i=1$ and the local time at 0 of $j_i^n(t)-j_{i-1}^{n-1}(t)$ otherwise, and $L_i^{n,-}(t)$ is 0 if $i=n$ and the local time at 0 of $j_i^{n-1}(t)-j_i^n(t)$ otherwise. Theorem 1.3, our second main result, states that a solution to (1.4) from a Gibbs initial condition defined by (1.3) provides a simultaneous coupling of Jacobi eigenvalues processes of different levels. It follows from Theorems 3.5 and 3.6 and Corollaries 3.7 and 3.8 in Section 3.3.

Theorem 1.3. The SDER (1.4) admits a unique strong solution, known as the Jacobi Warren process, for any Gibbs initial condition. This solution satisfies the following properties.

(a) Its projection to level n is Markovian and coincides in law with the Jacobi eigenvalues process with parameters (p,q) and level n.
(b) Its fixed time distribution at any $t>0$ is Gibbs.
(c) It may be started with invariant measure proportional to

$$\Delta(j^{\min\{p,q\}}) \prod_{i=1}^{\min\{p,q\}} (j_i^{\min\{p,q\}})^{p-\min\{p,q\}}(1-j_i^{\min\{p,q\}})^{q-\min\{p,q\}} \prod_{n=1}^{\min\{p,q\}} \prod_{i=1}^{n} dj_i^n.$$

Remark. The invariant measure of Theorem 1.3 corresponds to the joint distribution of eigenvalues at different levels of the Jacobi ensemble from random matrix theory. More precisely, let X and Y be matrices of standard complex Gaussians with infinitely many columns and p rows and q rows, respectively. Let X_n and Y_n be the first n columns of X and Y; then the measure of Theorem 1.3 is the joint density of eigenvalues of $X_n^*X_n(X_n^*X_n+Y_n^*Y_n)^{-1}$ for $1 \leq n \leq \min\{p,q\}$.

1.3. **Pal-Shkolnikov method of intertwining diffusions.** Our proof of Theorems 1.2 and 1.3 proceeds through the framework of intertwining diffusions introduced by Pal-Shkolnikov in [PS15], where it was shown that the Warren process of [War07] and the Whittaker process of [O'C12] fall into this framework. Let X and Y be diffusion processes with generators

$$\mathcal{A}^X := \frac{1}{2}\sum_{i=1}^m\sum_{j=1}^m a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^m b_i(x)\partial_{x_i}$$

$$\mathcal{A}^Y := \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \rho_{ij}(y)\partial_{y_i}\partial_{y_j} + \sum_{i=1}^n \gamma_i(y)\partial_{y_i}$$

on domains $\mathcal{X}$ and $\mathcal{Y}$. Fix a domain $D \subset \mathcal{X}\times\mathcal{Y}$, and define $D(y) := \{x \mid (x,y)\in D\}$. Let $\Lambda(y,x): D \to \mathbb{R}$ be a non-negative function defining the integral kernel

$$(Lf)(y) := \int_{D(y)} \Lambda(y,x)f(x)dx.$$

Under several technical conditions given in Assumptions 5.1, 5.2, and 5.3, Pal-Shkolnikov define an intertwining diffusion as follows.

Definition 5.4 ([PS15, Definition 2])**.** Suppose that D, X, Y, L satisfy Assumptions 5.1, 5.2, and 5.3. A process $Z=(Z_1,Z_2)$ valued in a domain D is an intertwining of diffusions X and Y with link operator L if:

(i) $Z_1 \overset{d}{=} X$ and $Z_2 \overset{d}{=} Y$, where $\overset{d}{=}$ denotes equality in law, and

$$\mathbb{E}[f(Z_1(0)) \mid Z_2(0)=y] = (Lf)(y),$$

for all bounded Borel measurable functions f on $D(y)$.

(ii) The transition semigroups P_t and Q_t of Z_1 and Z_2 are intertwined, meaning that $Q_t L = LP_t$ for all $t \geq 0$.

(iii) The process Z_1 is Markovian with respect to the joint filtration generated by (Z_1, Z_2).

(iv) For any $s \geq 0$, conditional on $Z_2(s)$, the random variable $Z_1(s)$ is independent of $\{Z_2(u), 0 \leq u \leq s\}$ and is conditionally distributed according to L.

(v) For any $t \geq 0$, conditional on $Z_2(0)$ and $Z_1(t)$, the random variables $Z_1(0)$ and $Z_2(t)$ are independent.

We give in Theorem 5.6 a new partial criterion for the existence of an intertwining diffusion with reflection on moving boundaries which extends Pal-Shkolnikov's criterion in [PS15, Theorem 3] to the case of non-Brownian diffusion terms. This criterion requires Assumption 5.5, in which conditions (5.1) and (5.2) are significant modifications from [PS15, Theorem 3] to handle non-Brownian diffusion terms. Checking the conditions of this criterion yields the desired couplings of Theorems 1.2 and 1.3. While we are not aware of other examples of applications of Theorem 5.6, we believe it may be of independent interest.

Theorem 5.6. Suppose that D, Λ, X, Y satisfy Assumptions 5.1, 5.2, 5.3, and 5.5. Suppose the SDER (5.3) has a weak solution D which is a Feller diffusion for which the intersection of $C_c^\infty(D)$ with the domain of $\mathcal{A}^Z$ in $C_0(D)$ is dense in the domain of $\mathcal{A}^Z$ in $C_0(D)$ and that for $(x, y) \in \partial D$, $U(x, y)$ intersects the faces of $\overline{D}$ containing (x, y) only at (x, y). If the resulting process satisfies the initial condition

$$\mathbb{P}(Z_1(0) \in B \mid Z_2(0) = y) = \int_B \Lambda(y, x) dx \text{ for Borel } B \subset D(y),$$

then Z satisfies parts (i) to (iv) of the condition to be an intertwining of X and Y with link L.

Remark. We believe that the hypotheses of Theorem 5.6 are sufficient for Z to be a full intertwining of X and Y with link L, but we have not yet performed the estimates necessary to check part (v) of the intertwining condition.

1.4. **Organization of the paper.** The remainder of this paper is organized as follows. In Section 2, we define the Laguerre and Jacobi eigenvalues processes, their realizations via Doob h-transform, and their entrance law and invariant measure. In Section 3, we define the Laguerre and Jacobi Warren processes and prove Theorems 3.1, 3.2, 3.5, and 3.6 showing that they give coupling of the Laguerre and Jacobi eigenvalues processes at different levels; in this section we make reference to tools developed in the next two sections. In Section 4, we collect results from the literature to prove Theorem 4.4 giving a criterion for strong uniqueness and existence for stochastic differential equations with reflection on time-dependent boundaries with Holder regular diffusion term. In Section 5, we introduce the Pal-Shkolnikov framework of intertwining diffusions and prove Theorem 5.6 partially extending their results to general diffusion terms and reflection on moving boundaries. Sections 4 and 5 develop tools which we apply in Section 3 and may be read independently.

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2. The Laguerre and Jacobi eigenvalues processes

In this section we introduce the Laguerre and Jacobi eigenvalues processes as Doob h-transforms of independent squared Bessel and univariate Jacobi processes and explain their realization as eigenvalues of certain matrix-valued processes.

2.1. **The Laguerre eigenvalues process.** Let $A(t)$ be an infinite matrix of complex Brownian motions with p columns and initial condition $A(0)$. Denote by $A_n(t)$ its top n rows. The *complex Wishart process* of rank p and level n is the process valued in $n \times n$ matrices given by $M_n(t) = A_n(t)A_n(t)^*$. The *Laguerre eigenvalues process* of rank p and level n consists of the $\min\{n, p\}$ largest eigenvalues

$$0 \leq \lambda_1^{(n)}(t) \leq \cdots \leq \lambda_{\min\{n,p\}}^{(n)}(t)$$

of $M_n(t)$. This process was introduced in the real case in [Bru91], studied in the complex case in [KO01], and the singular SDE it satisfies was analyzed in detail in [GM14]; we collect some of its properties below.

Proposition 2.1 ([Bru91, Theorem 1, Corollary 1], [KO01, Lemma 3.2], and [GM14, Corollary 6.2])**.** For any initial condition $\lambda^{(n)}(0)$, the Laguerre eigenvalues process $\lambda^{(n)}(t)$ of rank p and level n satisfies the following:

(a) The $\lambda^n(t)$ are Markovian, have all $\lambda_i^{(n)}(t)$ distinct for $t > 0$, and form a diffusion with generator

$$\mathsf{L}_n^p := \sum_{i=1}^{\min\{n,p\}} 2\lambda_i^{(n)}\partial_i^2 + \sum_{i=1}^{\min\{n,p\}} 2\Big(|n-p|+1\Big)\partial_i + \sum_{i=1}^{\min\{n,p\}} \sum_{j\neq i} \frac{4\lambda_i^{(n)}}{\lambda_i^{(n)} - \lambda_j^{(n)}}\partial_i.$$

Equivalently, $\lambda^{(n)}(t)$ is a solution to the SDE

$$d\lambda_i^{(n)}(t) = 2\sqrt{\lambda_i^{(n)}(t)}dB_i^{(n)}(t) + 2\Big(|n-p|+1\Big)dt + \sum_{j\neq i} \frac{4\lambda_i^{(n)}(t)}{\lambda_i^{(n)}(t) - \lambda_j^{(n)}(t)}dt$$

for standard Brownian motions $B_i^{(n)}(t)$.

(b) When started at $\lambda^{(n)}(0) = 0$, the Laguerre eigenvalues process has entrance law proportional to

$$\Delta(\lambda^{(n)})^2 \prod_{i=1}^{\min\{p,n\}} (\lambda_i^{(n)})^{|p-n|} e^{-\frac{\lambda_i^{(n)}}{2t}} d\lambda_i^{(n)}.$$

In [KO01], it was observed that the Laguerre eigenvalues process may be constructed from independent squared Bessel processes conditioned never to intersect. More precisely, let $\mathsf{BESQ}^d(t)$ denote the squared Bessel process of dimension d; it solves the stochastic differential equation

$$d\mathsf{BESQ}^d(t) = 2\sqrt{\mathsf{BESQ}^d(t)}dB(t) + d\,dt$$

and has generator $2x\partial + d\,\partial$. Let $\Delta(\lambda^{(n)})$ denote the Vandermonde determinant $\Delta(\lambda^{(n)}) := \prod_{i<j}(\lambda_i^{(n)} - \lambda_j^{(n)})$, and denote the generator of $\min\{n,p\}$ independent squared Bessel processes of dimension $2\Big(|n-p|+1\Big)$ by

$$L_n^p := \sum_{i=1}^{\min\{n,p\}} 2\lambda_i^{(n)}\partial_i^2 + \sum_{i=1}^{\min\{n,p\}} 2\Big(|n-p|+1\Big)\partial_i.$$

In [KO01], König-O'Connell realized the Laguerre eigenvalues process as a Doob h-transform as follows.

Proposition 2.2 ([KO01])**.** The function $\Delta(\lambda^{(n)})$ is harmonic with respect to L_n^p, and the Doob h-transform of $\min\{n,p\}$ independent squared Bessel processes of dimension $2\Big(|n-p|+1\Big)$ with respect to $\Delta(\lambda^{(n)})$ is the Laguerre eigenvalues process with rank p and level n.

2.2. **The Jacobi eigenvalues process.** Fix parameters (p,q) and $n \leq p,q$, and let $N = p+q$. Let $U_N(t)$ be a Brownian motion on the space of unitary $N \times N$ matrices, and let $X_n^{p,q}(t)$ denote the top left $n \times p$ submatrix of $U_N(t)$. In [Dou05], the *complex matrix Jacobi process* with parameters (p,q) and level n was defined to be the process valued in $n \times n$ matrices given by

$$J_n^{p,q}(t) := X_n^{p,q}(t)X_n^{p,q}(t)^*.$$

The *Jacobi eigenvalues process* with parameters (p,q) and level n consists of the eigenvalues

$$0 \leq \mu_1^{(n)}(t) \leq \cdots \leq \mu_n^{(n)}(t) \leq 1$$

of $J_n^{p,q}(t)$. This process was introduced in [Dou05] and analyzed in detail in [GM13, GM14]. We collect some of its properties below.

Proposition 2.3 ([Dou05, Proposition 9.4.7], [GM13, Corollary 9], [GM14, Corollary 6.7])**.** The Jacobi eigenvalues process $\mu^{(n)}(t)$ with parameters (p,q) and level n satisfies the following:

(a) When started at any initial condition $\mu^{(n)}(0)$, $\mu^{(n)}(t)$ is Markovian, has all $\mu_i^{(n)}(t)$ distinct for $t > 0$, and forms a diffusion with generator

$$\mathsf{J}_n^{p,q} := 2\sum_{i=1}^n \mu_i^{(n)}(1-\mu_i^{(n)})\partial_i^2 + 2\sum_{i=1}^n \Big((p-n+1) + (p+q-2n+2)\mu_i^{(n)}\Big)\partial_i + \sum_{i=1}^n \sum_{j\neq i} \frac{4\mu_i^{(n)}(1-\mu_i^{(n)})}{\mu_i^{(n)} - \mu_j^{(n)}}\partial_i.$$

Equivalently, $\mu^{(n)}(t)$ is a solution to the SDE

$$d\mu_i^{(n)}(t) = 2\sqrt{\mu_i^{(n)}(t)(1-\mu_i^{(n)}(t))}dB_i^{(n)}(t)+2\Big((p-n+1)+(p+q-2n+2)\mu_i^{(n)}(t)\Big)dt+\sum_{j\neq i}\frac{4\mu_i^{(n)}(t)(1-\mu_i^{(n)}(t))}{\mu_i^{(n)}(t)-\mu_j^{(n)}(t)}dt$$

for standard Brownian motions $B_i^{(n)}(t)$.

(b) The Jacobi eigenvalues process is a diffusion whose invariant measure on $[0,1]^n$ has density proportional to

$$\Delta(\mu^{(n)})^2\prod_{i=1}^n(\mu_i^{(n)})^{p-n}(1-\mu_i^{(n)})^{q-n}d\mu_i^{(n)}. \tag{2.1}$$

Remark. The density of (2.1) admits an alternate realization as follows. It is the probability density of the eigenvalues of $X^*X(X^*X+Y^*Y)^{-1}$, where X and Y are matrices of standard complex Gaussians of size $p\times n$ and $q\times n$, respectively.

In [Dou05], it was observed that the Jacobi eigenvalues process may be constructed from independent univariate Jacobi processes conditioned to never intersect. More precisely, let $\mathsf{JAC}^{a,b}(t)$ denote the univariate Jacobi process with parameters (a,b). It is the diffusion which solves the stochastic differential equation

$$d\mathsf{JAC}^{a,b}(t) = 2\sqrt{\mathsf{JAC}^{a,b}(t)(1-\mathsf{JAC}^{a,b}(t))}dB(t)+2\Big(a+1-(a+b+2)\mathsf{JAC}^{a,b}(t)\Big)dt$$

and has generator $2x(1-x)\partial^2+2(a+1-(a+b+2)x)\partial$. Denote the generator of n independent univariate Jacobi processes by

$$J_n^{a,b} := 2\sum_{i=1}^n\mu_i^{(n)}(1-\mu_i^{(n)})\partial_i^2+2\sum_{i=1}^n(a+1-(a+b+2)\mu_i^{(n)})\partial_i.$$

In [Dou05], Doumerc realized the Jacobi eigenvalues process via Doob h-transform as follows.

Proposition 2.4 ([Dou05, Proposition 9.4.7])**.** For p,q,n with $n\le\min\{p,q\}$, the function $\Delta(\mu^{(n)})$ is an eigenfunction $J_n^{p-n,q-n}$ with eigenvalue $\frac{n(n-1)(3p+3q-4n+2)}{3}$, and the Doob h-transform of n independent univariate Jacobi processes with parameters $(p-n,q-n)$ is the Jacobi eigenvalues process of with parameters (p,q) and level n.

3. The Laguerre and Jacobi Warren processes via intertwining diffusions

In this section, we define the Laguerre and Jacobi Warren processes and show that, when started at Gibbs initial conditions, their projections to each level recover the Laguerre and Jacobi eigenvalues processes. These are the main results of this paper.

3.1. The Laguerre Warren process.

In this section we define the Laguerre Warren process as the unique weak solution to a certain stochastic differential equation with reflection. This construction is the Laguerre analogue of the Warren process defined using Brownian motions in [War07]. Denote the positive Gelfand-Tsetlin cone of rank p and level m by

$$\mathbb{GT}_{m,p} := \{l_i^n \mid 0\le l_{i-1}^{n-1}\le l_i^n\le l_i^{n-1} \text{ and } l_1^m<\cdots<l_{\min\{m,p\}}^m\}_{1\le i\le\min\{n,p\},1\le n\le m}$$

and the positive Gelfand-Tsetlin polytope of rank p and level m subordinate to $\lambda=(\lambda_1<\cdots<\lambda_{\min\{m,p\}})$ by

$$\mathbb{GT}_{m,p}(\lambda) := \{\{l_i^n\}\in\mathbb{GT}_{m,p}\mid l^m=\lambda\}.$$

We say that a probability distribution ν on $\mathbb{GT}_{m,p}$ is *Gibbs* if for any $(\lambda_1<\cdots<\lambda_{\min\{m,p\}})$ and any Borel $B\subset\mathbb{GT}_{n,p}(\lambda)$, we have that

$$\mathbb{P}_\nu(\{l_i^n\}\in B\mid l^m=\lambda) = (\min\{m,p\}-1)!\frac{\mathrm{vol}(B)}{\Delta(\lambda)}.$$

Consider the system of stochastic differential equations with reflection with domain $\mathbb{GT}_{m,p}$ given by

$$dl_i^n(t) = 2\sqrt{l_i^n(t)}dB_i^n(t)+2\Big(|p-n|+1\Big)dt+\frac{1}{2}dL_i^{n,+}(t)-\frac{1}{2}dL_i^{n,-}(t),\ 1\le i\le\min\{n,p\},1\le n\le m, \tag{3.1}$$

where $B_i^n(t)$ are standard real Brownian motions, $L_i^{n,+}(t)$ is 0 if $i = 1$ and the local time at 0 of $l_i^n(t) - l_{i-1}^{n-1}(t)$ otherwise, and $L_i^{n,-}(t)$ is 0 if $i = n$ and the local time at 0 of $l_i^{n-1}(t) - l_i^n(t)$ otherwise.

Remark. Informally, a solution to (3.1) may be described as follows. At level n, it consists of independent squared Bessel processes with dimension $2\Big(|p-n|+1\Big)$ interlacing with and reflecting off the processes at level $n-1$. This differs from the Warren process by replacing Brownian motions by squared Bessel processes and introducing different parameters on each level.

The following two theorems, whose proofs are given in Section 3.2, are the first of our main results. We show that (3.1) admits a unique strong solution for any Gibbs initial condition and that this solution provides a coupling of Laguerre eigenvalues processes on each level. We call the resulting process the *Laguerre Warren process*. The key technical ingredient is our extension of the theory of intertwining diffusions of [PS15] given in Section 5.

Theorem 3.1. For any Gibbs initial condition $\{\lambda_i^n(0)\}$, if the SDER (3.1) admits a unique weak solution which is a Feller Markov process, then for $1 < n \leq m$, its projection to levels n and $n-1$ is Markovian and satisfies parts (i) to (iv) of the condition to be an intertwining of the Laguerre eigenvalues processes of rank p and levels n and $n-1$ in the sense of Definition 5.4.

Theorem 3.2. For any Gibbs initial condition $\{l_i^n(0)\}$, the SDER (3.1) admits a unique strong solution $\{l_i^n(t)\}_{1\leq i\leq \min\{n,p\}, 1\leq n\leq m}$ which is a Feller Markov process and which we call the Laguerre Warren process.

Corollary 3.3. For $1 \leq n \leq m$, the projection of the Laguerre Warren process to level n is Markovian and coincides in law with the Laguerre eigenvalues process of rank p and level n.

Proof. This follows by combining Theorem 3.2, Theorem 3.1, and the definition of intertwining. □

Corollary 3.4. The Laguerre Warren process may be started from $l_i^n(0) = 0$ with entrance law proportional to

$$\Delta(l^m) \prod_{i=1}^{\min\{m,p\}} (l_i^m)^{|p-m|} e^{-\frac{l_i^m}{2t}} \prod_{n=1}^{m} \prod_{i=1}^{\min\{n,p\}} dl_i^n.$$

Proof. This measure is the Gibbs measure associated to the entrance law of the Laguerre eigenvalues process from Proposition 2.1(b). By Theorem 3.1, the Laguerre Warren process preserves Gibbs measures and projects to the Laguerre eigenvalues process of rank p and level m, hence the claimed measure is a valid entrance law because the measure of Proposition 2.1(b) is. □

Remark. Consider the projection of $\{l_i^n(t)\}$ to the smallest particle on each level. The result is a Markovian particle process on

$$l_1^1(t) \geq l_1^2(t) \geq \cdots \geq l_1^m(t) \geq 0$$

solving the system of stochastic differential equations

$$dl_1^n(t) = 2\sqrt{l_1^n(t)}dB_1^n(t) + 2\Big(|p-n|+1\Big)dt - \frac{1}{2}dL_1^{n,-}(t), \tag{3.2}$$

where $L_1^{n,-}(t)$ is the local time at 0 of $l_1^{n-1}(t) - l_1^n(t)$. By Corollary 3.3, for each level n and time t, the smallest particle $l_1^n(t)$ on each level is equal in law to the smallest particle of a Laguerre eigenvalues process of rank p and level n. Therefore, the fixed time distribution at $t = 1$ has the law of the smallest particle of a Laguerre random matrix of shape $p \times n$, which is the hard edge of random matrix theory. On the other hand, (3.2) only involves interactions between neighboring particles, meaning that this construction gives a method to produce the hard edge from a process with purely local interactions.

3.2. **Proofs for the Laguerre Warren process.** In this section we collect proofs for Theorems 3.1 and 3.2. We first prove Theorem 3.1 by producing an intertwining using Theorem 5.6 via the Dixon-Anderson kernel

$$\Lambda(y,x) := (n-1)!\frac{\Delta(x)}{\Delta(y)}.$$

This kernel was shown via random corank 1 projections to correspond to the Gibbs property of fixed time distributions of random matrix eigenvalues in [FR05, FN11] and is thus the natural one to use.

Proof of Theorem 3.1. Let $\{l_i^n(t)\}$ be a weak solution to (3.1) with Gibbs initial condition. We prove the claim by induction on $1 \leq n \leq m$. Notice first that $l_1^1(t)$ is a free squared Bessel process of dimension $2p$, which coincides with the Laguerre eigenvalues process of rank p and level 1, giving the base case. We now consider the inductive step. Suppose the result holds for the projection to levels $(n-2, n-1)$. By the structure of (3.1), the projection of the process to levels $(n-1, n)$ is a Feller Markov process which solves the SDE with reflection

$$dl_i^{n-1}(t) = 2\sqrt{l_i^{n-1}(t)}dB_i^{n-1}(t) + 2\Big(|n-p-1|+1\Big)dt + \sum_{j\neq i}\frac{4l_i^{n-1}(t)}{l_i^{n-1}(t)-l_j^{n-1}(t)}dt \tag{3.3}$$

$$dl_k^n(t) = 2\sqrt{l_k^n(t)}dB_k^n(t) + 2\Big(|n-p|+1\Big)dt + \frac{1}{2}dL_k^{n,+}(t) - \frac{1}{2}dL_k^{n,-}(t)$$

for $1 \leq i \leq \min\{n-1, p\}$ and $1 \leq k \leq \min\{n, p\}$, where $L_k^{n,+}(t)$ is the local time of $l_k^n(t) - l_{k-1}^{n-1}(t)$ at 0 and $L_k^{n,-}(t)$ is the local time of $l_k^{n-1}(t) - l_k^n(t)$ at 0. We will apply Theorem 5.6 on domain

$$D := \begin{cases} \{(x,y) \mid 0 \leq y_1 \leq x_1 \leq y_2 \cdots \leq x_{n-1} \leq y_n \text{ and } y_1 < \cdots < y_n\} & n \leq p \\ \{(x,y) \mid 0 \leq y_1 \leq x_1 \leq y_2 \cdots \leq y_{\min\{n,p\}} \leq x_{\min\{n-1,p\}} \text{ and } y_1 < \cdots < y_{\min\{n,p\}}\} & n > p. \end{cases}$$

with the Dixon-Anderson kernel $\Lambda(y, x)$ to show that $(l^{n-1}(t), l^n(t))$ satisfies parts (i) to (iv) of the condition to be an intertwining of the Laguerre eigenvalues processes $X(t)$ and $Y(t)$ of rank p and levels $n-1$ and n. For this, we verify the hypotheses of Theorem 5.6 for D and Λ. For the verification, we use freely the notations of Section 5 in this setting; note that $X(t)$ and $Y(t)$ have no boundary in this case. In addition, we use $\mathsf{L}_n^{p,y}$ and $\mathsf{L}_{n-1}^{p,x}$ to denote generators acting in the y and x variables, respectively.

- **Assumption 5.1:** Note that the Laguerre eigenvalues process is Feller as a projection under the eigenvalue map of the Wishart process, which is Feller by [Bru91, Lemma 2].
- **Assumption 5.2:** Points (a) and (b) are evident. For (c), if $n \leq p$, then $D(y) = [y_1, y_2] \times \cdots \times [y_{n-1}, y_n]$, so that the faces of $\partial D(y)$ are contained in $\{x_i = y_i\}$ and $\{x_i = y_{i+1}\}$ for $1 \leq i \leq n-1$. If $n > p$, then $D(y) = [y_1, y_2] \times \cdots \times [y_{\min\{n,p\}-1}, y_{\min\{n,p\}}] \times [y_{\min\{n,p\}}, \infty)$ so that the faces of $\partial D(y)$ are contained in $\{x_i = y_i\}$ for $1 \leq i \leq \min\{n,p\}$ and $\{x_i = y_{i+1}\}$ for $1 \leq i \leq \min\{n,p\}-1$. In either case, as explained in [PS15, Section 5], on $\partial D(y)_k = \{x_i = y_i\}$ we have $\eta_k = -1_i$ and $\Psi_k^j = -\delta_{ij}\eta_k$, and on $\partial D(y)_k = \{x_i = y_{i+1}\}$ we have $\eta_k = 1_i$ and $\Psi_k^j = \delta_{i+1,j}\eta_k$, verifying the conditions of (c). Point (d) is ensured by the constraint $y_1 < \cdots < y_{\min\{n,p\}}$ on D.
- **Assumption 5.3:** Points (a) and (b) follow from the definition of $\Lambda(y, x)$. For (c), note that $\mathsf{L}_n^{p,y} = \Delta(y)^{-1} \circ L_n^{2(|p-n|+1),y} \circ \Delta(y)$ by Proposition 2.2, so the transition semigroup of Y is the conjugate of the transition semigroup of free squared Bessel processes by multiplication by $\Delta(y)$. Therefore, for any x, the function $\Lambda(-, x)$ is in the domain of $\mathsf{L}_n^{p,y}$ and $\mathsf{L}_n^{p,y}\Lambda(y, x) = 0$, hence is continuous and bounded, giving (c).
- **Assumption 5.5:** Point (a) is trivial because $Y(t)$ has no boundary reflection. Point (b) follows by definition of the Dixon-Anderson kernel. For (c), by Proposition 2.2, we have

  $$\mathsf{L}_n^{p,y} := \Delta(y)^{-1} \circ L_n^{2(|p-n|+1),y} \circ \Delta(y) \text{ and } (\mathsf{L}_{n-1}^{p,x})^* := \Delta(x) \circ L_{n-1}^{2(|p-n+1|+1),x} \circ \Delta(x)^{-1},$$

  which implies that $\mathsf{L}_n^{p,y}\Lambda(y,x) = 0 = (\mathsf{L}_{n-1}^{p,x})^*\Lambda(y,x)$. For (5.1), on $\partial D(y)_k = \{x_i = y_i\}$ the equality becomes

  $$2\Lambda\Big(-p+1-\sum_{j\neq i}\frac{x_i+x_j}{x_i-x_j}+\sum_{j\neq i}\frac{x_i}{x_i-x_j}\Big) = 2\Lambda\Big(-p-\sum_{j\neq i}\frac{y_i+y_j}{y_i-y_j}-\sum_{j\neq i}\frac{y_i}{x_i-x_j}+2\sum_{j\neq i}\frac{y_i}{y_i-y_j}\Big),$$

  and on $\partial D(y)_k = \{x_i = y_{i+1}\}$ it becomes

  $$2\Lambda\Big(p+\sum_{j\neq i}\frac{x_i+x_j}{x_i-x_j}-1-\sum_{j\neq i}\frac{x_i}{x_i-x_j}\Big) = 2\Lambda\Big(p+\sum_{j\neq i+1}\frac{y_{i+1}+y_j}{y_{i+1}-y_j}+\sum_{j\neq i}\frac{y_{i+1}}{x_i-x_j}-2\sum_{j\neq i+1}\frac{y_{i+1}}{y_{i+1}-y_j}\Big),$$

  both of which hold, establishing the boundary conditions of (c). Finally, for (d), on $\{x_i = y_i\}$, the relation (5.2) reduces to $-4x_i = -4y_i$ and on $\{x_i = y_{i+1}\}$ it reduces to $4x_i = 4y_{i+1}$, which both always hold.

We conclude that Assumptions 5.1, 5.2, 5.3, and 5.5 hold for $(X(t), Y(t))$. Notice now that the SDER of Theorem 5.6 is given by (3.3) and by the given has a weak solution which is a Feller diffusion whose Markov generator (5.4) is

$$\mathcal{A}^Z := \mathsf{L}^{p,x}_{n-1} + \mathsf{L}^{p,y}_{n} - \sum_{i=1}^{\min\{n,p\}} \sum_{j \neq i} \frac{4y_i}{y_i - y_j} \partial_{y_j}$$
$$= \mathsf{L}^{p,x}_{n-1} + \sum_{i=1}^{\min\{n,p\}} 2y_i \partial_{y_i}^2 + \sum_{i=1}^{\min\{n,p\}} 2\Big(|n-p|+1\Big)\partial_{y_i},$$

where we note that $\partial_{y_i} \log \Lambda(y,x) = -\sum_{j\neq i} \frac{1}{y_i - y_j}$. The domain of $\mathcal{A}^Z$ is given by imposing the Neumann boundary conditions

$$-4y_i \partial_{y_i} v = 0 \text{ on } \{x_i = y_i\} \qquad \text{and} \qquad 4y_{i+1}\partial_{y_{i+1}} v = 0 \text{ on } \{x_i = y_{i+1}\} \tag{3.4}$$

on $\partial D(y)$; these satisfy the cone condition of Theorem 5.6 and imply that $\mathsf{domain}(\mathcal{A}^Z)$ satisfies the domain condition of Theorem 5.6. Therefore, for Gibbs initial condition we may apply Theorem 5.6 to conclude that $(l^{n-1}(t), l^n(t)$ satisfies parts (i) to (iv) of the condition to be an intertwining of Laguerre eigenvalue processes of rank p and levels $n-1$ and n. □

We now prove Theorem 3.2 by applying Theorem 3.1 to reduce (3.1) to the case of adjacent levels, rephrasing that case in terms of a stochastic differential equation with reflection on a time dependent boundary as described in Section 4, and finally applying the the Yamada-Watanabe type exactness criterion for SDER's provided by Theorem 4.4. A crucial input is given by the fact proven in Proposition 2.1 that the Laguerre eigenvalues process has no collisions; this corresponds to the fact that the time-dependent boundaries in our SDER never touch.

Proof of Theorem 3.2. We proceed by induction on m. For the base case $m = 1$, equation (3.1) is the SDE with no reflection terms for a single copy of $\mathsf{BESQ}^{2p}(t)$, hence admits a unique strong solution which is Feller Markov. For the inductive step, suppose that the result holds for (3.1) for $m-1$ and consider (3.1) for m. By Theorem 3.1 applied to the solution $\{l_i^n(t)\}_{1 \le i \le \min\{n,p\}, 1 \le n \le m-1}$ for $m-1$, the projection to $\{l_i^{m-1}(t)\}_{1\le i \le \min\{m-1,p\}}$ is equal in law to the Laguerre eigenvalues process of rank p and level $m-1$. In particular, we have a.s. that $l_i^{m-1}(t) < l_{i+1}^{m-1}(t)$ for $1 \le i < \min\{m-1, p\} - 1$. Therefore, by Theorem 4.4, there exists a unique strong solution $\{l_i^m(t)\}_{1 \le i \le \min\{m,p\}}$ to the SDER

$$dl_i^m(t) = 2\sqrt{l_i^m(t)} dB_i^m(t) + 2\Big(|p-m|+1\Big)dt + d\Phi_i^m(t) - d\Psi_i^m(t)$$

with time-dependent boundaries given by $l_{i-1}^{m-1}(t) < l_i^{m-1}(t)$. To check that $\Phi_i^m(t) = \frac{1}{2}L_i^{m,+}(t)$, we note by the Itô-Tanaka formula of [RY91, Theorem VI.1.2] that

$$\frac{1}{2}L_i^{m,+}(t) = (l_i^m(t) - l_{i-1}^{m-1}(t))^+ - (l_i^m(0) - l_{i-1}^{m-1}(0))^+ - \int_0^t 1_{l_i^m(s) > l_{i-1}^{m-1}(s)} d(l_i^m(s) - l_{i-1}^{m-1}(s))$$
$$= l_i^m(t) - l_i^m(0) - \int_0^t 2\sqrt{l_i^m(s)} dB_i^m(s) - \int_0^t 2(|p-m|+1)ds + \Psi_i^m(t)$$
$$+ \int_0^t 1_{l_i^m(s) = l_{i-1}^{m-1}(s)} \Big(2\sqrt{l_i^m(s)} dB_i^m(s) + 2(|p-m|+1)dt\Big)$$
$$= \Phi_i^m(t) + \int_0^t 1_{l_i^m(s) = l_{i-1}^{m-1}(s)} \Big(2\sqrt{l_i^m(s)} dB_i^m(s) + 2(|p-m|+1)dt\Big).$$

Now, by the occupation time formula of [RY91, Corollary VI.1.6], we see that

$$\int_0^t 1_{l_i^m(s) = l_{i-1}^{m-1}(s)} (4l_i^m(s) + 4l_{i-1}^{m-1}(s))ds = \int_{-\infty}^{\infty} 1_{a=0}\, dL_t^a(l_i^m - l_{i-1}^{m-1}) = 0.$$

We conclude that

$$\mathbb{P}\Big(\int_0^t 1_{l_i^m(s) = l_{i-1}^{m-1}(s)} ds = 0\Big) = 1,$$

and therefore that

$$\int_0^t 1_{l_i^m(s)=l_{i-1}^{m-1}(s)}(2\sqrt{l_i^m(s)}dB_i^m(s) + 2(|p-m|+1)dt) = 0,$$

so $\Phi_i^m(t) = \frac{1}{2}L_i^{m,+}(t)$. A similar argument shows that $\Psi_i^m(t) = \frac{1}{2}L_i^{m,-}(t)$. Combining $\{l_i^m(t)\}$ with the previous solution $\{l_i^n(t)\}_{1\leq i\leq \min\{n,p\},1\leq n\leq m-1}$ yields the desired unique strong solution.

We now check that $\{l_i^n(t)\}_{1\leq i\leq \min\{n,p\},1\leq n\leq m}$ is a Feller process. Because $\{l_i^n(t)\}$ has continuous trajectories, to check that it is Feller, it suffices to check that its generator $\mathcal{A}$ preserves the space $C_0(\mathbb{GT}_{m,p})$ of continuous functions on $\mathbb{GT}_{m,p}$ vanishing at ∞. For this, we induct on m; for $m=1$, this follows because the squared Bessel process is Feller. For the inductive step, it suffices to show that for any $t\geq 0$ and any bounded Borel set $A\times B\subset \mathbb{GT}_{m+1,p}$ with $A\subset \mathbb{GT}_{m,p}$, we have

$$\lim_{\{\mu_i^n\}\to\infty} \mathbb{P}\Big(\{l_i^n(t)\}_{1\leq n\leq m+1} \in A\times B \mid l_i^n(0) = \mu_i^n\Big) = 0 \tag{3.5}$$

for any trajectory $\{\mu_i^n\}_{1\leq n\leq m+1}\to\infty$. If $\{\mu_i^n\}_{1\leq n\leq m}\to\infty$, (3.5) follows by the Feller property for A and $\{l_i^n(t)\}_{1\leq n\leq m}$ given by the inductive hypothesis. Otherwise, by the interlacing property, we must have $m+1\leq p$ and $\mu_{m+1}^{m+1}\to\infty$ along the trajectory; in this case, the law of $l_{m+1}^{m+1}(t)$ stochastically dominates the law of a squared Bessel process started at μ_{m+1}^{m+1}, so (3.5) follows by the Feller property for the squared Bessel process applied to the projection of B onto the last coordinate. This completes the proof that $\{l_i^n(t)\}$ is Feller. □

3.3. **The Jacobi Warren process.** In this section, we couple the Jacobi eigenvalue processes at different levels. This construction is the Laguerre analogue of the Warren process for Dyson Brownian motion of [War07]. Define the the $[0,1]$ Gelfand-Tsetlin polytope with parameters (p,q) by

$$\mathbb{GT}_{p,q}^{[0,1]} := \{j_i^n \mid 0\leq j_{i-1}^{n-1}\leq j_i^n\leq j_i^{n-1}\leq 1 \text{ and } j_1^{\min\{p,q\}} < \cdots < j_{\min\{p,q\}}^{\min\{p,q\}}\}_{1\leq i\leq n, 1\leq n\leq \min\{p,q\}}$$

and the $[0,1]$ Gelfand-Tsetlin polytope with parameters (p,q) subordinate to $\mu = (\mu_1 < \cdots < \mu_{\min\{p,q\}})$ by

$$\mathbb{GT}_{p,q}^{[0,1]}(\mu) := \{\{j_i^n\}\in \mathbb{GT}_{p,q} \mid j^{\min\{p,q\}} = \mu\}.$$

We say that a probability distribution ν on $\mathbb{GT}_{p,q}^{[0,1]}$ is Gibbs if for any $\mu = (\mu_1 < \cdots < \mu_{\min\{p,q\}})$ and any Borel $B\subset \mathbb{GT}_{p,q}^{[0,1]}(\mu)$, we have that

$$\mathbb{P}_\nu(\{j_i^n\}\in B \mid j^{\min\{p,q\}} = \mu) = (\min\{p,q\}-1)!\frac{\text{vol}(B)}{\Delta(\mu)}.$$

Consider the system of stochastic differential equations with reflection with domain $\mathbb{GT}_{p,q}^{[0,1]}$ given by

$$\begin{aligned} dj_i^n(t) &= 2\sqrt{j_i^n(t)(1-j_i^n(t))}dB_i^n(t) \\ &+ 2\Big((p-n+1) - (p+q-2n+2)j_i^n(t)\Big)dt + \frac{1}{2}dL_i^{n,+}(t) - \frac{1}{2}dL_i^{n,-}(t), \qquad 1\leq i\leq n, 1\leq n\leq p,q, \end{aligned} \tag{3.6}$$

where $B_i^n(t)$ are standard real Brownian motions, $L_i^{n,+}$ is 0 if $i=1$ and the local time of $j_i^n(t) - j_{i-1}^{n-1}(t)$ at 0 otherwise, and $L_i^{n,-}$ is 0 if $i=n$ and the local time of $j_i^{n-1}(t) - j_i^n(t)$ at 0 otherwise.

Remark. Informally, a solution to (3.6) may be described as follows. At level n, it consists of independent univariate Jacobi processes with parameters $(p-n, q-n)$ interlacing with and reflecting off the processes at level $n-1$. As with the SDER (3.1), this differs from the Warren process by replacing Brownian motions by univariate Jacobi processes and introducing different parameters on each level.

The following two theorems are analogues of Theorems 3.1 and 3.2 and are our second set of main results; proofs will be given in Section 3.4. We show that (3.6) admits a unique strong solution for any Gibbs initial condition and that this solution provides a coupling of the Jacobi eigenvalues processes on each level. We call the resulting process the *Jacobi Warren process*.

Theorem 3.5. For any Gibbs initial condition $\{j_i^n(0)\}$, if the SDER (3.6) admits a unique weak solution which is a Feller Markov process, then for $1 < n\leq \min\{p,q\}$, its projection to levels n and $n-1$ is Markovian and satisfies parts (i) to (iv) of the condition to be an intertwining of the Jacobi eigenvalues processes with parameters (p,q) and levels n and $n-1$ in the sense of Definition 5.4.

Theorem 3.6. For any Gibbs initial condition $\{j_i^n(0)\}$, the SDER (3.6) admits a unique strong solution $\{j_i^n(t)\}_{1\leq i\leq n, 1\leq n\leq \min\{p,q\}}$ which is a Feller Markov process and which we call the Jacobi Warren process.

Corollary 3.7. For $1 \leq n \leq \min\{p, q\}$, the projection of the Jacobi Warren process to level n is Markovian and coincides in law with the Jacobi eigenvalues process with parameters (p, q) and level n.

Proof. This follows by combining Theorem 3.6, Theorem 3.5, and the definition of intertwining. □

Corollary 3.8. The Jacobi Warren process admits an invariant measure proportional to

$$\Delta(j^{\min\{p,q\}}) \prod_{i=1}^{\min\{p,q\}} (j_i^{\min\{p,q\}})^{p-\min\{p,q\}}(1 - j_i^{\min\{p,q\}})^{q-\min\{p,q\}} \prod_{n=1}^{\min\{p,q\}} \prod_{i=1}^{n} dj_i^n.$$

Proof. This measure is the Gibbs measure associated to the invariant measure (2.1) for the Jacobi eigenvalues process with parameters (p, q) and level $\min\{p, q\}$ from Proposition 2.3(b). By Theorem 3.5, the Jacobi Warren process preserves Gibbs measures and projects to the Jacobi eigenvalues process. Therefore, the claimed measure is invariant for the Jacobi Warren process because its projection to level $\min\{p, q\}$ is invariant for the Jacobi eigenvalues process. □

3.4. **Proofs for the Jacobi Warren process.** In this section we collect proofs for Theorems 3.5 and 3.6. We first prove Theorem 3.5 by again applying Theorem 5.6 with the Dixon-Anderson kernel in a manner similar to the proof of Theorem 3.1.

Proof of Theorem 3.5. Let $\{j_i^n(t)\}$ be a weak solution to (3.6). We prove the claim by induction on $1 \leq n \leq \min\{p, q\}$. For the base case, notice that $j_1^1(t)$ is a univariate Jacobi process with parameters (p, q), which coincides with the Jacobi eigenvalues process with parameters (p, q) and level 1.

We now consider the inductive step. Suppose the result holds for the projection to levels $(n - 2, n - 1)$. By the structure of (3.6), the projection of the process to levels $(n - 1, n)$ is a Feller Markov process which solves the SDE with reflection

$$\begin{aligned} dj_i^{n-1}(t) &= 2\sqrt{j_i^{n-1}(t)(1 - j_i^{n-1}(t))}dB_i^{n-1}(t) \\ &\quad + 2\Big((p - n + 2) + (p + q - 2n + 4)j_i^{n-1}(t)\Big)dt + \sum_{j\neq i} \frac{4j_i^{n-1}(t)(1 - j_i^{n-1}(t))}{j_i^{n-1}(t) - j_j^{n-1}(t)}dt \qquad (3.7) \\ dj_k^n(t) &= 2\sqrt{j_k^n(t)(1 - j_k^n(t))}dB_k^n(t) + 2\Big((p - n + 1) + (p + q - 2n + 2)j_k^n(t)\Big)dt + \frac{1}{2}dL_k^{n,+}(t) - \frac{1}{2}dL_k^{n,-}(t) \end{aligned}$$

for $1 \leq i \leq n - 1$ and $1 \leq k \leq n$, where $L_k^{n,+}(t)$ is the local time of $j_k^n(t) - j_{k-1}^{n-1}(t)$ at 0 and $L_k^{n,-}(t)$ is the local time of $j_k^n(t) - j_k^{n-1}(t)$ at 0. We will apply Theorem 5.6 on the domain

$$D := \{(x, y) \mid 0 \leq y_1 \leq x_1 \leq y_2 \cdots \leq x_{n-1} \leq y_n \leq 1 \text{ and } y_1 < \cdots < y_n\}$$

with the Dixon-Anderson kernel $\Lambda(y, x)$ to show that $(j^{n-1}(t), j^n(t))$ satisfies parts (i) to (iv) of the condition to be an intertwining of the Jacobi eigenvalue processes $X(t)$ and $Y(t)$ with parameters (p, q) and levels $n-1$ and n, respectively. For this, we verify the hypotheses of Theorem 5.6 for D and Λ. For the verification, we use freely the notations of Section 5 in this setting; note that $X(t)$ and $Y(t)$ have no boundary in this case. As before, we use $\mathsf{J}_n^{p,q,y}$ and $\mathsf{J}_{n-1}^{p,q,x}$ to denote generators acting in the y and x variables, respectively.

- **Assumption 5.1:** Note that the Jacobi eigenvalues process is Feller because it has compact domain and continuous trajectories.
- **Assumption 5.2:** This follows in the same way as in the proof of Theorem 3.1.
- **Assumption 5.3:** Points (a) and (b) follow from the definition of $\Lambda(y, x)$. For (c), by Proposition 2.4, we have that

$$\mathsf{J}_n^{p,q,y} = \Delta(y)^{-1} \circ J_n^{p-n,q-n,y} \circ \Delta(y) + \frac{n(n - 1)(3p + 3q - 4n + 2)}{3},$$

  so Λ lies in its domain. Further, we see that

$$\mathsf{J}_n^{p,q,y}\Lambda(y, x) = \frac{n(n - 1)(3p + 3q - 4n + 2)}{3}\Lambda(y, x)$$

  is continuous and bounded on $\bigcup_{x\in K} D(x, -)$ for all compact K, giving (c).

- **Assumption 5.5:** Point (a) is trivial because $Y(t)$ has no reflecting boundary. Point (b) follows by definition of the Dixon-Anderson kernel. For (c), by Proposition 2.4, we see that

$$\frac{\mathsf{J}_n^{p,q,y}\Lambda(y,x)}{\Lambda(y,x)} = \frac{n(n-1)(3p+3q-4n+2)}{3}.$$

By directly computing adjoints, we have that

$$\begin{aligned}(\mathsf{J}_{n-1}^{p,q,x})^* &= \Delta(x)\circ J_{n-1}^{p-n+1,q-n+1,x}\circ\Delta(x)^{-1} \\ &\quad -4(n-1)+2(n-1)\Big((p-n+1)+(q-n+1)+2\Big)+\frac{(n-1)(n-2)(3p+3q-4n+6)}{3} \\ &= \Delta(x)\circ J_{n-1}^{p-n+1,q-n+1,x}\circ\Delta(x)^{-1}+\frac{n(n-1)(3p+3q-4n+2)}{3},\end{aligned}$$

from which we conclude that

$$\frac{(\mathsf{J}_{n-1}^{p,q,x})^*\Lambda(y,x)}{\Lambda(y,x)} = \frac{n(n-1)(3p+3q-4n+2)}{3}.$$

We conclude that $\mathsf{J}_n^{p-n,q-n,y}\Lambda(y,x) = (\mathsf{J}_{n-1}^{p-n+1,q-n+1,x})^*\Lambda(y,x)$. For (5.1), on $\partial D(y)_k = \{x_i = y_i\}$ we have that

$$\Lambda\langle b,\eta_k\rangle - \frac{1}{2}\Lambda\langle \mathrm{div}_x a,\eta_k\rangle - \frac{1}{2}\langle\langle a,\eta_k\rangle,\nabla_x\Lambda\rangle = \Lambda\Big(2-4x_i-2(p-n+2)+2(p+q-2n+4)x_i-2\sum_{j\neq i}\frac{x_i(1-x_i)}{x_i-x_j}\Big)$$

and

$$\sum_j\Lambda\gamma_j\langle\Psi_k^j,\eta_k\rangle+\frac{1}{2}\sum_{i,j}\rho_{ij}\Big(\langle\nabla_x\Lambda,\Psi_k^j\rangle+2\partial_{y_j}\Lambda\Big)\langle\Psi_k^i,\eta_k\rangle = \Lambda\Big(-2(p-n+1)+2(p+q-2n+2)y_i-2\sum_{j\neq i}\frac{y_i(1-y_i)}{x_i-x_j}\Big),$$

which are equal. On $\partial D(y)_k = \{x_i = y_{i+1}\}$ we have that

$$\Lambda\langle b,\eta_k\rangle - \frac{1}{2}\Lambda\langle \mathrm{div}_x a,\eta_k\rangle - \frac{1}{2}\langle\langle a,\eta_k\rangle,\nabla_x\Lambda\rangle = \Lambda\Big(-(2-4x_i)+2(p-n+2)-2(p+q-2n+4)x_i+2\sum_{j\neq i}\frac{x_i(1-x_i)}{x_i-x_j}\Big)$$

and

$$\sum_j\Lambda\gamma_j\langle\Psi_k^j,\eta_k\rangle+\frac{1}{2}\sum_{i,j}\rho_{ij}\Big(\langle\nabla_x\Lambda,\Psi_k^j\rangle+2\partial_{y_j}\Lambda\Big)\langle\Psi_k^i,\eta_k\rangle = \Lambda\Big(2(p-n+1)-2(p+q-2n+2)y_{i+1}+2\sum_{j\neq i}\frac{y_{i+1}(1-y_{i+1})}{x_i-x_j}\Big),$$

which are again equal, establishing (c). For (d), the condition of (5.2) holds because it is equivalent to $-4x_i(1-x_i) = -4y_i(1-y_i)$ on $\{x_i = y_i\}$ and $4x_i(1-x_i) = 4y_{i+1}(1-y_{i+1})$ on $\{x_i = y_{i+1}\}$.

We conclude that Assumptions 5.1, 5.2, 5.3, and 5.5 hold for the two Jacobi eigenvalues processes $X(t)$ and $Y(t)$ with parameters (p,q) and levels $n-1$ and n, respectively. Now, the SDER of Theorem 5.6 is given by (3.7) and by the given has a weak solution which is a Feller diffusion whose Markov generator (5.4) is

$$\begin{aligned}\mathcal{A}^Z &:= \mathsf{J}_{n-1}^{p,q,x}+\mathsf{J}_n^{p,q,y}-\sum_{i=1}^n\sum_{j\neq i}\frac{4y_i(1-y_i)}{y_i-y_j} \\ &= \mathsf{J}_{n-1}^{p,q,x}+\sum_{i=1}^n 2y_i(1-y_i)\partial_{y_i}^2+\sum_{i=1}^n 2(p-n+1-(p+q-2n+2)y_i)\partial_{y_i}.\end{aligned}$$

The domain of $\mathcal{A}^Z$ is given by imposing the Neumann boundary conditions

$$-4y_i(1-y_i)\partial_{y_i}v = 0 \text{ on } \{x_i = y_i\} \qquad\text{and}\qquad 4y_{i+1}(1-y_{i+1})\partial_{y_{i+1}}v = 0 \text{ on } \{x_i = y_{i+1}\} \tag{3.8}$$

on $\partial D(y)$; these satisfy the cone condition of Theorem 5.6 and imply that $\mathsf{domain}(\mathcal{A}^Z)$ satisfies the domain condition of Theorem 5.6. Therefore, for Gibbs initial conditions we may apply Theorem 5.6 to conclude that $(j^{n-1}(t), j^n(t))$ satisfies parts (i) to (iv) of the condition to be an intertwining between Jacobi eigenvalues processes with parameters (p,q) and levels $n-1$ and n. □

We now prove Theorem 3.6 in parallel to the proof of Theorem 3.2. Again, we require the fact from Proposition 2.3 that the Jacobi eigenvalues process has no collisions and hence the time-dependent boundaries in our SDER never touch.

Proof of Theorem 3.6. We proceed by induction on n to show that a strong solution exists for the first n levels of (3.6). For $n = 1$, the equation is the SDE with no reflection terms for a single copy of $\mathsf{JAC}^{p-1,q-1}(t)$, hence admits a unique strong solution which is Feller Markov. For the inductive step, suppose that the result holds for (3.6) for $n-1$ and consider (3.6) for n. By Theorem 3.5 applied to the solution $\{j_i^k(t)\}_{1\leq i\leq k, 1\leq k\leq n-1}$ for $n-1$, the projection to $\{j_i^{n-1}(t)\}$ is equal in law to the Jacobi eigenvalues process with parameters (p, q) and level $n-1$. In particular, we have a.s. that $j_i^{n-1}(t) < j_{i+1}^{n-1}(t)$ for $1 \leq i \leq n-2$. Therefore, by Theorem 4.4 there exists a unique strong solution $\{j_i^n(t)\}$ to the SDER

$$dj_i^n(t) = 2\sqrt{j_i^n(t)(1-j_i^n(t))}dB_i^n(t) + 2\Big((p-n+1) + (p+q-2n+2)j_i^n(t)\Big)dt + d\Phi_i^n(t) - d\Psi_i^n(t)$$

with time-dependent boundaries given by $j_{i-1}^{n-1}(t) < j_i^{n-1}(t)$. The proof that $\Phi_i^n(t) = \frac{1}{2}L_i^{m,+}(t)$ and $\Psi_i^n(t) = \frac{1}{2}l_i^{m,-}(t)$ now follows in the exact same way as in the proof of Theorem 3.2. Combining $\{j_i^n(t)\}$ with $\{j_i^k(t)\}_{1\leq i\leq k,1\leq k\leq n-1}$ yields the desired strong solution. The Feller property follows because the state space of $\{j_i^k(t)\}$ is compact and trajectories are continuous. □

4. Strong existence and uniqueness for SDER's with time-dependent boundary

In this section, we gather existing results to prove a criterion for strong existence and uniqueness of solutions to one-dimensional SDE's with reflection on two time-dependent boundaries. In particular, we handle the case of Lipschitz drift and Holder diffusion coefficient which is used in our applications.

4.1. **Statement of the SDER.** By a *context*, we mean a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with an increasing family $\{\mathcal{F}_t\}_{t\geq 0}$ of sub-σ-fields of $\mathcal{F}$ and an $\mathcal{F}_t$-adapted Brownian motion B_t. Given such a context, let (x_0, L_t, U_t) be $\mathcal{F}_0$-measurable random variables on $\mathbb{R}\times C[0,\infty)\times C[0,\infty)$ so that $L_t < U_t$ and $L_0 < x_0 < U_0$ a.s.. For diffusion and drift coefficients σ and b, we consider the SDE with reflection

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt + d\Phi_t - d\Psi_t. \tag{4.1}$$

A strong solution to (4.1) with initial condition x_0 is a triple of continuous $\mathcal{F}_t$-adapted processes (X_t, Φ_t, Ψ_t) so that:

- $X_t = \int_0^t \sigma(X_s)dB_s + \int_0^t b(X_s)ds + \Phi_t - \Psi_t$ and $X_0 = x_0$;
- $L_t \leq X_t \leq U_t$ for all t;
- Φ_t and Ψ_t are non-decreasing with bounded variation and $\Phi_0 = \Psi_0 = 0$;
- $\int_0^\infty 1_{X_t\neq L_t}d\Phi_t = \int_0^\infty 1_{X_t\neq U_t}d\Psi_t = 0$.

We say that the SDER (4.1) is *exact* if for every choice of initial-boundary conditions (x_0, L_t, U_t) there is a unique strong solution to (4.1).

Theorem 4.1 ([SW13, Theorem 3.3] and [BKR09, Corollary 2.4])**.** If σ and b are Lipschitz, then (4.1) is exact.

Remark. We allow $U_t \equiv \infty$ or $L_t \equiv -\infty$, in which case we have $\Psi_t \equiv 0$ or $\Phi_t \equiv 0$, respectively, and this setting coincides with that of [Sou01, Definition III.1.6].

Remark. This definition is a generalization of [Sou01, Definition III.1.6] and a specialization of the definition in [SW13, Section 3] to the case where $L_t < U_t$ for all t via [BKR09, Corollary 2.4]. In our setting, the random variables L_t and U_t do not depend on the past trajectory of X_t, hence are constant Lipschitz operators in the sense of [SW13, Section 3].

4.2. **Pathwise uniqueness and strong existence of solutions.** We have the following result on pathwise uniqueness for (4.1) following the modifications to the proof of [Sou01, Proposition III.5.2] discussed in [Sou01, Section IV.3.1],

Proposition 4.2. Let $\rho : \mathbb{R}_+ \to \mathbb{R}$ be a function so that $\int_{0+} \frac{1}{\rho(s)}ds = \infty$. Suppose that σ satisfies

$$|\sigma(x) - \sigma(x')|^2 \leq \rho(|x - x'|)$$

for $x, x' \in \mathbb{R}$ and b is Lipschitz. Then pathwise uniqueness holds for (4.1).

Proof. If $L_t \equiv -\infty$ or $U_t \equiv \infty$, this is exactly [Sou01, Proposition III.5.2]. Consider now the case where $-\infty < L_t < U_t < \infty$. Suppose that (X_t, Φ_t, Ψ_t) and (X'_t, Φ'_t, Ψ'_t) are two solutions to (4.1) defined on the same context. By Tanaka's formula applied to $X_t - X'_t$, we find that

$$\begin{aligned}(X_t - X'_t)_+ &= \int_0^t 1_{X_s > X'_s} d(X - X')_s + \frac{1}{2} L_t^0 \\ &= \int_0^t 1_{X_s > X'_s} (\sigma(X_s) - \sigma(X'_s)) dB_s + \int_0^t 1_{X_s > X'_s} (b(X_s) - b(X'_s)) ds \\ &\quad + \int_0^t 1_{X_s > X'_s} (d\Phi_s - d\Phi'_s) - \int_0^t 1_{X_s > X'_s} (d\Psi_s - d\Psi'_s) + \frac{1}{2} L_t^0,\end{aligned} \tag{4.2}$$

where L_t^a is the local time of $X_t - X'_t$ at a. We first claim that $L_t^0 = 0$ for all t. By the occupation time formula, we have that

$$\int_0^\infty \frac{1}{\rho(a)} L_t^a da = \int_0^t 1_{X_s > X'_s} \frac{1}{\rho(X_s - X'_s)} d\langle X - X', X - X' \rangle_s = \int_0^t 1_{X_s > X'_s} \frac{(\sigma(X_s) - \sigma(X'_s))^2}{\rho(X_s - X'_s)} ds \leq t.$$

On the other hand, if $L_t^0 > \varepsilon$ occurs with positive probability for some t and some $\varepsilon > 0$, then because $\lim_{a \to 0} L_t^a = L_t^0$, we may find $\delta > 0$ sufficiently small so that $L_t^a > \varepsilon/2$ for $a < \delta$ with positive probability. This implies that with positive probability we have

$$t \geq \int_0^\infty \frac{1}{\rho(a)} L_t^a da > \frac{\varepsilon}{2} \int_0^\delta \frac{1}{\rho(a)} da,$$

a contradiction. We conclude that $L_t^0 = 0$ for all t.

Now, notice that $X_s > X'_s$ implies that $X_s > L_s$ and that $X'_s < U_s$, hence we find that

$$\int_0^t 1_{X_s > X'_s} d\Phi_s = \int_0^t 1_{X_s > X'_s} d\Psi'_s = 0.$$

On the other hand, because Φ_t, Φ'_t, Ψ_t, and Ψ'_t are non-decreasing, we have that

$$\int_0^t 1_{X_s > X'_s} d\Phi'_s \geq 0 \qquad \text{and} \qquad \int_0^t 1_{X_s > X'_s} d\Psi'_s \geq 0.$$

Since $-\infty < L_s < U_s < \infty$, for $0 \leq s \leq t$, we see that $\sigma(X_s) - \sigma(X'_s)$ is bounded on $[0, t]$ and hence $\int_0^t 1_{X_s > X'_s} (\sigma(X_s) - \sigma(X'_s)) dB_s$ is a martingale. Taking expectations in (4.2), we conclude that

$$\begin{aligned}\mathbb{E}[(X_t - X'_t)_+] &\leq \mathbb{E}\left[\int_0^t 1_{X_s > X'_s} |b(X_s) - b(X'_s)| ds\right] \\ &\leq K\, \mathbb{E}\left[\int_0^t 1_{X_s > X'_s} |X_s - X'_s| ds\right] = K\, \mathbb{E}\left[\int_0^t (X_s - X'_s)_+ ds\right],\end{aligned}$$

where K is a Lipschitz constant for b. By Gronwall's Lemma, we conclude that $\mathbb{E}[(X_t - X'_t)_+] = 0$ and hence that $X_t \leq X'_t$ a.s.. On the other hand, exchanging the roles of X_t and X'_t implies that $X'_t \leq X_t$ and hence $X_t = X'_t$ a.s.. Since Φ_t and Ψ_t never increase at the same time, they are determined by X_t, hence we conclude $(X_t, \Phi_t, \Psi_t) = (X'_t, \Phi'_t, \Psi'_t)$ a.s., as desired. □

We now give a criterion for strong existence of solutions to (4.1) based on localization and Theorem 4.1. Let $D \subset \mathbb{R}$ be a connected domain. We say that (4.1) has a strong solution in D if for any $x_0 \in D$ and L_t, U_t lying in D for all t, there is a strong solution to (4.1) for which X_t lies in D for all t.

Proposition 4.3. Suppose σ and b are locally Lipschitz on D and pathwise uniqueness in D holds for (4.1) for all initial-boundary conditions. We have the following.

(a) For any initial-boundary condition (x_0, L_t, U_t) there exists a strong solution to (4.1) in D up to some explosion time τ_E;

(b) If strong existence in D holds for the classical SDE

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt \qquad X_0 = x_0$$

for each $x_0 \in D$, then we have $\tau_E = \infty$ a.s.

Proof. For (a), let $D_1 \subset D_2 \subset \cdots D$ be a sequence of connected subdomains which exhaust D and on which σ, b are Lipschitz. Let $D_k = (l_k, u_k)$, and define the modified functions

$$\sigma^k(x) := \begin{cases} \sigma(x) & x \in D_k \\ \sigma(l_k) & x \leq l_k \\ \sigma(u_k) & x \geq u_k \end{cases} \text{ and } b^k(x) := \begin{cases} b(x) & x \in D_k \\ b(l_k) & x \leq l_k \\ b(u_k) & x \geq u_k. \end{cases}$$

Observe that σ^k, b^k are Lipschitz on $\mathbb{R}$, so by Theorem 4.1 for every initial-boundary condition, there is a unique strong solution $(X_t^k, \Phi_t^k, \Psi_t^k)$ to the reflected SDE with Lipschitz coefficients

$$dX_t^k = \sigma^k(X_t^k)dB_t + b^k(X_t^k)dt + d\Phi_t^k - d\Psi_t^k \tag{4.3}$$

and initial-boundary condition (x_0, L_t, U_t). Define the stopping times $\tau_k := \inf\{t \mid X_t^k \notin D_k\}$ so that $\tau_1 \leq \tau_2 \leq \cdots$. By strong uniqueness, the restriction of $(X_t^k, \Phi_t^k, \Psi_t^k)$ to times in $[0, \tau_l)$ for $l \leq k$ coincides with $(X_t^l, \Phi_t^l, \Psi_t^l)$, so we may glue together all such solutions to obtain a strong solution $(X_t^*, \Phi_t^*, \Psi_t^*)$ in D valid until the explosion time $\tau_E := \lim_{k\to\infty} \tau_k$.

For (b), if $U_t \equiv \infty$ or $L_t \equiv -\infty$, then this follows from [Sou01, Lemma III.4.4]. Otherwise, for any $T > 0$, we may find some k so that $L_t, U_t \in D_k$ for $t \in [0, T]$. The solution $(X_t^k, \Phi_t^k, \Psi_t^k)$ to (4.3) given by Theorem 4.1 satisfies $L_t \leq X_t^k \leq U_t$, meaning that $X_t^k \in D_k$ for $0 \leq t \leq T$ and hence that $\tau_k > T$ and hence $\tau_E > T$. We conclude that $\tau_E = \infty$ a.s., as desired. ☐

4.3. **An exactness criterion for reflected SDE's.** We now assemble the previous results to prove the goal of this section, the following refinement of Theorem 4.1 which provides a criterion of Yamada-Watanabe type for exactness of SDER's applicable to the setting of squared Bessel and univariate Jacobi generators.

Theorem 4.4. Suppose that b is Lipschitz and for $x \neq x' \in \mathbb{R}$ the diffusion σ satisfies

$$|\sigma(x) - \sigma(x')|^2 \leq \rho(|x - x'|)$$

for some $\rho : \mathbb{R}_+ \to \mathbb{R}$ so that $\int_{0+} \frac{1}{\rho(s)} ds = \infty$. If σ is locally Lipschitz and for all $x \in \mathbb{R}$ we have

$$|\sigma(x)|^2 + |b(x)|^2 \leq K(1 + |x|^2)$$

for some constant K, then (4.1) is exact.

Proof. First, pathwise uniqueness holds by Proposition 4.2. Now, strong existence holds up to some explosion time τ_E by Proposition 4.3(a). By [IW81, Theorem IV.2.4], under these assumptions strong existence holds for the classical SDE associated to σ, b for all initial conditions, so we may apply Proposition 4.3(b) to conclude that $\tau_E = \infty$, giving strong existence and hence exactness. ☐

5. An extension of the Pal-Shkolnikov approach to intertwining diffusions

The goal of this section is to introduce the Pal-Shkolnikov approach to intertwining diffusions given in [PS15] and to prove Theorem 5.6, which provides an extension of [PS15, Theorem 3] to non-Brownian diffusion terms required in our work.

5.1. **The setting of Pal-Shkolnikov.** Consider two diffusions X and Y on domains $\mathcal{X} \subset \mathbb{R}^m$ and $\mathcal{Y} \subset \mathbb{R}^n$ with generators

$$\mathcal{A}^X = \frac{1}{2}\sum_{i=1}^m \sum_{j=1}^m a_{ij}(x)\partial_{x_i}\partial_{x_j} + \sum_{i=1}^m b_i(x)\partial_{x_i}$$

$$\mathcal{A}^Y = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \rho_{ij}(y)\partial_{y_i}\partial_{y_j} + \sum_{i=1}^n \gamma_i(y)\partial_{y_i},$$

where $(a_{ij}), (\rho_{ij})$ are continuous on the interiors of $\mathcal{X}$ and $\mathcal{Y}$ and take values in $m \times m$ and $n \times n$ positive semidefinite matrices and b, γ are continuous on the interiors of $\mathcal{X}$ and $\mathcal{Y}$. Suppose further that each of X and Y satisfy one of the following assumptions which generalize [PS15, Assumption 1]. Denote by $C_0(\mathcal{X})$ the set of continuous functions $f : \mathcal{X} \to \mathbb{R}$ so that $\lim_{|x|\to\infty} f(x) = 0$ and $C_c^\infty(\mathcal{X})$ the space of smooth compactly supported functions on $\mathcal{X}$.

Assumption 5.1. One of the following assumptions holds for processes X and Y.

(a) **No boundary conditions:** The martingale problem corresponding to $\mathcal{A}^X$ on $\mathcal{X}$ with no boundary conditions is well-posed in the sense of [SV69a, SV69b]. Moreover, the solution X of the martingale problem is a Feller process.

(b) **Neumann boundary conditions:** The domain $\mathcal{X}$ is smooth with $n(x)$ denoting the set of inward normal vectors to $\partial\mathcal{X}$ at $x \in \partial\mathcal{X}$. Let $U_1 : \partial\mathcal{X} \to \mathbb{R}^m$ be a smooth and nowhere vanishing vector field so that $\langle U_1(x), n(x)\rangle > 0$ for each $x \in \partial\mathcal{X}$. The submartingale problem corresponding to $\mathcal{A}^X$ and reflection direction defined by U_1 is well-posed in the sense of [SV71]. Moreover, the solution is a Feller process, and $\mathcal{A}^X$ is regular in the sense that the intersection of $C_c^\infty(\mathcal{X})$ with the domain of $\mathcal{A}^X$ in $C_0(\mathcal{X})$ is dense with respect to the uniform norm on $C_0(\mathcal{X})$.

For Y, replace $\mathcal{A}^X$ by $\mathcal{A}^Y$, $\mathcal{X}$ by $\mathcal{Y}$, and U_1 by U_2.

Let $D \subset \mathbb{R}^{m+n}$ be a domain with polyhedral closure, and let $D(x,-) := \{y \mid (x,y) \in D\}$ and $D(y) := \{x \mid (x,y) \in D\}$. We suppose that the following conditions hold.

Assumption 5.2. We consider the following assumptions on the domain D.

(a) The projection of D on $\mathbb{R}^m$ is $\mathcal{X}$, and the projection of D on $\mathbb{R}^n$ is $\mathcal{Y}$.

(b) For each $y \in \mathcal{Y}$, the domain $D(y) := \{x \mid (x,y) \in D\}$ has piecewise smooth boundary $\partial D(y)$. Denote the pieces by $\partial D(y)_i$ so that $\partial D(y) = \bigcup_k \partial D(y)_k$. On each $\partial D(y)_k$, the x-coordinate of $\partial D(y)$ can be parametrized as the diffeomorphic image $(x_k(y,\xi), y)$ of a smooth function $x_k(y,-)$.

(c) At each point $x \in \partial D(y)_k$, the directional derivatives Ψ_k^j of the boundary point $x = x_k(y,\xi)$ with respect to changes in the coordinates y_j exist and are piecewise constant in (x,y). In addition, for η_k the unit outward normal vector on $\partial D(y)_k$, we have $\eta_k = \sum_{j=1}^n \Psi_k^j \langle \Psi_k^j, \eta_k\rangle$ on $\partial D(y)_k$.

(d) Each dimension $m+n-d$ face of $\overline{D}$ with non-empty intersection with D lies in at most d dimension $m+n-1$ faces of $\overline{D}$.

Let $\Lambda(y,x) : D \to \mathbb{R}$ a non-negative function defining the integral kernel

$$(Lf)(y) := \int_{D(y)} f(x)\Lambda(y,x)dx$$

which maps $C_0(\mathcal{X})$ to $C_0(\mathcal{Y})$ so that the following conditions hold.

Assumption 5.3. We consider the following assumptions on the link function Λ.

(a) Λ is continuously differentiable in x on a neighborhood of $\partial\mathcal{X} \times Y \cup \partial D$;

(b) Λ is twice continuously differentiable in y on a neighborhood U_∂ of ∂D in $\mathcal{X} \times \mathcal{Y}$;

(c) for every $x \in \mathcal{X}$, $\Lambda(-,x)$ can be extended to a function on $\mathcal{Y}$ in the domain of $\mathcal{A}^Y$ in $C_0(\mathcal{Y})$ with $\mathcal{A}^Y\Lambda$ being continuous on D and bounded on $\bigcup_{x\in K} D(x,-)$ for any compact $K \subset \mathcal{X}$;

We now recall the notion of an intertwining of diffusions given in [PS15].

Definition 5.4 ([PS15, Definition 2]). Suppose that D, X, Y, L satisfy Assumptions 5.1, 5.2, and 5.3. A process $Z = (Z_1, Z_2)$ valued in a domain D is an intertwining of diffusions X and Y with link operator L if:

(i) $Z_1 \overset{d}{=} X$ and $Z_2 \overset{d}{=} Y$, where $\overset{d}{=}$ denotes equality in law, and

$$\mathbb{E}[f(Z_1(0)) \mid Z_2(0) = y] = (Lf)(y),$$

for all bounded Borel measurable functions f on $D(y)$.

(ii) The transition semigroups P_t and Q_t of Z_1 and Z_2 are intertwined, meaning that $Q_tL = LP_t$ for all $t \geq 0$.

(iii) The process Z_1 is Markovian with respect to the joint filtration generated by (Z_1, Z_2).

(iv) For any $s \geq 0$, conditional on $Z_2(s)$, the random variable $Z_1(s)$ is independent of $\{Z_2(u), 0 \leq u \leq s\}$ and is conditionally distributed according to L.

(v) For any $t \geq 0$, conditional on $Z_2(0)$ and $Z_1(t)$, the random variables $Z_1(0)$ and $Z_2(t)$ are independent.

5.2. **Existence of intertwiners for general diffusions.** We give conditions on D, Λ, X, Y under which a general theorem on existence of intertwiners holds. We consider the following compatibility condition.

Assumption 5.5. We consider the following compatibility conditions between X and Y given D and Λ.

(a) We have $\sum_{j=1}^n \langle \Psi_k^j, \eta_k\rangle U_{2,j} = 0$ on $\partial D(y)_k$ for $y \in \partial\mathcal{Y}$, where $U_{2,j}$ are the coordinates of U_2.

(b) For every $y \in \mathcal{Y}$, $\Lambda(y, \cdot)$ is a probability density on $D(y)$.
(c) The function Λ is twice continuously differentiable on $\overline{D}$ and solves

$$(\mathcal{A}^X)^* \Lambda = \mathcal{A}^Y \Lambda$$

pointwise with boundary conditions $\langle \nabla_y \Lambda, U_2 \rangle = 0$ on $\partial \mathcal{Y}$ and

$$\Lambda \langle b, \eta_k \rangle - \frac{1}{2} \Lambda \langle \mathrm{div}_x a, \eta_k \rangle - \frac{1}{2} \langle \langle a, \eta_k \rangle, \nabla_x \Lambda \rangle = \sum_j \Lambda \gamma_j \langle \Psi_k^j, \eta_k \rangle + \frac{1}{2} \sum_{i,j=1}^n \rho_{ij} \Big(\langle \nabla_x \Lambda, \Psi_k^j \rangle + 2 \partial_{y_j}(\Lambda) \Big) \langle \Psi_k^i, \eta_k \rangle \text{ on } \partial D(y)_k \text{ for each } y \in \mathcal{Y}. \tag{5.1}$$

(d) The diffusion terms satisfy the compatibility

$$\langle a, \eta_k \rangle = \sum_{i,j=1}^n \rho_{ij} \Psi_k^j \langle \Psi_k^i, \eta_k \rangle, \qquad \text{on } \partial D(y)_k \text{ for } y \in \mathcal{Y}. \tag{5.2}$$

Remark. In Assumption (5.5), condition (5.1) is modified from [PS15, Equation (2.29)] to account for non-Brownian diffusion terms, and condition (5.2) is new to our setting, as it is trivial in the case of Brownian diffusion terms.

We now formulate a SDER for which a solution will provide an intertwining diffusion of X and Y with link function L. Define the set-valued mapping $U(x, y)$ on $\partial D(x, -)$ to be the cone with vertex at 0 spanned by the vectors

$$\sum_{i,j=1}^n \rho_{ij}(y) \langle \Psi_k^i, \eta_k \rangle 1_j$$

for all k so that $x \in \partial D(y)_k$. Consider the SDER on domain D for $Z = (Z_1, Z_2)$ given by

$$\begin{aligned} dZ_1(t) &= \sigma_X(Z_1(t)) dB_X(t) + b(Z_1(t)) dt + d\Phi_1(t) \\ dZ_2(t) &= \sigma_Y(Z_2(t)) dB_Y(t) + \Big(\gamma(Z_2(t)) + \langle \rho(Z_2(t)), \nabla_{z_2}[\log \Lambda(Z_2(t), Z_1(t))] \rangle \Big) dt + d\Phi_2(t) + d\Phi(t), \end{aligned} \tag{5.3}$$

with the quantities satisfying

- $a(x) = \sigma_X(x) \sigma_X(x)^T$;
- $\rho(y) = \sigma_Y(y) \sigma_Y(y)^T$;
- $\Phi_1(t)$ is 0 if Assumption 5.1(a) holds and otherwise is a bounded variation process with an auxiliary function $\phi_1(s)$ so that $\Phi_1(0) = 0$, $\int_0^\infty 1_{Z_1(t) \notin \mathcal{X}} d|\Phi_1|(t) = 0$, $\Phi_1(t) = \int_0^t \phi_1(s) d|\Phi_1|(s)$, and $\phi_1(s)$ is in the same direction as $U_1(Z_1(s))$;
- $\Phi_2(t)$ satisfies the same conditions as $\Phi_1(t)$ with U_1 replaced by U_2 and $\mathcal{X}$ replaced by $\mathcal{Y}$;
- $\Phi(t)$ is a bounded variation process with an auxiliary function $\phi(s)$ so that it satisfies $\Phi(0) = 0$, $\int_0^\infty 1_{Z_2(t) \notin \partial D(Z_1(t), -)} d|\Phi|(t) = 0$, $\Phi(t) = \int_0^t \phi(s) d|\Phi|(s)$, and $\phi(s) \in U(Z_1(s), Z_2(s))$.

The SDER (5.3) corresponds to the formulation of the Skorokhod problem given in [KR14] and if well-posed its solution is a diffusion process with generator given by

$$\mathcal{A}^Z := \mathcal{A}^X + \mathcal{A}^Y + \sum_{i=1}^n \sum_{j=1}^n \rho_{ij}(y) \partial_{y_i} [\log \Lambda(y, x)] \partial_{y_j}, \tag{5.4}$$

with domain in $C_0(D)$ satisfying Neumann boundary conditions of X on $\partial \mathcal{X} \times \mathcal{Y} \cap D$, Neumann boundary conditions of Y on $\mathcal{X} \times \partial \mathcal{Y} \cap D$, and Neumann boundary conditions on $\partial D(x, -)$ given by

$$\sum_{i,j=1}^n \rho_{ij} \langle \Psi_k^i, \eta_k \rangle \partial_{y_j} v(y) = 0 \text{ for } x \in \partial D(y)_k. \tag{5.5}$$

Theorem 5.6. Suppose that D, Λ, X, Y satisfy Assumptions 5.1, 5.2, 5.3, and 5.5. Suppose the SDER (5.3) has a weak solution D which is a Feller diffusion for which the intersection of $C_c^\infty(D)$ with the domain of

$\mathcal{A}^Z$ in $C_0(D)$ is dense in the domain of $\mathcal{A}^Z$ in $C_0(D)$ and that for $(x,y) \in \partial D$, $U(x,y)$ intersects the faces of $\overline{D}$ containing (x,y) only at (x,y). If the resulting process satisfies the initial condition

$$\mathbb{P}(Z_1(0) \in B \mid Z_2(0) = y) = \int_B \Lambda(y,x)dx \text{ for Borel } B \subset D(y),$$

then Z satisfies parts (i) to (iv) of the condition to be an intertwining of X and Y with link L.

Remark. We believe that part (v) of the condition to be an intertwining of X and Y with link L is also satisfied under the hypotheses of Theorem 5.6, but have not yet performed the necessary estimates.

Remark. Theorem 5.6 partially generalizes [PS15, Theorem 3], which addresses the case where $a_{ij} = \delta_{ij}$ and $\rho_{kl} = \delta_{kl}$.

Proof of Theorem 5.6. We follow the same steps as in the proof of [PS15, Theorem 3]. We work over a probability space on $C([0,\infty), D)$, the space of continuous paths $[0,\infty) \to D$, with the standard Borel σ-algebra and a probability measure $\mathbb{P}$ given by the law of Z. Filter this probability space by $\{\mathcal{F}_t, t \geq 0\}$, the filtration generated by the coordinate maps and augmented by the common null sets under $(\mathbb{P}_z, z \in D)$. We consider also $\{\mathcal{F}_t^X, t \geq 0\}$ and $\{\mathcal{F}_t^Y, t \geq 0\}$, the right-continuous complete subfiltrations of $\{\mathcal{F}_t, t \geq 0\}$ generated by the first m and last n coordinate processes in $C([0,\infty), D)$, respectively. We first prove three lemmas. The first, Lemma 5.7, generalizes Step 2 in the proof of [PS15, Theorem 3]. The next two, Lemmas 5.8 and 5.9, provide a technical construction of uniformly bounded smooth approximations of certain indicator functions functions.

Lemma 5.7. For any $f \in C_c^\infty(D) \cap \mathsf{domain}(\mathcal{A}^Z)$, the functions

$$u(t,y) := \mathbb{E}\left[f(Z_1(t), Z_2(t)) \mid Z_2(0) = y\right], \qquad (t,y) \in [0,\infty) \times \mathcal{Y}$$

and

$$u'(t,y) := \mathbb{E}\left[\int_{D(Y(t))} \Lambda(Y(t),x) f(x, Y(t)) dx \mid Y(0) = y\right], \qquad (t,y) \in [0,\infty) \times \mathcal{Y}$$

coincide.

Proof. We will check that both $u(t,y)$ and $u'(t,y)$ lie in the domain of $\mathcal{A}^Y$, are continuously differentiable with respect to the uniform norm on $C_0(\mathcal{Y})$, and solve the Kolmogorov forward equation $\partial_t u = \mathcal{A}^Y u$ with $u(0,y) = \int_{D(y)} \Lambda(y,x) f(x,y) dx$. The desired equality will then follow from the uniqueness of [EN00, Proposition II.6.2].

Fix some $f \in C_c^\infty(D) \cap \mathsf{domain}(\mathcal{A}^Z)$. For $u'(t,y)$, notice that $\widetilde{f}(y) := \int_{D(y)} \Lambda(y,x) f(x,y) dx$ lies in $\mathsf{domain}(\mathcal{A}^Y)$ by the boundary conditions on f and since $\langle \nabla_y \Lambda, U_2 \rangle = 0$ on $\partial \mathcal{Y}$. Therefore, we see that $u'(t,y) := Q_t \widetilde{f}(y)$, hence $u'(t,y) \in \mathsf{domain}(\mathcal{A}^Y)$ is continuously differentiable and satisfies $\partial_t u' = \mathcal{A}^Y u'$ and $u'(0,y) = \int_{D(y)} \Lambda(y,x) f(x,y) dx$.

For $u(t,y)$, define $v(t,x,y) := \mathbb{E}[f(Z_1(t), Z_2(t)) \mid Z_1(0) = x, Z_2(0) = y]$. By the choice of initial condition for Z, we have that

$$u(t,y) = \int_{D(y)} \Lambda(y,x) v(t,x,y) dx, \tag{5.6}$$

which for $t = 0$ yields $u(0,y) = \int_{D(y)} \Lambda(y,x) f(x,y) dx$. Further, this shows that $u(t,y) \in \mathsf{domain}(\mathcal{A}^Y)$ by again noting that $\langle \nabla_y \Lambda, U_2 \rangle = 0$ on $\partial \mathcal{Y}$. It remains to check that $\partial_t u = \mathcal{A}^Y u$; we compute both sides using the fact that v solves the Kolmogorov forward equation $\partial_t v = \mathcal{A}^Z v$.

Computing the time derivative: By the given, we may find $v_l \in C_c^\infty(D)$ in the domain of $\mathcal{A}^Z$ so that $v_l(t) \to v(t)$ uniformly in $C_0(D)$. Because $\mathcal{A}^Z$ is closed, we conclude that

$$\partial_t u = \int_{D(y)} \Lambda(y,x) \partial_t v(t,x,y) dx = \int_{D(y)} \Lambda(y,x) \mathcal{A}^Z v(t,x,y) dx = \lim_{l \to \infty} \int_{D(y)} \Lambda(y,x) \mathcal{A}^Z v_l(t,x,y) dx$$

Observe now that

$$\int_{D(y)} \Lambda(\mathcal{A}^X v_l) dx = \int_{D(y)} \Lambda \langle b, \nabla_x v_l \rangle dx + \frac{1}{2} \sum_{i,j=1}^m \int_{D(y)} \Lambda\, a_{ij} \partial_{x_i} \partial_{x_j} (v_l) dx,$$

where, denoting by $d\theta(x)$ the Lebesgue surface measure on $D(y)$, repeated application of the divergence theorem yield

$$\int_{D(y)} \Lambda\langle b, \nabla_x v_l\rangle dx = -\int_{D(y)} \Lambda v_l(\nabla\cdot b)dx - \int_{D(y)} v_l\langle b, \nabla_x\Lambda\rangle dx + \sum_k \int_{\partial D(y)_k} v_l\Lambda\langle b, \eta_k\rangle d\theta(x)$$

and

$$\begin{aligned}\frac{1}{2}\sum_{i,j=1}^m \int_{D(y)} \Lambda\, a_{ij}\partial_{x_i}\partial_{x_j}(v_l)dx = \frac{1}{2}\int_{D(y)} v_l\langle a, \nabla_x^2\Lambda\rangle dx + \frac{1}{2}\int_{D(y)} \Lambda v_l(\nabla\cdot\nabla\cdot a)dx \\ + \int_{D(y)} v_l\langle\nabla\cdot a, \nabla_x\Lambda\rangle dx - \frac{1}{2}\sum_k\int_{\partial D(y)_k} v_l\Lambda\langle\nabla\cdot a, \eta_k\rangle d\theta(x) \\ - \frac{1}{2}\sum_k\int_{\partial D(y)_k} v_l\langle\langle a, \eta_k\rangle, \nabla_x\Lambda\rangle d\theta(x) + \frac{1}{2}\sum_k\int_{\partial D(y)_k} \Lambda\langle\langle a, \eta_k\rangle, \nabla_x v_l\rangle d\theta(x).\end{aligned}$$

Putting these together, we find that

$$\begin{aligned}\int_{D(y)} \Lambda(\mathcal{A}^X v_l)dx = \int_{D(y)} ((\mathcal{A}^X)^*\Lambda)\, v_l\, dx + \sum_k\int_{\partial D(y)_k} v_l\Lambda\langle b, \eta_k\rangle d\theta(x) \\ - \frac{1}{2}\sum_k\int_{\partial D(y)_k} v_l\Lambda\langle\nabla\cdot a, \eta_k\rangle d\theta(x) - \frac{1}{2}\sum_k\int_{\partial D(y)_k} v_l\langle\langle a, \eta_k\rangle, \nabla_x\Lambda\rangle d\theta(x) \\ + \frac{1}{2}\sum_k\int_{\partial D(y)_k} \Lambda\langle\langle a, \eta_k\rangle, \nabla_x v_l\rangle d\theta(x).\end{aligned}$$

and therefore that

$$\begin{aligned}(5.7)\quad \partial_t u = \lim_{l\to\infty}\int_{D(y)} \Big(((\mathcal{A}^X)^*\Lambda)\, v_l + \Lambda\mathcal{A}^Y v_l + \Lambda\sum_{i,j=1}^n \rho_{ij}\partial_{y_i}[\log\Lambda]\partial_{y_j}v_l\Big)dx \\ + \sum_k\int_{\partial D(y)_k} v_l\Lambda\langle b, \eta_k\rangle d\theta(x) - \frac{1}{2}\sum_k\int_{\partial D(y)_k} v_l\Lambda\langle\nabla\cdot a, \eta_k\rangle d\theta(x) \\ - \frac{1}{2}\sum_k\int_{\partial D(y)_k} v_l\langle\langle a, \eta_k\rangle, \nabla_x\Lambda\rangle d\theta(x) + \frac{1}{2}\sum_k\int_{\partial D(y)_k} \Lambda\langle\langle a, \eta_k\rangle, \nabla_x v_l\rangle d\theta(x).\end{aligned}$$

Computing the action of the generator: Choose now a sequence of functions $\Lambda_q \in C_c^\infty(D)$ so that $\Lambda_q \to \Lambda$, $\nabla_y\Lambda_q \to \nabla_y\Lambda$, $\mathcal{A}^Y\Lambda_q \to \mathcal{A}^Y\Lambda$ uniformly on compact subsets of D, $\nabla_x\Lambda_q \to \nabla_x\Lambda$ uniformly on compact subsets of the neighborhood of ∂D on which $\nabla_x\Lambda$ is defined, and $\langle\nabla_y\Lambda_q, U_2\rangle = 0$ on $\partial\mathcal{Y}$. Note that the final condition implies that $\int_{D(y)}\Lambda_q - dx$ maps $\mathsf{domain}(\mathcal{A}^Z)$ and $\mathsf{domain}(\mathcal{A}^Y)$ to $\mathsf{domain}(\mathcal{A}^Y)$. By the multidimensional Leibniz rule, we have

$$\begin{aligned}\partial_{y_j}\int_{D(y)} \Lambda_q v_l dx = \sum_k\int_{\partial D(y)_k} \Lambda_q v_l\langle\Psi_k^j, \eta_k\rangle d\theta(x) + \int_{D(y)} \partial_{y_j}(\Lambda_q v_l)dx \\ \partial_{y_i}\partial_{y_j}\int_{D(y)} \Lambda_q v_l dx = \sum_k\int_{\partial D(y)_k} \mathrm{div}_x(\Lambda_q v_l\langle\Psi_k^j, \eta_k\rangle\Psi_k^i)d\theta(x) + \sum_k\int_{\partial D(y)_k} \partial_{y_i}(\Lambda_q v_l\langle\Psi_k^j, \eta_k\rangle)d\theta(x) \\ + \sum_k\int_{\partial D(y)_k} \partial_{y_j}(\Lambda_q v_l)\langle\Psi_k^i, \eta_k\rangle d\theta(x) + \int_{D(y)} \partial_{y_i}\partial_{y_j}(\Lambda_q v_l)dx.\end{aligned}$$

We conclude that

$$\begin{aligned}\mathcal{A}^Y u = \lim_{l\to\infty}\lim_{q\to\infty}\mathcal{A}^Y\int_{D(y)} \Lambda_q v_l dx + \sum_{j=1}^n \gamma_j\sum_k\int_{\partial D(y)_k} \Lambda_q v_l\langle\Psi_k^j, \eta_k\rangle d\theta(x) \\ + \frac{1}{2}\sum_{i,j=1}^n \rho_{ij}\sum_k\int_{\partial D(y)_k} \Big(\mathrm{div}_x(\Lambda_q v_l\langle\Psi_k^j, \eta_k\rangle\Psi_k^i) + 2\partial_{y_j}(\Lambda_q v_l\langle\Psi_k^i, \eta_k\rangle)\Big)d\theta(x).\end{aligned}$$

Note that

$$\mathcal{A}^Y(\Lambda_q v_l) = (\mathcal{A}^Y \Lambda_q)v_l + \Lambda_q(\mathcal{A}^Y v_l) + \sum_{i,j=1}^n \rho_{ij}\partial_{y_i}\Lambda_q \partial_{y_j} v_l$$
$$= (\mathcal{A}^Y \Lambda_q)v_l + \Lambda_q(\mathcal{A}^Y v_l) + \Lambda_q \sum_{i,j=1}^n \rho_{ij}\partial_{y_i}[\log \Lambda_q]\partial_{y_j} v_l,$$

which implies by the definition of Λ_q and the fact that $\mathcal{A}^Y$ is closed that

$$\text{(5.8)} \quad \mathcal{A}^Y u = \lim_{l\to\infty} \mathcal{A}^Y \int_{D(y)} \Lambda v_l dx + \sum_{j=1}^n \gamma_j \sum_k \int_{\partial D(y)_k} \Lambda v_l \langle \Psi_k^j, \eta_k\rangle d\theta(x)$$
$$+ \frac{1}{2}\sum_{i,j=1}^n \rho_{ij} \sum_k \int_{\partial D(y)_k} \Big(\mathrm{div}_x(\Lambda v_l \langle \Psi_k^j, \eta_k\rangle \Psi_k^i) + 2\partial_{y_j}(\Lambda v_l \langle \Psi_k^i, \eta_k\rangle)\Big) d\theta(x).$$

Comparing the two sides: First, note that

$$\mathcal{A}^Y(\Lambda v_l) = (\mathcal{A}^Y \Lambda)v_l + \Lambda(\mathcal{A}^Y v_l) + \sum_{i,j=1}^n \rho_{ij}\partial_{y_i}\Lambda\partial_{y_j} v_l = (\mathcal{A}^Y \Lambda)v_l + \Lambda(\mathcal{A}^Y v_l) + \Lambda \sum_{i,j=1}^n \rho_{ij}\partial_{y_i}[\log\Lambda]\partial_{y_j} v_l,$$

which implies that the first line of (5.7) is equal to $\int_{D(y)} \mathcal{A}^Y(\Lambda v_l)dx$, the first term in (5.8). Expanding out, the other terms (5.8) are given by

$$\sum_{j=1}^n \gamma_j \sum_k \int_{\partial D(y)_k} \Lambda v_l \langle \Psi_k^j, \eta_k\rangle d\theta(x)$$
$$+ \frac{1}{2}\sum_{i,j=1}^n \rho_{ij}\sum_k \int_{\partial D(y)_k} \Big(\langle \nabla_x \Lambda, \Psi_k^j\rangle v_l + \Lambda v_l \mathrm{div}_x(\Psi_k^j) + 2\partial_{y_j}(\Lambda) v_l\Big)\langle \Psi_k^i, \eta_k\rangle d\theta(x)$$
$$+ \frac{1}{2}\sum_{i,j=1}^n \rho_{ij}\sum_k \int_{\partial D(y)_k} \Big(\Lambda\langle \nabla_x v_l, \Psi_k^j\rangle + 2\Lambda\partial_{y_j} v\Big)\langle \Psi_k^i, \eta_k\rangle d\theta(x).$$

Comparing with the remaining terms in (5.7), we see that the terms containing a factor of v_l are equal by (5.4), the terms containing $\nabla_x(v_l)$ are equal by (5.2), and the terms containing $\nabla_y(v_l)$ are equal due to the boundary condition (5.5) for $v_l \in \mathsf{domain}(\mathcal{A}^Z)$. We conclude that

$$\partial_t u = \lim_{l\to\infty} \mathcal{A}^Y \int_{D(y)} \Lambda v_l dx.$$

Again applying closedness of $\mathcal{A}^Y$ implies that $\partial_t u = \mathcal{A}^Y u$, as desired. ☐

Lemma 5.8. Let $K \subset U \subset D$ be subsets of D with K compact and U open with smooth boundary so that $U - K$ contains no vertices of $\overline{D}$ which lie in D. For any $g \in C^\infty(K) \cap \mathsf{domain}(\mathcal{A}^Z|_{C^\infty(K)})$, there exists $f \in C_c^\infty(D) \cap \mathsf{domain}(\mathcal{A}^Z)$ supported on U so that $f|_K \equiv g$ and $|f| \leq \max_{\partial K}|g|$ on the faces of minimal dimension intersecting $U - K$.

Proof. If no face of $\overline{D}$ which lies in D intersects $U - K$, this follows from ordinary existence of smooth cutoff functions, since $\mathsf{domain}(\mathcal{A}^Z)$ imposes only Neumann boundary conditions on ∂D. Otherwise, let the lowest dimensional face of $\overline{D}$ which lies in D which intersects $U - K$ have dimension $m + n - d$. We induct on d with trivial base case $d = 0$. We suppose the claim is known for some $d - 1$ and prove it for d.

Let $\{F_{l,i}\}_{l\leq d}$ be the collection of dimension $m + n - l$ faces of $\overline{D}$ which lie in D and intersect $U - K$ so that F_0 is D itself. We claim by downwards induction on $l \leq d$ that we may find some smooth $f_{l,i}$ on $F_{l,i}$ supported on $F_{l,i} \cap U$ so that $f_{l,i}|_{K\cap F_{l,i}} \equiv g$, $|f_{d,i}| \leq \max_{\partial(K\cap F_{d,i})}|g|$, and $f_{l,i}$ and all derivatives which exist agree on all intersections. For $l = d$, this follows by ordinary existence of smooth cutoff functions since none of the $F_{d,i}$ intersect in $U - K$. For the inductive step $l < d$, this follows by applying the Whitney extension theorem to the existing $f_{l',i}$, the closed set

$$F_{l,i} \cap \Big((D - U) \cup K \cup \bigcup_{l' > l, i} F_{l',i}\Big),$$

and Whitney data determined on $(D-U)$ by the fact that $f_{l,i} \equiv 0$, on K by the fact that $f_{l,i} \equiv g$, and on $F_{l',i}$ by derivatives of $f_{l',i}$ on $F_{l',i}$ and the condition that the derivative of $f_{l',i}$ vanishes along every direction in $U(x,y)$; note that this data is non-trivial by Assumption 5.2(d) and the given condition on $U(x,y)$. We conclude as desired that there is a smooth function $f \in C_c^\infty(D) \cap \mathsf{domain}(\mathcal{A}^Z)$ supported on U so that $f|_K \equiv 1$ and $|f|_{F_{d,i}}| \leq \max_{\partial(K\cap F_{d,i})} |g|$. ☐

Lemma 5.9. Let $U \subset D$ be a (relatively) open set. There exists a uniformly bounded sequence $f_n \in C_c^\infty(D) \cap \mathsf{domain}(\mathcal{A}^Z)$ so that $f_n \to 1_U$ pointwise.

Proof. Let the lowest dimensional face of $\overline{D}$ lying in D which intersects U but is not contained in U have dimension $m+n-d$. If $d=0$, the result follows from ordinary existence of smooth cutoff functions, since $\mathsf{domain}(\mathcal{A}^Z)$ is defined only by Neumann boundary conditions. Otherwise, choose an increasing sequence of sets $K_1 \subset V_1 \subset K_2 \subset V_2 \subset \cdots$ with K_i compact and V_i open with smooth boundary so that $\bigcup_{n=1}^\infty K_n = C$ and K_1 contains all faces of $\overline{D}$ contained in U.

We claim by downward induction on $l \leq d$ that there exists $g_n^l \in C_c^\infty(D) \cap \mathsf{domain}(\mathcal{A}^Z)$ supported in V_n so that $g_n^l \equiv 1$ on K_n and $|g_n^l| \leq 3^{d-l}$ on faces of dimension at most $m+n-l$ intersecting U but not contained in U. For $l=d$, this follows from Lemma 5.8 applied to 1_{K_n} and $K_n \subset V_n$. For the inductive step, define the compact set $K_n' := \{g_n^l \geq 3^{d-l+1}\} \subset V_n - K_n$ and notice that the lowest dimensional face of $\overline{D}$ lying in D which intersects U_n' but is not contained in U_n' has dimension greater than $m+n-l$. By Lemma 5.8 applied to $g_n^l - 3^{d-l+1}$ on $K_n' \subset V_n - K_n$, there exists $h_n^l \in C_c^\infty(D) \cap \mathsf{domain}(\mathcal{A}^Z)$ with support in $V_n - K_n$ so that $h_n^l \equiv g_n^l - 3^{d-l+1}$ on K_n' and $|h_n^l| \leq 3^{d-l+1}$ on the faces of dimension at most $m+n-l+1$. Similarly, for $K_n'' := \{g_n^l \leq -3^{d-l+1}\} \subset V_n - K_n$, we can find $r_n^l \in C_c^\infty(D) \cap \mathsf{domain}(\mathcal{A}^Z)$ supported in $V_n - K_n$ so that $r_n^l \equiv g_n^l + 3^{d-l+1}$ on K_n'' and $|r_n^l| \leq 3^{d-l+1}$ on the faces of dimension at most $m+n-l+1$. Define now

$$g_n^{l-1} := g_n^l - h_n^l - r_n^l$$

and notice that $g_n^{l-1} \equiv 1$ on K_n and $|g_n^{l-1}| \leq 3^{d-l+1}$ on faces of dimension at most $m+n-l+1$ intersecting U but not contained in U. This completes the induction. Setting $f_n := g_n^0$ gives the desired sequence of functions. ☐

We now verify parts (i) to (iv) of the intertwining relation.

Claim 1: We claim that Z_1 is Markov with respect to its own filtration and that $Z_1 \overset{d}{=} X$. The proof is the same as in Step 1 of [PS15, Theorem 1].

Claim 2: We now establish (iii). By the monotone class and convergence theorems, it suffices to check that for $f \in C_c^\infty(\mathcal{X}) \cap \mathsf{domain}(\mathcal{A}^X)$, we have

$$\mathbb{E}[f(Z_1(t)) \mid Z_1(0) = x, Z_2(0) = y] = \mathbb{E}[f(Z_1(t)) \mid Z_1(0) = x], \qquad (t,x,y) \in [0,\infty) \times D.$$

Write $v(t,x,y)$ for the LHS and $u(t,x)$ for the RHS. Viewed as a function on D, f lies in $\mathsf{domain}(\mathcal{A}^Z)$, since the Neumann boundary conditions hold trivially. Therefore, we see that $v(t,x,y) \in C_0(D) \cap \mathsf{domain}(\mathcal{A}^Z)$ solves the Kolmogorov forward equation $\partial_t v = \mathcal{A}^Z v$ with initial condition $v(0,x,y) = f(x)$. On the other hand, $u(t,x) \in C_0(\mathcal{X}) \cap \mathsf{domain}(\mathcal{A}^X)$ solves the Kolmogorov forward equation $\partial_t u = \mathcal{A}^X u$ with initial condition $u(0,x) = f(x)$. By density of $C_c^\infty(\mathcal{X})$ in $\mathsf{domain}(\mathcal{A}^X)$, we may choose a uniformly converging sequence $u_l \to u$ with $u_l \in C_c^\infty(\mathcal{X})$ so that $\mathcal{A}^X u_l \to \mathcal{A}^X u$ in $C_0(\mathcal{X})$; we then have that $\mathcal{A}^Z u = \lim_{l\to\infty} \mathcal{A}^Z u_l = \lim_{l\to\infty} \mathcal{A}^X u_l = \mathcal{A}^X u$ by closedness of $\mathcal{A}^Z$ and $\mathcal{A}^X$, meaning that $\widetilde{u}(t,x,y) = u(t,x)$ solves the Kolmogorov forward equation $\partial_t \widetilde{u} = \mathcal{A}^Z \widetilde{u}$ with initial condition $\widetilde{u}(0,x,y) = f(x)$. We conclude that $v(t,x,y) = u(t,x)$ by uniqueness of solutions to the Kolmogorov forward equation given by [EN00, Proposition II.6.2].

Claim 3: We now establish (ii). It suffices to check that $Q_t L f = L P_t f$ for all $f \in C_0(\mathcal{X}) \cap \mathsf{domain}(\mathcal{A}^X)$ and $t \geq 0$. By approximating f by a uniform limit of functions in $C_c^\infty(\mathcal{X}) \cap \mathsf{domain}(\mathcal{A}^X)$, it suffices to check the relation for $f \in C_c^\infty(\mathcal{X}) \cap \mathsf{domain}(\mathcal{A}^X)$. By (5.6) and Claim 2, we find that

$$\begin{aligned} Q_t L f = \mathbb{E}\left[\int_{D(Z_2(t))} \Lambda(Z_2(t),x) f(x) dx \mid Z_2(0) = y\right] &= \int_{D(y)} \Lambda(y,x) \mathbb{E}[f(Z_1(t)) \mid Z_1(0) = x, Z_2(0) = y] dx \\ &= \int_{D(y)} \Lambda(y,x) \mathbb{E}[f(Z_1(t)) \mid Z_1(0) = x] dx = L P_t f. \end{aligned}$$

Claim 4: We now establish (iv). It suffices to check that for any $k \in \mathbb{N}$ and distinct times $0 = t_0 < t_1 < \cdots < t_k = t$, we have

$$\mathbb{E}[f(Z_1(t_k)) \mid Z_2(t_0), \ldots, Z_2(t_k)] = (Lf)(Z_2(t_k)).$$

This is equivalent to the fact that for any bounded Borel measurable function g on $\mathcal{Y}^{k+1}$, we have

$$\mathbb{E}[f(Z_1(t_k))g(Z_2(t_0), \ldots, Z_2(t_k))] = \mathbb{E}[(Lf)(Z_2(t_k))g(Z_2(t_0), \ldots, Z_2(t_k))].$$

We proceed by induction on k. For $k = 1$, by the monotone class theorem, it suffices to replace f and g by indicator functions $f(x) = 1_A(x)$ and $g(y, \widetilde{y}) = 1_B(\widetilde{y})$ of open boxes $A \subset \mathcal{X}$ and $B \subset \mathcal{Y}$. Now choose by Lemma 5.9 applied to $1_{(A\times B)\cap D}$ a uniformly bounded sequence of functions h_n in $C_c^\infty(D) \cap \mathsf{domain}(\mathcal{A}^Z)$ converging pointwise to $(x, \widetilde{y}) \mapsto 1_A(x)1_B(\widetilde{y})$; applying Lemma 5.7 to h_n and using the dominated convergence theorem implies that

$$\mathbb{E}[1_A(Z_1(t))1_B(Z_2(t)) \mid Z_2(0) = y] = \mathbb{E}[(L1_A)(Z_2(t))1_B(Z_2(t)) \mid Z_2(0) = y],$$

as needed. For the inductive step, notice that

$$\begin{aligned}\mathbb{E}[f(Z_1(t_{k+1})) \mid Z_2(t_0), \ldots, Z_2(t_{k+1})] &= \int_{D(Z_2(t_k))} \Lambda(Z_2(t_k), x)\mathbb{E}[f(Z_1(t_{k+1})) \mid Z_1(t_k) = x, Z_2(t_k), Z_2(t_{k+1})]dx \\ &= \mathbb{E}[f(Z_1(t_{k+1})) \mid Z_2(t_k), Z_2(t_{k+1})] \\ &= (Lf)(Z_2(t_{k+1})).\end{aligned}$$

where in the first step we use that the law of $Z_1(t_k)$ given $Z_2(t_0), \ldots, Z_2(t_k)$ is $\Lambda(Z_2(t_k), \cdot)$ by the inductive hypothesis and in the last step we use time homogeneity and the $k = 1$ case.

Claim 5: We show that $Z_2 \overset{d}{=} Y$. It suffices to check that for any $k \in \mathbb{N}$ and distinct times $0 = t_0 < t_1 < \cdots < t_k = t$ we have

$$\mathbb{E}[f(Z_2(t)) \mid Z_2(t_0), \ldots, Z_2(t_{k-1})] = Q_{t-t_{k-1}}f(Z_2(t_{k-1})).$$

We induct on k. For $k = 1$, by the monotone class theorem, it suffices to check the condition for $f(y) = 1_A(y)$ for any open $A \subset \mathcal{Y}$. Choose by Lemma 5.9 a uniformly bounded sequence of functions $f_n \in C_c^\infty(D) \cap \mathsf{domain}(\mathcal{A}^Z)$ converging pointwise to 1_A; for each such f_n, by Lemma 5.7 we have

$$\mathbb{E}[f_n(Z_1(t), Z_2(t)) \mid Z_2(t_0) = y] = \mathbb{E}\left[\int_{D(Y(t))} \Lambda(Y(t), x)f_n(x, Y(t))dx \mid Y(t_0) = y\right].$$

Taking the limit as $n \to \infty$, by the dominated convergence theorem we obtain the desired

$$\begin{aligned}\mathbb{E}[1_A(Z_2(t)) \mid Z_2(t_0) = y] &= \mathbb{E}\left[\int_{D(Y(t))} \Lambda(Y(t), x)1_A(Y(t))dx \mid Y(t_0) = y\right] \\ &= \mathbb{E}[1_A(Y(t)) \mid Y(t_0) = y] = Q_{t-t_0}1_A(y).\end{aligned}$$

For the inductive step, notice that

$$\begin{aligned}\mathbb{E}[f(Z_2(t)) \mid Z_2(t_0), \ldots, Z_2(t_k)] &= \int_{D(Z_2(t_k))} \Lambda(Z_2(t_k), x)\mathbb{E}[f(Z_2(t)) \mid Z_1(t_k) = x, Z_2(t_0), \ldots, Z_2(t_k)] \\ &= \int_{D(Z_2(t_k))} \Lambda(Z_2(t_k), x)\mathbb{E}[f(Z_2(t)) \mid Z_1(t_k) = x, Z_2(t_k)] \\ &= \mathbb{E}[f(Z_2(t)) \mid Z_2(t_k)] \\ &= Q_{t-t_k}f(Z_2(t_k)),\end{aligned}$$

where in the first equality we apply Claim 4 and in the second we use the Markov property of Z. □

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