

PROBABILISTIC CONFORMAL BLOCKS FOR LIOUVILLE CFT ON THE TORUS

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ABSTRACT. Liouville theory is a fundamental example of a conformal field theory (CFT) first introduced by Polyakov in the context of string theory. Conformal blocks are objects underlying the integrable structure of CFT via the conformal bootstrap equation. The present work provides a probabilistic construction of the 1-point toric conformal block of Liouville theory in terms of a Gaussian multiplicative chaos measure corresponding to a one-dimensional log-correlated field. We prove that our probabilistic conformal block satisfies Zamolodchikov's recursion, and we relate it to the instanton part of Nekrasov's partition function by the Alday-Gaiotto-Tachikawa correspondence. Our proof rests upon an analysis of Belavin-Polyakov-Zamolodchikov differential equations, operator product expansions, and Dotsenko-Fateev type integrals.

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1. INTRODUCTION

The present work gives a probabilistic representation for the 1-point torus conformal blocks of Liouville conformal field theory (LCFT). They enable computation of 1-point torus correlation functions in terms of 3-point sphere correlation functions via the conjectured modular bootstrap equation

$$(1.1) \quad \langle e^{\alpha\phi(0)} \rangle_{\mathbb{T}} = \frac{1}{|\eta(q)|^2} \int_{-\infty}^{\infty} C_{\gamma}(\alpha, Q - \mathbf{i}P, Q + \mathbf{i}P) |q|^{P^2} \mathcal{F}_{\gamma, P}^q(\alpha) \mathcal{F}_{\gamma, P}^{\bar{q}}(\alpha) dP,$$

where $\mathcal{F}_{\gamma, P}^q(\alpha)$ denotes the conformal block, $\langle e^{\alpha\phi(0)} \rangle_{\mathbb{T}}$ is the 1-point torus correlation function, $\eta(q)$ is the Dedekind eta function, and $C_{\gamma}(\alpha_1, \alpha_2, \alpha_3)$ is the DOZZ formula for the 3-point function of Liouville theory on the sphere proposed in [DO94, ZZ96]. The fundamental parameter of the theory is $\gamma \in (0, 2)$, related to central charge c of LCFT by $c = 1 + 6Q^2$ with $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$. We also denote $q = e^{i\pi\tau}$ with τ being the modular parameter of the torus \mathbb{T} , and α the insertion weight of the 1-point function $\langle e^{\alpha\phi(0)} \rangle_{\mathbb{T}}$ computed using $\mathcal{F}_{\gamma, P}^q(\alpha)$. As a formal q -series, $\mathcal{F}_{\gamma, P}^q(\alpha)$ is the unique solution to the recursion relation

$$(1.2) \quad \mathcal{F}_{\gamma, P}^q(\alpha) = \sum_{n, m=1}^{\infty} q^{2mn} \frac{R_{\gamma, m, n}^q(\alpha)}{P^2 - P_{m, n}^2} \mathcal{F}_{\gamma, P-m, n}^q(\alpha) + q^{\frac{1}{12}} \eta(q)^{-1},$$

where $R_{\gamma, m, n}^q(\alpha)$ and $P_{m, n}$ are explicit constants defined in (2.24) and (2.25). The AGT conjecture stated in [AGT09] and proven in [FL10] shows that it may be represented explicitly in terms of the instanton part of the Nekrasov partition function $\mathcal{Z}_{\gamma, P}^q(\alpha)$ as

$$(1.3) \quad \mathcal{F}_{\gamma, P}^q(\alpha) = q^{-\frac{1}{12}(1-\alpha(Q-\frac{\alpha}{2}))} \eta(q)^{1-\alpha(Q-\frac{\alpha}{2})} \mathcal{Z}_{\gamma, P}^q(\alpha).$$

Here, $\mathcal{Z}_{\gamma, P}^q(\alpha)$ is a formal series coming from a certain four-dimensional $SU(2)$ gauge theory given by

$$(1.4) \quad \mathcal{Z}_{\gamma, P}^q(\alpha) := 1 + \sum_{k=1}^{\infty} q^k \sum_{\substack{(Y_1, Y_2) \text{ Young diagrams} \\ |Y_1| + |Y_2| = k}} \prod_{i, j=1}^2 \prod_{s \in Y_i} \frac{(E_{ij}(s, P) - \alpha)(Q - E_{ij}(s, P) - \alpha)}{E_{ij}(s, P)(Q - E_{ij}(s, P))},$$

where $E_{ij}(s, P)$ is an explicit product given by (2.22). Our probabilistic construction of conformal blocks relies on a certain Gaussian multiplicative chaos (GMC) $e^{\frac{\gamma}{2} Y_{\tau}(x)} dx$ on $[0, 1]$. This object is a random measure defined as the regularized exponential of the Gaussian field $Y_{\tau}(x)$ on $[0, 1]$ with covariance

$$\mathbb{E}[Y_{\tau}(x)Y_{\tau}(y)] = -2 \log |\Theta_{\tau}(x - y)| + 2 \log |q^{\frac{1}{6}} \eta(q)|,$$

where $\Theta_{\tau}(x)$ is the Jacobi theta function (see Appendix A). For $\gamma \in (0, 2)$, $\alpha \in (-\frac{4}{\gamma}, Q)$, $q \in (0, 1)$, and $P \in \mathbb{R}$, define the probabilistic 1-point toric conformal block by

$$(1.5) \quad \mathcal{G}_{\gamma, P}^q(\alpha) := \frac{1}{Z} \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_{\tau}(x)} \Theta_{\tau}(x)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma}} \right],$$

where Z is a constant such that $\lim_{q \rightarrow 0} \mathcal{G}_{\gamma, P}^q(\alpha) = 1$, $\lim_{P \rightarrow +\infty} \mathcal{G}_{\gamma, P}^q(\alpha) = q^{\frac{1}{12}} \eta(q)^{-1}$, and is explicitly given in Definition 2.9. Our main result Theorem 2.12 shows that our probabilistic construction coincides with the definition of the 1-point toric Liouville conformal block from mathematical physics. This resolves a conjecture of Felder-Müller-Lennert from [FML18] on the convergence of the q -series (1.4).¹ The remainder of this introduction gives additional motivation and background for our results and outlines our methods. All notations and results will be reintroduced in full detail in later sections.

Theorem 2.12. *For $\gamma \in (0, 2)$, $\alpha \in (-\frac{4}{\gamma}, Q)$, and $P \in \mathbb{R}$, the formal q -series for $\mathcal{F}_{\gamma, P}^q(\alpha)$ converges for $|q| < r_{\alpha}$ and satisfies*

$$\mathcal{F}_{\gamma, P}^q(\alpha) = \mathcal{G}_{\gamma, P}^q(\alpha),$$

where

$$r_{\alpha} = \begin{cases} 1 & \alpha \in [0, Q) \\ 1 \wedge \left(1 + \frac{\gamma\alpha}{4}\right) \sqrt{\frac{2}{\gamma|\alpha|}} & \alpha \in (-\frac{4}{\gamma}, 0). \end{cases}$$

¹More precisely, they state their conjecture for the 4-point spherical conformal block. In light of [FLNO09, Pog09], the 1-point toric conformal block is a special case of the 4-point spherical conformal block under a parameter change.

1.1. Relation to probabilistic Liouville theory. In the probabilistic setting, the construction of Liouville CFT was first performed on the Riemann sphere by David-Kupiainen-Rhodes-Vargas in [DKRV16] and later on the complex torus in [DRV16] (see also the companion works [HRV18, GRV19] for the case of other topologies). Those papers used the definition of GMC as a renormalized exponential of the 2D Gaussian free field to rigorously construct the Liouville correlation functions and prove their conformal covariance. Our work extends the spirit of these constructions to Liouville conformal blocks. The main innovation is to replace the 2D GMC with a 1D GMC on the unit interval, which corresponds heuristically to the factorization of correlation functions into the chiral and anti-chiral sectors.

This probabilistic framework suggests some hope for solving Liouville CFT at a mathematical level of rigor. First in [KRV19b], Kupiainen-Rhodes-Vargas proved that the BPZ equations translating the constraints of local conformal invariance of a CFT hold for correlation functions on the sphere with a degenerate insertion. Building upon this work, the same authors proved in [KRV19a] the DOZZ formula for the 3-point function of LCFT on the sphere, first proposed in physics in [DO94, ZZ96]. Similar methods were used in the recent works [Rem20, RZ18, RZ20] to study LCFT on simply connected domain with boundary and solve several open problems about the distribution of one-dimensional GMC measures. The next step in this program is to prove a bootstrap statement such as (1.1) for the torus. We hope to leverage the present construction to achieve this goal in a future work; see Section 1.4 below for more details.

1.2. Relation to existing approaches to Liouville conformal blocks in mathematical physics. Liouville conformal blocks have been studied from many different perspectives in mathematical physics, beginning with their definition in the seminal work of Belavin-Polyakov-Zamolodchikov in [BPZ84]. We now relate our results to a few directions in the literature, although we do not attempt to provide a complete survey of this vast space.

- **Zamolodchikov’s recursion:** In [Zam84, Zam87], Zamolodchikov gave a recursive relation for the 4-point conformal block on the sphere uniquely specifying its formal series expansion. In [Pog09], Poghossian conjectured the analogous recursion (1.2) for the toric case, which was proven for 1-point toric conformal blocks in [HJS10] and for N -point toric conformal blocks in [CCY19]. Our proof establishes an analogue of (1.2) for the Dotsenko-Fateev integral expression of the probabilistic block (1.5) when $N = -\frac{\alpha}{\gamma}$ is an integer and uses it as an input into later arguments at general N .
- **Dotsenko-Fateev integrals:** When a certain combination of parameters equals a positive integer N , the early papers [DF84, DF85] of Dotsenko-Fateev proposed expressions for Liouville correlation functions on the sphere in terms of certain N -fold integrals over the complex plane known as Dotsenko-Fateev integrals. Following a suggestion of [DV09], similar expressions involving N -dimensional real integrals were proposed for conformal blocks on the sphere in [FLNO09, MMS10] and on the torus in [MY11, MMS11]. In the toric case, we have $N = -\frac{\alpha}{\gamma}$, and the corresponding Dotsenko-Fateev type integral is

$$\left(\int_0^1\right)^N \prod_{1 \leq i < j \leq N} |\Theta_\tau(x_i - x_j)|^{-\frac{\gamma^2}{4}} \prod_{i=1}^N \Theta_\tau(x_i)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x_i} \prod_{i=1}^N dx_i,$$

which by Fubini’s theorem and direct Gaussian computation is recovered up to a constant by the numerator of our GMC expression for the conformal block in (1.5) when N is an integer. Our probabilistic expression for conformal blocks may therefore be viewed as an extension of the Dotsenko-Fateev type integral expression to parameter ranges where the number of integrals is not an integer.

- **AGT correspondence:** In [AGT09], Alday-Gaiotto-Tachikawa conjectured a general relation between N -point Liouville conformal blocks on the sphere and torus on the one hand and certain quantities called Nekrasov partition functions arising in $\mathcal{N} = 2$ supersymmetric gauge theory. In our setting of 1-point blocks on the torus, the correspondence was proven in [FL10] and provides an explicit q -series expression (1.4) for the conformal block. At the level of these series coefficients, Theorem 2.11 provides explicit expressions for certain expectations over GMC in terms of linear combinations of coefficients of the Nekrasov partition function. It also resolves a conjecture of [FML18] on analyticity of the Nekrasov partition function (1.4) in q .

1.3. Summary of method. Our method proceeds by characterizing the q -series coefficients of both the Liouville conformal block (1.3) and our probabilistic GMC expression (1.5) as solutions to the coupled system

of two difference equations (6.2). These *shift equations* are inhomogenous first order difference equations with difference 2χ for $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$. Similar homogeneous versions were proposed for the DOZZ formula in [Tes95] and used in its proof in [KRV19a], while other versions have played a role in the recent works [Rem20, RZ18, RZ20]. To find the desired result from the shift equations, we use the fact that the equality holds when $N := -\frac{\alpha}{\gamma}$ is an integer and that solutions to the shift equations are unique up to a constant factor.

To establish the shift equations for the GMC expression $\mathcal{G}_{\gamma,P}^q(\alpha)$ of (1.5), for $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$ we define deformed GMC expressions $\psi_\chi^\alpha(u, q)$ in (3.3) corresponding to degenerate insertions with weight χ at the additional parameter u . We then prove in Theorem 3.4 that $\psi_\chi^\alpha(u, q)$ satisfies the *BPZ equation*, which for $l_\chi = \frac{\chi^2}{2} - \frac{\alpha\chi}{2}$ is the PDE

$$(1.6) \quad \left(\partial_{uu} - l_\chi(l_\chi + 1)\wp(u) + 2i\pi\chi^2\partial_\tau \right) \psi_\chi^\alpha(u, q) = 0$$

relating variation in the modular parameter τ and the additional parameter u . This equation was shown for Dotsenko-Fateev type integral expressions for conformal blocks in [FLNO09] and coincides with the KZB heat equation described in [Ber88] for the WZW model on the torus.

We then apply separation of variables to the BPZ equation (1.6), obtaining that the q -series coefficients of $\psi_\chi^\alpha(u, q)$ satisfy a system of coupled inhomogeneous hypergeometric ODEs after a proper normalization. Each ODE in this system has a two dimensional solution space, and we obtain the shift equations in Theorem 6.1 by analyzing the solution space near $u = 0$ and $u = 1$ using the *operator product expansions* (OPEs) of Theorem 5.4, which characterize the behavior of the deformed blocks $\psi_\chi^\alpha(u, q)$ near $u = 0, 1$. This argument is a generalization of the one used in [KRV19a] to prove the DOZZ formula, although that case only involved a single homogeneous hypergeometric ODE. We mention also that the OPE for $\chi = \frac{2}{\gamma}$ requires an intricate reflection argument making use of the results and the techniques of [RZ20].

Finally, to show that the Liouville conformal block $\mathcal{F}_{\gamma,P}^q(\alpha)$ satisfies the shift equations, we leverage the Dotsenko-Fateev type integral expression for $\mathcal{G}_{\gamma,P}^q(\alpha)$ at integer $N := -\frac{\alpha}{\gamma}$. This expression allows us to check that the GMC expression $\mathcal{G}_{\gamma,P}^q(\alpha)$ satisfies Zamolodchikov's recursion (1.2) and therefore equals $\mathcal{F}_{\gamma,P}^q(\alpha)$ at integer N as a formal q -series. This implies that $\mathcal{F}_{\gamma,P}^q(\alpha)$ satisfies the shift equations with $\chi = \frac{\gamma}{2}$ on a sequence of γ 's limiting to 0 by virtue of its equality with $\mathcal{G}_{\gamma,P}^q(\alpha)$. An analytic argument based on the meromorphicity of q -series coefficients of $\mathcal{F}_{\gamma,P}^q(\alpha)$ in γ then shows that the shift equation for $\chi = \frac{\gamma}{2}$ holds for all values of γ . Finally, the shift equations for $\chi = \frac{2}{\gamma}$ follow from the fact that $\mathcal{F}_{\gamma,P}^q(\alpha)$ is invariant under the exchange $\frac{\gamma}{2} \leftrightarrow \frac{2}{\gamma}$, yielding both shift equations for the conformal block $\mathcal{F}_{\gamma,P}^q(\alpha)$ and completing our proof. This procedure is carried out in detail in Section 6.

1.4. Outlook: the modular conformal bootstrap for Liouville theory. In the bootstrap approach to conformal field theory, conformal blocks are building blocks allowing any N -point correlation function on any Riemann surface to be computed from a combination of 3-point functions on the sphere and *bootstrap equations* such as (1.1) corresponding to the gluing of punctured surfaces. The *modular bootstrap equation* (1.1) corresponds to the gluing of two points of a 3-punctured sphere together to obtain a singly punctured torus and is one of the key steps to rigorously establish consistency of probabilistic Liouville theory in the bootstrap approach. It was previously shown to hold in the $\tau \rightarrow i\infty$ limit by Baverez in [Bav19]; however, in this limit the conformal blocks in (1.1) degenerate to constants, meaning the full modular bootstrap equation has significant additional complexity. As a future direction of study, we plan to use our probabilistic knowledge of the 1-point toric conformal block to prove the modular bootstrap equation (1.1). We also hope to adapt our methods to propose a probabilistic definition of 4-point spherical conformal blocks and to thereby understand the conformal bootstrap equation for the 4-point correlation function of LCFT on the sphere.

1.5. Organization of the paper. The remainder of this paper is organized as follows. In Section 2, we define our candidate probabilistic expression for the conformal block in terms of Gaussian multiplicative chaos and characterize it analytically. We then state the main result Theorem 2.12. In Section 3, we define deformed versions of our 1-point conformal blocks, characterize their analytic properties, and prove the BPZ equations stated in Theorem 3.4. In Section 4, we perform separation of variables for the deformed conformal block and derive from the BPZ equations a system of coupled inhomogeneous hypergeometric equations. In

Section 5, we state the operator product expansions (OPEs) for these deformed conformal blocks in Theorem 5.4, and perform an analytic continuation in α leveraging crucially a reflection principle. In Section 6, we use the results derived in Sections 4 and 5 to obtain two shift equations on series coefficients of our probabilistic conformal blocks in Theorem 6.1. We then put everything together to prove Theorem 2.11 by deriving Theorem 6.4 giving of Zamolodchikov's recursion for our probabilistic conformal block. Appendices A, B, C and D collect facts and conventions on theta functions, theorems from probability used throughout the text, the Gauss hypergeometric equation, and the proof of the OPE statements used in the main text.

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2. PROBABILISTIC CONSTRUCTION OF THE CONFORMAL BLOCK

In this section, we state our main result giving a probabilistic construction of the 1-point toric conformal block and verifying Zamolodchikov's recursion for it. We begin by introducing Gaussian multiplicative chaos (GMC), the probabilistic object which will enable our construction.

2.1. Definition of Gaussian multiplicative chaos. Let $\{\alpha_n\}_{n \geq 1}$, $\{\beta_n\}_{n \geq 1}$, $\{\alpha_{n,m}\}_{n,m \geq 1}$, $\{\beta_{n,m}\}_{n,m \geq 1}$ be sequences of i.i.d. standard real Gaussians. For τ purely imaginary and $q = e^{i\pi\tau}$, define the Gaussian fields $Y_\infty(x)$ and $Y_\tau(x)$ on $[0, 1]$ by

$$(2.1) \quad Y_\infty(x) := \sum_{n \geq 1} \sqrt{\frac{2}{n}} \left(\alpha_n \cos(2\pi n x) + \beta_n \sin(2\pi n x) \right)$$

$$(2.2) \quad Y_\tau(x) := Y_\infty(x) + 2 \sum_{n,m \geq 1} \frac{q^{nm}}{\sqrt{n}} \left(\alpha_{n,m} \cos(2\pi n x) + \beta_{n,m} \sin(2\pi n x) \right).$$

It will also be convenient to use the notation $F_\tau(x) := Y_\tau(x) - Y_\infty(x)$.

Lemma 2.1. *For $x \neq y$ in $[0, 1]$, these fields satisfy*

$$(2.3) \quad \mathbb{E}[Y_\infty(x)Y_\infty(y)] = \mathbb{E}[Y_\tau(x)Y_\infty(y)] = -2 \log |2 \sin(\pi(x-y))|$$

$$(2.4) \quad \mathbb{E}[Y_\tau(x)Y_\tau(y)] = -2 \log |\Theta_\tau(x-y)| + 2 \log |q^{1/6} \eta(q)|$$

where Θ_τ is the Jacobi theta function given by (A.1).

Proof. For the first covariance, notice that

$$\mathbb{E}[Y_\infty(x)Y_\infty(y)] = \mathbb{E}[Y_\tau(x)Y_\infty(y)] = \sum_{n \geq 1} \frac{2}{n} \cos(2\pi n(x-y)) = -2 \log |2 \sin(\pi(x-y))|,$$

where the last equality follows by computing Fourier series. For the second covariance, notice that

$$\begin{aligned} \mathbb{E}[Y_\tau(x)Y_\tau(y)] &= \mathbb{E}[Y_\infty(x)Y_\infty(y)] + \sum_{n,m \geq 1} \frac{4q^{2nm}}{n} \cos(2\pi n(x-y)) \\ &= -2 \log |2 \sin(\pi(x-y))| - 2 \sum_{m \geq 1} \log |(1 - q^{2m} e^{2i\pi(x-y)})(1 - q^{2m} e^{-2i\pi(x-y)})| \\ &= -2 \log |\Theta_\tau(x-y)| + 2 \log |q^{1/6} \eta(q)|. \end{aligned} \quad \square$$

Remark. Let X be the field on the unit circle such that $Y_\infty(x) = X(e^{2\pi i x})$. By (2.3), X is the restriction to the unit circle of a Gaussian free field on \mathbb{D} with free boundary (see [DMS14, Section 4.1.4]). Similarly, let X_τ be the Gaussian free field on the torus \mathbb{T} (see definition in [DRV16, Section 3.2]). For $x \in [0, 1]$, $Y_\tau(x) + \mathcal{N}(0, -\frac{1}{3} \log |q|)$ has the same covariance as $\sqrt{2}X_\tau(x)$, where $X_\tau(x)$ is seen as the restriction of the torus GFF to the loop parametrized by $x \in [0, 1]$.

Lemma 2.2. For $x \neq y$ in $[0, 1]$, we have

$$\mathbb{E}[\partial_\tau Y_\tau(x)Y_\tau(y)] = \mathbb{E}[\partial_\tau F_\tau(x)F_\tau(y)] = \frac{\mathbf{i}\pi}{6} - \frac{\partial_\tau \Theta_\tau(x-y)}{\Theta_\tau(x-y)} + \frac{\partial_\tau \eta(q)}{\eta(q)},$$

and for $x \in [0, 1]$ we have

$$\mathbb{E}[\partial_\tau Y_\tau(x)Y_\tau(x)] = \mathbb{E}[\partial_\tau F_\tau(x)F_\tau(x)] = \frac{\mathbf{i}\pi}{6} - \frac{2}{3} \frac{\partial_\tau \Theta'_\tau(0)}{\Theta'_\tau(0)}.$$

Proof. Computing using (2.2), we find that

$$\begin{aligned} \mathbb{E}[\partial_\tau Y_\tau(x)Y_\tau(y)] &= 4\pi\mathbf{i} \sum_{m,n=1}^{\infty} m q^{2nm} \cos(2\pi n(x-y)) = \frac{1}{2} \partial_\tau \mathbb{E}[Y_\tau(x)Y_\tau(y)] \\ &= -\partial_\tau \log \left| q^{-\frac{1}{6}} \frac{\Theta_\tau(x-y)}{\eta(q)} \right| = \frac{\mathbf{i}\pi}{6} - \frac{\partial_\tau \Theta_\tau(x-y)}{\Theta_\tau(x-y)} + \frac{\partial_\tau \eta(q)}{\eta(q)}. \end{aligned}$$

The second claim is a direct consequence. \square

The following observation is straightforward.

Lemma 2.3. $\mathbb{E}[F_\tau(x)^2] = 4 \sum_{n,m \geq 1} \frac{q^{2nm}}{n} = -4 \log |q^{-1/12} \eta(q)|$ for all $x \in [0, 1]$. Almost surely, the function $(0, 1) \ni q \mapsto F$ can be analytically extended to the unit disk. Moreover, $\lim_{\tau \rightarrow \mathbf{i}\infty} Y_\tau(x) = Y_\infty(x)$.

We now introduce the Gaussian Multiplicative Chaos (GMC) measures $e^{\frac{\gamma}{2} Y_\infty(x)} dx$ and $e^{\frac{\gamma}{2} Y_\tau(x)} dx$ on $[0, 1]$ for τ purely imaginary. Because the fields $Y_\infty(x)$ and $Y_\tau(x)$ live in the space of distributions, exponentiating them requires a regularization procedure, which we perform as follows. For $N \in \mathbb{N}$, define

$$\begin{aligned} Y_{\infty,N}(x) &= \sum_{n=1}^N \sqrt{\frac{2}{n}} \left(\alpha_n \cos(2\pi n x) + \beta_n \sin(2\pi n x) \right) \\ Y_{\tau,N}(x) &= Y_{\infty,N}(x) + 2 \sum_{n,m=1}^{\infty} \frac{q^{nm}}{\sqrt{n}} \left(\alpha_{n,m} \cos(2\pi n x) + \beta_{n,m} \sin(2\pi n x) \right). \end{aligned}$$

Definition 2.4 (Gaussian Multiplicative Chaos). For $\gamma \in (0, 2)$ and $\tau \in \mathbf{i}\mathbb{R}_{>0}$, we define the Gaussian multiplicative chaos measures $e^{\frac{\gamma}{2} Y_\infty(x)} dx$ and $e^{\frac{\gamma}{2} Y_\tau(x)} dx$ to be the weak limits of measures in probability

$$\begin{aligned} e^{\frac{\gamma}{2} Y_\infty(x)} dx &:= \lim_{N \rightarrow \infty} e^{\frac{\gamma}{2} Y_{\infty,N}(x) - \frac{\gamma^2}{8} \mathbb{E}[Y_{\infty,N}(x)^2]} dx \\ e^{\frac{\gamma}{2} Y_\tau(x)} dx &:= \lim_{N \rightarrow \infty} e^{\frac{\gamma}{2} Y_{\tau,N}(x) - \frac{\gamma^2}{8} \mathbb{E}[Y_{\tau,N}(x)^2]} dx. \end{aligned}$$

More precisely, for any continuous test function $f : [0, 1] \rightarrow \mathbb{R}$, we have in probability that

$$\begin{aligned} \int_0^1 f(x) e^{\frac{\gamma}{2} Y_\infty(x)} dx &= \lim_{N \rightarrow \infty} \int_0^1 f(x) e^{\frac{\gamma}{2} Y_{\infty,N}(x) - \frac{\gamma^2}{8} \mathbb{E}[Y_{\infty,N}(x)^2]} dx \\ \int_0^1 f(x) e^{\frac{\gamma}{2} Y_\tau(x)} dx &= \lim_{N \rightarrow \infty} \int_0^1 f(x) e^{\frac{\gamma}{2} Y_{\tau,N}(x) - \frac{\gamma^2}{8} \mathbb{E}[Y_{\tau,N}(x)^2]} dx. \end{aligned}$$

By Definition 2.4 and Lemma 2.3, we have

$$(2.5) \quad e^{\frac{\gamma}{2} Y_\tau(x)} dx = e^{-\frac{\gamma^2}{8} \mathbb{E}[F_\tau(0)^2]} e^{\frac{\gamma}{2} F_\tau(x)} e^{\frac{\gamma}{2} Y_\infty(x)} dx.$$

2.2. A GMC construction for the 1-point toric conformal block. We provide an analytic construction for the 1-point toric conformal block in terms of certain expectations against GMC. Throughout this section, we will fix $\gamma \in (0, 2)$, and set $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$. We first record a basic fact on analyticity of random functions.

Lemma 2.5. Let $D \subset \mathbb{C}$ be a domain and $\mathcal{X}(z)$ random analytic function on D . If $\mathbb{E}[|\mathcal{X}(z)|]$ is bounded on each compact subset of D , then $\mathbb{E}[\mathcal{X}(z)]$ is analytic on D .

Proof. Let \mathcal{C} be a contour in D . Since $\mathbb{E}[|\mathcal{X}(z)|] < \infty$, by Fubini's Theorem, we have $\oint \mathbb{E}[\mathcal{X}(z)] dz = \mathbb{E}[\oint \mathcal{X}(z) dz] = 0$, giving the analyticity of $\mathbb{E}[\mathcal{X}(z)]$. \square

Lemma 2.6. For $\gamma \in (0, 2)$, $\alpha \in (-\frac{4}{\gamma}, Q)$, $q \in (0, 1)$, and $\text{Im}(P) \in (-\frac{1}{2\gamma}, \frac{1}{2\gamma})$, we have

$$(2.6) \quad \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma}} \right] < \infty,$$

where we interpret the $-\frac{\alpha}{\gamma}$ power using a branch cut along $(-\infty, 0]$. Moreover, the function $P \mapsto \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma}} \right]$ is holomorphic on the domain $\{P \in \mathbb{C} : \text{Im}(P) \in (-\frac{1}{2\gamma}, \frac{1}{2\gamma})\}$.

Proof. First, since $\text{Im}(\gamma\pi Px) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, for all x the integrand and hence the integral in

$$\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma Px} dx$$

almost surely has positive real part, meaning we can take its $-\frac{\alpha}{\gamma}$ power using a branch cut along $(-\infty, 0]$, so the expression in (2.6) is well defined. Let $M(q) := 2 \sum_{n,m=1}^N \frac{q^{nm}}{\sqrt{n}} (|\alpha_{n,m}| + |\beta_{n,m}|)$. Then the expectation in Lemma 2.6 is upper bounded by $C \mathbb{E}[e^{\frac{\alpha}{2} M(q)}] \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\infty(x)} \sin(\pi x) e^{-\frac{\alpha\gamma}{2} x} dx \right)^{-\frac{\alpha}{\gamma}} \right]$ for some (P, q) -dependent constant C . Now (2.6) follows from Lemma B.1, and Lemma 2.5 yields the analyticity in P . \square

Recall that $\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} = e^{-\frac{\alpha\gamma}{2} g(x)}$ where $g = \log \Theta_\tau$ is as in Appendix A. In particular, we have

$$(2.7) \quad \Theta_\tau(x)^{-\alpha\gamma/2} = e^{-i\pi\alpha\gamma/2} |\Theta_\tau(x)|^{-\alpha\gamma/2} \quad \text{for each } x \in [0, 1].$$

For $\beta \in \mathbb{R}$, we interpret $\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma Px} dx \right)^\beta$ via

$$(2.8) \quad \left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma Px} dx \right)^\beta := e^{-i\pi\alpha\gamma\beta/2} \left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma Px} dx \right)^\beta.$$

For $\gamma \in (0, 2)$, $\alpha \in (-\frac{4}{\gamma}, Q)$, $q \in (0, 1)$, and $\text{Im}(P) \in (-\frac{1}{2\gamma}, \frac{1}{2\gamma})$, define

$$(2.9) \quad \mathcal{A}_{\gamma,P}^q(\alpha) := q^{\frac{1}{12}(-\alpha\gamma - \frac{2\alpha}{\gamma} + 2)} \eta(q)^{\alpha\gamma + \frac{2\alpha}{\gamma} - \frac{3}{2}\alpha^2 - 2} \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma}} \right].$$

Lemma 2.7. The quantity $\mathcal{A}_{\gamma,P}^q(\alpha)$ satisfies the following properties.

(a) The function $q \mapsto \mathcal{A}_{\gamma,P}^q(\alpha)$ admits a holomorphic extension on $\{q \in \mathbb{C} : |q| < r_\alpha\}$ where

$$(2.10) \quad r_\alpha := 1 \text{ for } \alpha \in [0, Q) \text{ and } r_\alpha := 1 \wedge \left(1 + \frac{\gamma\alpha}{4} \right) \sqrt{\frac{2}{\gamma|\alpha|}} \text{ for } \alpha \in (-\frac{4}{\gamma}, 0).$$

(b) In light of (a), for $\alpha \in (-\frac{4}{\gamma}, Q)$ and $\text{Im}(P) \in (-\frac{1}{2\gamma}, \frac{1}{2\gamma})$, we define $\mathcal{A}_{\gamma,P,n}(\alpha)$ by requiring

$$(2.11) \quad \mathcal{A}_{\gamma,P}^q(\alpha) = \sum_{n=0}^{\infty} \mathcal{A}_{\gamma,P,n}(\alpha) q^n \quad \text{for } |q| \text{ sufficiently small.}$$

As functions of P , $\mathcal{A}_{\gamma,P}^q(\alpha)$ and $\mathcal{A}_{\gamma,P,n}(\alpha)$ are holomorphic on $\{P \in \mathbb{C} : \text{Im}(P) \in (-\frac{1}{2\gamma}, \frac{1}{2\gamma})\}$.

(c) For $P \in \mathbb{R}$ and $n \in \mathbb{N}$, the function $\alpha \mapsto \mathcal{A}_{\gamma,P,n}(\alpha)$ can be analytically extended to an open set of \mathbb{C} containing $(-\frac{4}{\gamma}, Q)$.

Proof. For (a), notice the definition (2.9) is originally only valid for $q \in (0, 1)$. To find the analytic continuation in q , we will apply Girsanov's theorem (Theorem B.2) to rewrite (2.9) so that taking q complex produces a holomorphic function. For this, notice that

$$(2.12) \quad \mathbb{E}[\alpha_n Y_\infty(x)] = \sqrt{\frac{2}{n}} \cos(2\pi n x) \quad \text{and} \quad \mathbb{E}[\beta_n Y_\infty(x)] = \sqrt{\frac{2}{n}} \sin(2\pi n x).$$

In the following computation, we will use the decomposition $Y_\tau(x) = Y_\infty(x) + F_\tau(x)$. Notice that Y_∞ and F_τ are independent. By Girsanov's theorem (Theorem B.2), Lemma 2.1 and (2.5), we can write

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma}} \right] \\ &= \left(q^{1/6} \eta(q) \right)^{\frac{\alpha^2}{2}} \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x) + \frac{\gamma}{2} \mathbb{E}[Y_\tau(x) : \frac{\alpha}{2} F_\tau(0)]} (2 \sin(\pi x))^{-\alpha\gamma/2} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma}} \right] \\ &= \left(q^{1/6} \eta(q) \right)^{\frac{\alpha^2}{2}} e^{-\frac{\alpha^2}{8} \mathbb{E}[F_\tau(0)^2]} \mathbb{E} \left[e^{\frac{\alpha}{2} F_\tau(0)} \left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} (2 \sin(\pi x))^{-\alpha\gamma/2} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma}} \right] \\ &= \left(q^{1/6} \eta(q) \right)^{\frac{\alpha^2}{2}} e^{\frac{\alpha\gamma}{8} - \frac{\alpha^2}{8} \mathbb{E}[F_\tau(0)^2]} \hat{\mathcal{A}}_{\gamma,P}^q(\alpha), \end{aligned}$$

where $\hat{\mathcal{A}}_{\gamma,P}^q(\alpha) := \mathbb{E} \left[e^{\frac{\alpha}{2} F_\tau(0)} \left(\int_0^1 e^{\frac{\gamma}{2} Y_\infty(x) + \frac{\gamma}{2} F_\tau(x)} (2 \sin(\pi x))^{-\alpha\gamma/2} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma}} \right]$.

We claim the following lemma with its proof postponed, and conclude the proof of (a) right after.

Lemma 2.8. *Assertion (a) in Lemma 2.7 holds with $\hat{\mathcal{A}}_{\gamma,P}^q(\alpha)$ in place of $\mathcal{A}_{\gamma,P}^q(\alpha)$.*

Recall from (A.4) that $q^{-\frac{1}{12}} \eta(q)$ is analytic and nonzero on the unit disk \mathbb{D} . Therefore, the function

$$q^{\frac{1}{12}(-\alpha\gamma - \frac{2\alpha}{\gamma} + 2)} \eta(q)^{\alpha\gamma + \frac{2\alpha}{\gamma} - \frac{3}{2}\alpha^2 - 2} \left(q^{1/6} \eta(q) \right)^{\frac{\alpha^2}{2}} = \left(q^{-\frac{1}{12}} \eta(q) \right)^{\alpha\gamma + \frac{2\alpha}{\gamma} - \alpha^2 - 2}$$

is analytic on \mathbb{D} . By the definition of $\mathcal{A}_{\gamma,P}^q(\alpha)$, (2.8), and Lemmas 2.3 and 2.8, we conclude the proof of (a).

For (b), the analyticity for $\mathcal{A}_{\gamma,P}^q(\alpha)$ follows from Lemma 2.6. Applying the operator $\bar{\partial}_P$ to both sides of (2.9), we get $\bar{\partial}_P \mathcal{A}_{\gamma,P,n}(\alpha) = 0$ for each n . This gives the desired analyticity for $\mathcal{A}_{\gamma,P,n}(\alpha)$.

For (c), the analyticity in α of moments of Gaussian multiplicative chaos has already been shown to hold in several works such as [KRV19a, RZ20]. To reduce our GMC to the one studied in [RZ20], one can map the unit disk \mathbb{D} to the upper-half plane \mathbb{H} by the map $z \mapsto -\mathbf{i} \frac{z-1}{z+1}$. The circle parametrized by $x \in [0, 1]$ becomes the real line \mathbb{R} and the point x goes to $y = -\mathbf{i} \frac{e^{2\pi i x} - 1}{e^{2\pi i x} + 1}$. The field $Y_\infty(x)$ is mapped to the restriction to the real line of the Gaussian field $X_{\mathbb{H}}$ with covariance given by

$$\mathbb{E}[X_{\mathbb{H}}(y)X_{\mathbb{H}}(y')] = \log \frac{1}{|y - y'| |y - \bar{y}'|} - \log |y + \mathbf{i}|^2 - \log |y' + \mathbf{i}|^2 + 2 \log 2$$

for $y, y' \in \mathbb{H}$. The field F_τ is also mapped to a continuous field \tilde{F}_τ on \mathbb{R} . By performing this change of variable one gets that

$$\mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma}} \right] = \mathbb{E} \left[\left(\int_{\mathbb{R}} e^{\frac{\gamma}{2} X_{\mathbb{H}}(y) + \frac{\gamma}{2} \tilde{F}_\tau(y)} |y|^{-\frac{\alpha\gamma}{2}} f_1(y) dy \right)^{-\frac{\alpha}{\gamma}} \right]$$

for a continuous bounded function $f_1 : \mathbb{R} \mapsto (0, \infty)$ defined by a change of variable through the identity $|\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma Px} dx = |y|^{-\frac{\alpha\gamma}{2}} f_1(y) dy$. The right hand side is now complex analytic in α on a complex neighborhood of any compact $K \subset (-\frac{4}{\gamma}, Q)$ by an argument similar to the proof of [RZ20, Lemma 5.6]. \square

Proof of Lemma 2.8. Using (2.12) again, we have

$$(2.13) \quad F_\tau = \sqrt{2} \sum_{m,n} q^{nm} (\alpha_{n,m} \mathbb{E}[\alpha_n Y_\infty(x)] + \beta_{n,m} \mathbb{E}[\beta_n Y_\infty(x)]).$$

Applying Girsanov's theorem (Theorem B.2) to Y_∞ while conditioning on $\{\alpha_{m,n}, \beta_{m,n}\}$, we obtain

$$\begin{aligned} \hat{\mathcal{A}}_{\gamma,P}^q(\alpha) &= \mathbb{E} \left[e^{\frac{\alpha}{2} F_\tau(0)} \left(\int_0^1 e^{\frac{\gamma}{2} Y_\infty(x) + \frac{\gamma}{\sqrt{2}} \sum_{m,n} q^{nm} (\alpha_{n,m} \mathbb{E}[\alpha_n Y_\infty(x)] + \beta_{n,m} \mathbb{E}[\beta_n Y_\infty(x)])} (\sin(\pi x))^{-\alpha\gamma/2} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma}} \right] \\ (2.14) \quad &= \mathbb{E} \left[e^{\frac{\alpha}{2} F_\tau(0)} e^{\sqrt{2} \sum_{m,n} q^{nm} (\alpha_{n,m} \alpha_n + \beta_{n,m} \beta_n)} e^{-\sum_{m,n} q^{2nm} (\alpha_{m,n}^2 + \beta_{m,n}^2)} \left(\int_0^1 e^{\frac{\gamma}{2} Y_\infty(x)} (\sin(\pi x))^{-\alpha\gamma/2} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma}} \right]. \end{aligned}$$

By Holder's inequality, for $p_1, p_2, p_3 \in (1, \infty)$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, we have

$$\begin{aligned} & \mathbb{E} \left[e^{\frac{\alpha}{2} F_\tau(0)} e^{\sqrt{2} \sum_{m,n} q^{nm} (\alpha_{n,m} \alpha_n + \beta_{n,m} \beta_n)} e^{-\sum_{m,n} q^{2nm} (\alpha_{m,n}^2 + \beta_{m,n}^2)} \left(\int_0^1 e^{\frac{\gamma}{2} Y_\infty(x)} (\sin(\pi x))^{-\alpha\gamma/2} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma}} \right] \\ & \leq \mathbb{E} \left[e^{\frac{p_1\alpha}{2} F_\tau(0)} \right]^{\frac{1}{p_1}} \mathbb{E} \left[e^{\sqrt{2} p_2 \sum_{m,n} q^{nm} (\alpha_{n,m} \alpha_n + \beta_{n,m} \beta_n)} e^{-p_2 \sum_{m,n} q^{2nm} (\alpha_{m,n}^2 + \beta_{m,n}^2)} \right]^{\frac{1}{p_2}} \\ & \quad \times \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\infty(x)} (\sin(\pi x))^{-\alpha\gamma/2} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha p_3}{\gamma}} \right]^{\frac{1}{p_3}}. \end{aligned}$$

We first suppose that $\alpha \in [0, Q)$. Since the GMC has negative moments of all orders and the expectation of the exponential of a Gaussian random variable is finite, for all $q \in (0, 1)$, $p_1 \in (1, \infty)$, $p_3 \in (1, \infty)$ we have

$$(2.15) \quad \mathbb{E} \left[e^{\frac{p_1\alpha}{2} F_\tau(0)} \right]^{\frac{1}{p_1}} < \infty, \quad \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\infty(x)} (\sin(\pi x))^{-\alpha\gamma/2} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha p_3}{\gamma}} \right]^{\frac{1}{p_3}} < \infty.$$

To deal with the term involving p_2 , we use the independence of the Gaussians to obtain

$$\begin{aligned} & \mathbb{E} \left[e^{\sqrt{2} p_2 \sum_{m,n} q^{nm} (\alpha_{n,m} \alpha_n + \beta_{n,m} \beta_n)} e^{-p_2 \sum_{m,n} q^{2nm} (\alpha_{m,n}^2 + \beta_{m,n}^2)} \right] \\ & = \prod_{m,n} \mathbb{E} \left[e^{\sqrt{2} p_2 q^{nm} \alpha_{n,m} \alpha_n - p_2 q^{2nm} \alpha_{m,n}^2} \right] \mathbb{E} \left[e^{\sqrt{2} p_2 q^{nm} \beta_{n,m} \beta_n - p_2 q^{2nm} \beta_{m,n}^2} \right] \\ & = \prod_{m,n} \mathbb{E} \left[e^{(p_2^2 - p_2) q^{2nm} \alpha_{n,m}^2} \right] \mathbb{E} \left[e^{(p_2^2 - p_2) q^{2nm} \beta_{n,m}^2} \right]. \end{aligned}$$

Now, for a standard Gaussian \mathcal{N} and $\mu < \frac{1}{2}$, we recall that

$$(2.16) \quad \mathbb{E} \left[e^{\mu \mathcal{N}^2} \right] = \frac{1}{\sqrt{1 - 2\mu}}.$$

Since p_1 and p_3 can be arbitrarily large, we can choose p_2 close enough to 1 so that $(p_2^2 - p_2) q^{2nm} < \frac{1}{2}$. In this case, (2.16) yields

$$\prod_{m,n} \mathbb{E} \left[e^{(p_2^2 - p_2) q^{2nm} \alpha_{n,m}^2} \right] \mathbb{E} \left[e^{(p_2^2 - p_2) q^{2nm} \beta_{n,m}^2} \right] = \prod_{m,n} \frac{1}{1 - 2(p_2^2 - p_2) q^{2nm}} < \infty.$$

By Lemma 2.5, this completes the proof in the case of $\alpha \in [0, Q)$.

Let us now move to the case $\alpha \in (-\frac{4}{\gamma}, 0)$. By Lemma B.1, (2.15) holds when $p_1 \in (1, \infty)$ and $p_3 \in (1, -\frac{4}{\gamma\alpha})$. Therefore, we can only choose p_2 within $(1, \frac{1}{1 + \frac{2\alpha}{4}})$. By (2.16), we also need $(p_2^2 - p_2) q^2 < \frac{1}{2}$. Such a p_2 exists when $q < (1 + \frac{\gamma\alpha}{4}) \sqrt{\frac{2}{\gamma|\alpha|}}$. \square

Remark. For $\alpha \in [0, Q)$ we are able to show convergence for $|q| < 1$, which is expected to be the optimal radius of convergence. This range of α is the most natural from the perspective of LCFT because it corresponds to the case where the 1-point correlation function obeys the Seiberg bounds (see [DRV16] for details). For $\alpha \in (-\frac{4}{\gamma}, 0)$, the function $q \mapsto A_{\gamma,P}^q(\alpha)$ is still well defined for $q \in (0, 1)$. However, we are only able to show it is analytic in q in a smaller range depending on α . We do not know the precise radius of convergence in this case. We note that $\alpha \in (-\frac{4}{\gamma}, 0)$ is less natural from the perspective of LCFT, as it requires a meromorphic extension of the correlation functions.

Define normalized versions of $\mathcal{A}_{\gamma,P}^q$ and $\mathcal{A}_{\gamma,P,n}$ from Lemma 2.7 by

$$(2.17) \quad \tilde{\mathcal{A}}_{\gamma,P}^q(\alpha) := \frac{\mathcal{A}_{\gamma,P}^q(\alpha)}{\mathcal{A}_{\gamma,P,0}(\alpha)} \quad \text{and} \quad \tilde{\mathcal{A}}_{\gamma,P,n}(\alpha) := \frac{\mathcal{A}_{\gamma,P,n}(\alpha)}{\mathcal{A}_{\gamma,P,0}(\alpha)}.$$

Definition 2.9. Let $\gamma \in (0, 2)$, $\alpha \in (-\frac{4}{\gamma}, Q)$, $\text{Im}(P) \in (-\frac{1}{2\gamma}, \frac{1}{2\gamma})$. For $q \in (0, 1)$, we define the 1-point toric GMC conformal block by

$$(2.18) \quad \mathcal{G}_{\gamma,P}^q(\alpha) := q^{-\frac{1}{12}(1 - \alpha(Q - \frac{\alpha}{2}))} \eta(q)^{1 - \alpha(Q - \frac{\alpha}{2})} \tilde{\mathcal{A}}_{\gamma,P}^q(\alpha).$$

In other words, one has

$$(2.19) \quad \mathcal{G}_{\gamma,P}^q(\alpha) = \frac{1}{Z} \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma}} \right]$$

for the normalization constant

$$Z := q^{\frac{1}{12}(\frac{\alpha\gamma}{2} + \frac{\alpha^2}{2} - 1)} \eta(q)^{\alpha^2 + 1 - \frac{\alpha\gamma}{2}} \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\infty(x)} [-2 \sin(\pi x)]^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma}} \right]$$

which has a further explicit evaluation given by (6.6).

Remark. Although Definition 2.9 only uses the law of Y_τ , throughout this paper we assume that different $\{Y_\tau\}$'s and Y_∞ are coupled as in Section 2.1.

2.3. 1-point toric conformal block and Nekrasov partition function. The AGT conjecture of [AGT09] postulates a general relation between Liouville conformal blocks and an object called the Nekrasov partition function occurring in four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory. We use it to give a mathematically rigorous definition of the 1-point toric conformal block $\mathcal{F}_{\gamma,P}^q(\alpha)$ as a formal series. For this, we first define the 1-point Nekrasov partition function on the torus as the formal q -series

$$(2.20) \quad \mathcal{Z}_{\gamma,P}^q(\alpha) := 1 + \sum_{k=1}^{\infty} \mathcal{Z}_{\gamma,P,k}(\alpha) q^k,$$

where

$$(2.21) \quad \mathcal{Z}_{\gamma,P,k}(\alpha) := \sum_{\substack{(Y_1, Y_2) \text{ Young diagrams} \\ |Y_1| + |Y_2| = k}} \prod_{i,j=1}^2 \prod_{s \in Y_i} \frac{(E_{ij}(s, P) - \alpha)(Q - E_{ij}(s, P) - \alpha)}{E_{ij}(s, P)(Q - E_{ij}(s, P))}$$

for

$$(2.22) \quad E_{ij}(s, P) := \begin{cases} P - \frac{\gamma}{2} H_{Y_j}(s) + \frac{2}{\gamma} (V_{Y_i}(s) + 1) & i = 1, j = 2 \\ -\frac{\gamma}{2} H_{Y_j}(s) + \frac{2}{\gamma} (V_{Y_i}(s) + 1) & i = j \\ -P - \frac{\gamma}{2} H_{Y_j}(s) + \frac{2}{\gamma} (V_{Y_i}(s) + 1) & i = 2, j = 1 \end{cases}$$

with $H_Y(s)$ and $V_Y(s)$ the horizontal and vertical distances from the square s to the edge of diagram Y . In these terms, we define the 1-point toric conformal block as the formal q -series

$$(2.23) \quad \mathcal{F}_{\gamma,P}^q(\alpha) := q^{-\frac{1}{12}(1 - \alpha(Q - \frac{\alpha}{2}))} \eta(q)^{1 - \alpha(Q - \frac{\alpha}{2})} \mathcal{Z}_{\gamma,P}^q(\alpha).$$

The toric conformal block was characterized in [HJS10, FL10] in terms of a recursive relation which is a toric analogue of Zamolodchikov's recursion from [Zam84]. Define the quantity

$$(2.24) \quad R_{\gamma,m,n}^q(\alpha) := \frac{2 \prod_{j=-m}^{m-1} \prod_{l=-n}^{n-1} (Q - \frac{\alpha}{2} + \frac{j\gamma}{2} + \frac{2l}{\gamma})}{\prod_{(j,l) \in S_{m,n}} (\frac{j\gamma}{2} + \frac{2l}{\gamma})}$$

for $S_{m,n} := \{(j, l) \mid 1 - m \leq j \leq m, 1 - n \leq l \leq n, (j, l) \notin \{(0, 0), (m, n)\}\}$ and

$$(2.25) \quad P_{m,n} := \frac{2in}{\gamma} + \frac{i\gamma m}{2}.$$

Notice that the q -series expansion of $\mathcal{F}_{\gamma,P}^q(\alpha)$ may be computed from (2.26).

Proposition 2.10 ([HJS10, FL10]). *As a formal q -series, the conformal block $\mathcal{F}_{\gamma,P}^q(\alpha)$ satisfies*

$$(2.26) \quad \mathcal{F}_{\gamma,P}^q(\alpha) = \sum_{n,m=1}^{\infty} q^{2mn} \frac{R_{\gamma,m,n}^q(\alpha)}{P^2 - P_{m,n}^2} \mathcal{F}_{\gamma,P-m,n}^q(\alpha) + q^{\frac{1}{12}} \eta(q)^{-1}.$$

Remark. The references [HJS10] and [FL10] are mathematical physics papers, while our paper is a fully rigorous mathematical paper. In this paper, we treat (2.23) as a mathematical definition of the conformal block as a formal series and use only Proposition 2.10, which is proven in a mathematically rigorous way for this definition in [FL10]. Combining our result in Corollary 2.12 with Lemma 2.7(a) implies that this formal series actually converges, resolving a conjecture of [FML18].

2.4. Statement of the main results. Our main result is Theorem 2.11, which gives a GMC expression for the Nekrasov partition function. Its direct consequence Theorem 2.12 is our probabilistic construction of the 1-point toric conformal block. The proof of Theorem 2.11 will occupy the remainder of this paper.

Theorem 2.11. *For $\gamma \in (0, 2)$, $\alpha \in (-\frac{4}{\gamma}, Q)$, and $P \in \mathbb{R}$, the formal q -series for $\mathcal{Z}_{\gamma, P}^q(\alpha)$ converges for $|q| < r_\alpha$ and satisfies*

$$(2.27) \quad \mathcal{Z}_{\gamma, P}^q(\alpha) = \tilde{\mathcal{A}}_{\gamma, P}^q(\alpha).$$

Theorem 2.12. *For $\gamma \in (0, 2)$, $\alpha \in (-\frac{4}{\gamma}, Q)$, and $P \in \mathbb{R}$, the formal q -series for $\mathcal{F}_{\gamma, P}^q(\alpha)$ converges for $|q| < r_\alpha$ and satisfies*

$$(2.28) \quad \mathcal{F}_{\gamma, P}^q(\alpha) = \mathcal{G}_{\gamma, P}^q(\alpha).$$

Proof. This follows from Theorem 2.11, (2.23), and Definition 2.9. \square

3. BPZ EQUATION FOR DEFORMED CONFORMAL BLOCKS

In this section we establish Theorem 3.4, which gives the BPZ equations for certain deformations of the conformal block corresponding to degenerate insertions. Throughout Sections 3–6.1, we view γ and P as fixed parameters such that $\gamma \in (0, 2)$ and $P \in \mathbb{R}$. For $\alpha \in (-\frac{4}{\gamma}, Q)$, and $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$, define

$$(3.1) \quad l_\chi = \frac{\chi^2}{2} - \frac{\alpha\chi}{2}.$$

Recall the definition $\mathfrak{B} := \{z : 0 < \text{Im}(z) < \frac{3}{4} \text{Im}(\tau)\}$ from Appendix A and q_0 in Lemma A.2. Let

$$(3.2) \quad \nu(dx) := e^{\frac{\gamma}{2} Y_\tau(x)} |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma Px} dx.$$

Fix $q \in (0, q_0)$. Recall Lemma A.4 and set $c = \frac{\gamma\chi}{2}$ there. For $u \in \mathfrak{B}$, we have

$$f_\nu(u) := \int_0^1 e^{\frac{\gamma\chi}{2} g(u+x)} \nu(dx) = \int_0^1 \Theta_\tau(u+x)^{\frac{\gamma\chi}{2}} \nu(dx).$$

By Lemma A.4, f_ν is almost surely analytic and nonzero on \mathfrak{B} , meaning we can define its fractional power according to Definition A.5. The next lemmas deal with the deformed block up to an explicit prefactor.

Lemma 3.1. *For $\alpha \in (-\frac{4}{\gamma} + \chi, Q)$ and $q \in (0, q_0)$, we have $\mathbb{E} \left[|f_\nu(u)|^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] < \infty$. Moreover, the function $u \mapsto \mathbb{E} \left[(f_\nu(u))^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right]$ is analytic on \mathfrak{B} . Finally, we have*

$$\mathbb{E} \left[(f_\nu(1))^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] = e^{-\pi i l_\chi} \mathbb{E} \left[(f_\nu(0))^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right].$$

Proof. The finiteness of the moment of $|f_\nu(u)|$ comes from Lemma B.1 recalled in appendix. Now, Lemma 2.5 gives the desired analyticity in u . By Lemma A.4 and the computation $\frac{\gamma\chi}{2}(-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}) = l_\chi$, we have $\mathbb{E} \left[(f_\nu(1))^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] = e^{-\pi i l_\chi} \mathbb{E} \left[(f_\nu(0))^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right]$. \square

Recall r_α defined in (2.10), and define the domain $D_\chi^\alpha := \{(q, u) : |q| < r_{\alpha-\chi}, u \in \mathfrak{B}\}$. We defer the proof of the following proposition to Section 3.1.

Proposition 3.2. *For $\alpha \in (-\frac{4}{\gamma} + \chi, Q)$ and $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$, let*

$$(3.3) \quad \hat{\psi}_\chi^\alpha(u, q) := q^{\frac{\gamma l_\chi}{12\chi} - \frac{1}{6} \frac{l_\chi^2}{\chi^2} - \frac{1}{6\chi^2} l_\chi(l_\chi+1)} \Theta'_\tau(0)^{-\frac{2l_\chi^2}{3\chi^2} + \frac{l_\chi}{3} + \frac{4l_\chi}{3\gamma\chi}} e^{\chi P u \pi} \Theta_\tau(u)^{-l_\chi} e^{-\frac{1}{2} i \pi \alpha \gamma (-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma})} \mathbb{E} \left[(f_\nu(u))^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right],$$

where $q \in (0, q_0)$ and $u \in \mathfrak{B}$. The function $\hat{\psi}_\chi^\alpha$ has a bi-holomorphic extension to D_χ^α .

Definition 3.3 (*u*-deformed conformal block). For (u, τ) such that $(u, e^{i\pi\tau}) \in D_\chi^\alpha$, define the *u*-deformed conformal block by

$$(3.4) \quad \psi_\chi^\alpha(u, q) = e^{\left(\frac{P^2}{2} + \frac{1}{6\chi^2} l_\chi(l_\chi + 1)\right) i\pi\tau} \hat{\psi}_\chi^\alpha(u, e^{i\pi\tau}),$$

where we extend $\hat{\psi}_\chi^\alpha$ in Proposition 3.2 as a bi-holomorphic function on D_χ^α .

Remark. More explicitly, Definition 3.3 yields the expression

$$\begin{aligned} \psi_\chi^\alpha(u, q) &= q^{\frac{P^2}{2} + \frac{\gamma l_\chi}{12\chi} - \frac{1}{6} \frac{l_\chi^2}{\chi^2}} \Theta'_\tau(0) e^{-\frac{2l_\chi^2}{3\chi^2} + \frac{l_\chi}{3} + \frac{4l_\chi}{3\gamma\chi}} e^{\chi P u \pi} \Theta_\tau(u) e^{-l_\chi} \\ &\quad \times \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x) e^{-\frac{\alpha\gamma}{2}} \Theta_\tau(u+x) e^{\frac{\gamma}{2} \chi} e^{\pi\gamma P x} dx \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] \end{aligned}$$

for the *u*-deformed block, where the arguments of the complex numbers appearing are interpreted by the procedure given above.

In the definition of $\psi_\chi^\alpha(u, q)$, the prefactor of $\mathbb{E} \left[(f_\nu(u))^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right]$ is chosen for the following *BPZ equation* to hold. Its proof is given in Section 3.2.

Theorem 3.4. *Recalling the definition of the Weierstrass \wp function from Section A, we have*

$$(3.5) \quad \left(\partial_{uu} - l_\chi(l_\chi + 1)\wp(u) + 2i\pi\chi^2\partial_\tau \right) \psi_\chi^\alpha(u, q) = 0 \quad \text{for } (u, e^{i\pi\tau}) \in D_\chi^\alpha.$$

3.1. Proof of Proposition 3.2. In a manner similar to the proof of Lemma 2.7(a), by (2.12) and Girsanov's theorem (Theorem B.2), we have

$$\begin{aligned} \mathbb{E} \left[(f_\nu(u))^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] &= \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} \Theta_\tau(u+x) e^{\frac{\gamma}{2} \chi} e^{\pi\gamma P x} dx \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] \\ &= \left(q^{1/6} \eta(q) \right)^{\frac{\alpha(\alpha-\chi)}{2}} e^{(\frac{\alpha\gamma}{8} - \frac{\gamma\chi}{8} - \frac{\alpha^2}{8}) \mathbb{E}[F_\tau(0)^2]} \\ &\quad \times \mathbb{E} \left[e^{\frac{\alpha}{2} F_\tau(0)} \left(\int_0^1 e^{\frac{\gamma}{2} Y_\infty(x)} e^{\frac{\gamma}{2} F_\tau(x)} (2 \sin(\pi x))^{-\frac{\alpha\gamma}{2}} \Theta_\tau(x+u) e^{\frac{\chi\gamma}{2}} e^{\pi\gamma P x} dx \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right]. \end{aligned}$$

By (2.13) and Girsanov's theorem (Theorem B.2), we get the following analog of (2.14)

$$\begin{aligned} &\mathbb{E} \left[(f_\nu(u))^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] \\ &= \left(q^{1/6} \eta(q) \right)^{\frac{\alpha(\alpha-\chi)}{2}} e^{(\frac{\alpha\gamma}{8} - \frac{\gamma\chi}{8} - \frac{\alpha^2}{8}) \mathbb{E}[F_\tau(0)^2]} \mathbb{E} \left[e^{\frac{\alpha}{2} F_\tau(0)} e^{\sqrt{2} \sum_{m,n} q^{nm} (\alpha_{n,m} \alpha_n + \beta_{n,m} \beta_n)} e^{-\sum_{m,n} q^{2nm} (\alpha_{m,n}^2 + \beta_{m,n}^2)} \right. \\ &\quad \left. \times \left(\int_0^1 e^{\frac{\gamma}{2} Y_\infty(x)} (2 \sin(\pi x))^{-\frac{\alpha\gamma}{2}} \Theta_\tau(x+u) e^{\frac{\chi\gamma}{2}} e^{\pi\gamma P x} dx \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right]. \end{aligned}$$

For $q \in \mathbb{D}$ and $u \in \mathfrak{B}$, define

$$(3.6) \quad \mathcal{X}(u, q) := -\chi \sum_{n,m=1}^{\infty} \frac{1}{\sqrt{2m}} \left((\alpha_m + i\beta_m) q^{(2n-2)m} e^{2\pi i u m} + (\alpha_m - i\beta_m) q^{2nm} e^{-2\pi i u m} \right).$$

Since $|q|^{3/2} < |e^{2\pi i u}| < 1 < |e^{-2\pi i u}| < |q|^{-3/2}$ when $u \in \mathfrak{B}$, the series converges almost surely in $q \in \mathbb{D}$. Moreover, $e^{\mathcal{X}(u,q)}$ has finite moments of all orders. We claim that

$$(3.7) \quad \Theta_\tau(u+x) = -i e^{-i\pi u} q^{\frac{1}{6}} \eta(q) e^{\frac{1}{\chi} \mathbb{E}[Y_\infty(x) \mathcal{X}(u,q)]}.$$

To see (3.7), set $u' = u - \frac{\tau}{2}$. By (A.5), we have

$$(3.8) \quad \Theta_\tau(u+x) = -i e^{-i\pi u} q^{\frac{1}{6}} \eta(q) \prod_{n=1}^{\infty} (1 - q^{2n-1} e^{2\pi i(u'+x)}) (1 - q^{2n-1} e^{-2\pi i(u'+x)}).$$

Using $1 - z = \exp\{\sum_{m=1}^{\infty} \frac{z^m}{m}\}$ for $|z| < 1$ and recalling (2.12), we have

$$\begin{aligned} \prod_{n=1}^{\infty} (1 - q^{2n-1} e^{2\pi i(u'+x)}) (1 - q^{2n-1} e^{-2\pi i(u'+x)}) &= \exp \left\{ -2 \sum_{n,m=1}^{\infty} \frac{q^{(2n-1)m}}{m} \cos(2\pi(x+u')m) \right\} \\ &= \exp \left\{ -\sqrt{2} \sum_{n,m=1}^{\infty} \frac{q^{(2n-1)m}}{\sqrt{m}} (\cos(2\pi u'm) \mathbb{E}[\alpha_m Y_{\infty}(x)] - \sin(2\pi u'm) \mathbb{E}[\beta_m Y_{\infty}(x)]) \right\}. \end{aligned}$$

Now, (3.7) follows from the observation that

$$(3.9) \quad \mathcal{X}(u, q) = -\chi \sqrt{2} \sum_{n,m=1}^{\infty} \frac{q^{(2n-1)m}}{\sqrt{m}} (\cos(2\pi u'm) \alpha_m - \sin(2\pi u'm) \beta_m).$$

Moreover, (3.9) also implies that

$$(3.10) \quad \mathcal{X}(u, q) \in \mathbb{R} \text{ and } \mathbb{E}[\mathcal{X}(u, q)^2] = 2\chi^2 \sum_{n,m=1}^{\infty} \frac{q^{2(2n-1)m}}{m} \quad \text{if } \text{Im } u = \frac{1}{2} \text{Im } \tau.$$

Now, we assume $\text{Im } u = \frac{1}{2} \text{Im } \tau$ so that $\mathcal{X}(u, q) \in \mathbb{R}$. By (3.7), we have

$$\begin{aligned} &\left(\int_0^1 e^{\frac{\gamma}{2} Y_{\infty}(x)} (2 \sin(\pi x))^{-\frac{\alpha\gamma}{2}} \Theta_{\tau}(x+u)^{\frac{\chi\gamma}{2}} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \\ &= \left(-\mathbf{i} e^{-\mathbf{i}\pi u} q^{1/6} \eta(q) \right)^{\frac{\chi}{2}(\chi-\alpha)} \left(\int_0^1 e^{\frac{\gamma}{2} Y_{\infty}(x)} (2 \sin(\pi x))^{-\frac{\alpha\gamma}{2}} e^{\frac{\gamma}{2} \mathbb{E}[Y_{\infty}(x) \mathcal{X}(u, q)]} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}}. \end{aligned}$$

Setting $\mathcal{Q}(q) = e^{\sqrt{2} \sum_{m,n} q^{nm} (\alpha_{n,m} \alpha_n + \beta_{n,m} \beta_n)} e^{-\sum_{m,n} q^{2nm} (\alpha_{m,n}^2 + \beta_{m,n}^2)}$, we have

$$\begin{aligned} \mathbb{E} \left[(f_{\nu}(u))^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] &= \left(q^{1/6} \eta(q) \right)^{\frac{\alpha(\alpha-\chi)}{2}} e^{(\frac{\alpha\gamma}{8} - \frac{\gamma\chi}{8} - \frac{\alpha^2}{8}) \mathbb{E}[F_{\tau}(0)^2]} \left(-\mathbf{i} e^{-\mathbf{i}\pi u} q^{1/6} \eta(q) \right)^{\frac{\chi}{2}(\chi-\alpha)} \\ &\times \mathbb{E} \left[e^{\frac{\alpha}{2} F_{\tau}(0)} \mathcal{Q}(q) \left(\int_0^1 e^{\frac{\gamma}{2} Y_{\infty}(x)} (2 \sin(\pi x))^{-\frac{\alpha\gamma}{2}} e^{\frac{\gamma}{2} \mathbb{E}[Y_{\infty}(x) \mathcal{X}(u, q)]} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right]. \end{aligned}$$

Applying Girsanov's theorem (Theorem B.2) gives

$$(3.11) \quad \mathbb{E} \left[e^{\frac{\alpha}{2} F_{\tau}(0)} \mathcal{Q}(q) \left(\int_0^1 e^{\frac{\gamma}{2} Y_{\infty}(x)} (2 \sin(\pi x))^{-\frac{\alpha\gamma}{2}} e^{\frac{\gamma}{2} \mathbb{E}[Y_{\infty}(x) \mathcal{X}(u, q)]} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] \\ = \mathbb{E} \left[e^{\frac{\alpha}{2} F_{\tau}(0)} \mathcal{Q}(q) e^{\mathcal{X}(u, q) - \frac{1}{2} \mathbb{E}[\mathcal{X}(u, q)^2]} \left(\int_0^1 e^{\frac{\gamma}{2} Y_{\infty}(x)} (2 \sin(\pi x))^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right].$$

By the same Holder's inequality argument as in the proof of Lemma 2.7(a), the right side of (3.11) is finite for $|q| < r_{\alpha-\chi}$. Here, the shift $\alpha \rightarrow \alpha - \chi$ is due to the exponent $-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}$. By Lemma 2.5, the right side of (3.11) is bi-holomorphic in (u, q) on D_{χ}^{α} . On the other hand, by Lemma 2.3 and (A.4), for $q \in (0, 1)$ the quantity

$$\begin{aligned} &q^{\frac{\gamma l_{\chi}}{12\chi} - \frac{1}{6} \frac{l_{\chi}^2}{\chi^2} - \frac{1}{6\chi^2} l_{\chi}(l_{\chi}+1)} \Theta'_{\tau}(0) - \frac{2l_{\chi}^2}{3\chi^2} + \frac{l_{\chi}}{3} + \frac{4l_{\chi}}{3\gamma\chi} e^{\chi P u \pi} \Theta_{\tau}(u)^{-l_{\chi}} \\ &\quad \times \left(q^{1/6} \eta(q) \right)^{\frac{\alpha(\alpha-\chi)}{2}} e^{(\frac{\alpha\gamma}{8} - \frac{\gamma\chi}{8} - \frac{\alpha^2}{8}) \mathbb{E}[F_{\tau}(0)^2]} \left(-\mathbf{i} e^{-\mathbf{i}\pi u} q^{1/6} \eta(q) \right)^{\frac{\chi}{2}(\chi-\alpha)} \end{aligned}$$

equals $q^{\frac{\gamma}{12\chi} - \frac{1}{6\chi^2} + \frac{1}{3\chi\gamma} - \frac{1}{6}}$ multiplied by a power series in q which converges in \mathbb{D} . By using the special values $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$, one can check that $q^{\frac{\gamma}{12\chi} - \frac{1}{6\chi^2} + \frac{1}{3\chi\gamma} - \frac{1}{6}} = 1$. This concludes the proof of Proposition 3.2.

3.2. Proof of Theorem 3.4. Since $q^{-\frac{P^2}{2} - \frac{1}{6\chi^2}l_\chi(l_\chi+1)}\psi_\chi^\alpha(u, q)$ is bi-holomorphic in (u, q) , it suffices to verify (3.5) for $q \in (0, q_0)$ and $u \in \mathfrak{B}$ where (2.18) applies. Define $s := -\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}$ and introduce the notations

$$\begin{aligned} \mathcal{T}(u, x) &:= \frac{\Theta(x)^{-\frac{\alpha\gamma}{2}} \Theta_\tau(u+x)^{\frac{\gamma}{2}\chi}}{\Theta_\tau(u)^{\frac{\gamma\chi}{2}}} \\ \mathcal{V}_1(u, y)dy &:= \mathbb{E} \left[e^{\frac{\gamma}{2}Y_\tau(y)} \Theta_\tau(u)^{\frac{\gamma\chi(1-s)}{2}} \left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)} \Theta(x)^{-\frac{\alpha\gamma}{2}} \Theta_\tau(u+x)^{\frac{\gamma}{2}\chi} e^{\pi\gamma Px} dx \right)^{s-1} \right] dy \\ \mathcal{V}_2(u, y, z)dydz &:= \mathbb{E} \left[e^{\frac{\gamma}{2}Y_\tau(y)} e^{\frac{\gamma}{2}Y_\tau(z)} \Theta_\tau(u)^{\frac{\gamma\chi(2-s)}{2}} \left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)} \Theta(x)^{-\frac{\alpha\gamma}{2}} \Theta_\tau(u+x)^{\frac{\gamma}{2}\chi} e^{\pi\gamma Px} dx \right)^{s-2} \right] dydz \\ \mathcal{W}(q) &:= q^{\frac{P^2}{2} + \frac{\gamma l_\chi}{12\chi} - \frac{1}{6} \frac{l_\chi^2}{\chi^2}} \Theta'_\tau(0)^{-\frac{2l_\chi^2}{3\chi^2} + \frac{l_\chi}{3} + \frac{4l_\chi}{3\gamma\chi}}. \end{aligned}$$

We start by computing derivatives with respect to u ; by direct differentiation, we have

$$\begin{aligned} \partial_u \psi_\chi^\alpha(u, q) &= \chi P \pi \psi_\chi^\alpha(u, q) + s \mathcal{W}(q) e^{\pi\chi Pu} \int_0^1 \partial_u \mathcal{T}(u, y) e^{\pi\gamma Py} \mathcal{V}_1(u, y) dy \\ \partial_{uu} \psi_\chi^\alpha(u, q) &= (\chi P \pi)^2 \psi_\chi^\alpha(u, q) + 2\chi P \pi s \mathcal{W}(q) e^{\pi\chi Pu} \int_0^1 \partial_u \mathcal{T}(u, y) e^{\pi\gamma Py} \mathcal{V}_1(u, y) dy \\ &\quad + s \mathcal{W}(q) e^{\pi\chi Pu} \int_0^1 \partial_{uu} \mathcal{T}(u, y) e^{\pi\gamma Py} \mathcal{V}_1(u, y) dy \\ &\quad + s(s-1) \mathcal{W}(q) e^{\pi\chi Pu} \int_0^1 \int_0^1 \partial_u \mathcal{T}(u, y) \partial_u \mathcal{T}(u, z) e^{\pi\gamma P(y+z)} \mathcal{V}_2(u, y, z) dydz. \end{aligned}$$

We now compute the derivative in τ , whose derivation is slightly more involved.

Lemma 3.5. *For $q \in (0, q_0)$ and $u \in \mathfrak{B}$, we have*

$$\begin{aligned} (3.12) \quad \partial_\tau \psi_\chi^\alpha(u, q) &= i\pi \left(\frac{P^2}{2} + \frac{\gamma l_\chi}{12\chi} - \frac{1}{6} \frac{l_\chi^2}{\chi^2} \right) \psi_\chi^\alpha(u, q) \\ &\quad + \left(-\frac{2l_\chi^2}{3\chi^2} + \frac{l_\chi}{3} + \frac{2}{3}s \right) \frac{\partial_\tau \Theta'_\tau(0)}{\Theta'_\tau(0)} \psi_\chi^\alpha(u, q) + s \mathcal{W}(q) e^{\pi\chi Pu} \int_0^1 \partial_\tau \mathcal{T}(u, y) e^{\pi\gamma Py} \mathcal{V}_1(u, y) dy \\ &\quad + \frac{\gamma^2 s(s-1)}{4} \mathcal{W}(q) e^{\pi\chi Pu} \int_0^1 \int_0^1 \left(\frac{i\pi}{6} - \frac{\partial_\tau \Theta_\tau(y-z)}{\Theta_\tau(y-z)} + \frac{1}{3} \frac{\partial_\tau \Theta'_\tau(0)}{\Theta'_\tau(0)} \right) \mathcal{T}(u, y) \mathcal{T}(u, z) e^{\pi\gamma Py + \pi\gamma Pz} \mathcal{V}_2(u, y, z) dydz. \end{aligned}$$

Proof. Taking the τ -derivative, we obtain

$$\begin{aligned} \partial_\tau \psi_\chi^\alpha(u, q) &= i\pi \left(\frac{P^2}{2} + \frac{\gamma l_\chi}{12\chi} - \frac{1}{6} \frac{l_\chi^2}{\chi^2} \right) \psi_\chi^\alpha(u, q) + \left(-\frac{2l_\chi^2}{3\chi^2} + \frac{l_\chi}{3} + \frac{2}{3}s \right) \frac{\partial_\tau \Theta'_\tau(0)}{\Theta'_\tau(0)} \psi_\chi^\alpha(u, q) \\ &\quad + s \mathcal{W}(q) e^{\pi\chi Pu} \int_0^1 \partial_\tau \mathcal{T}(u, y) e^{\pi\gamma Py} \mathcal{V}_1(u, y) dy \\ &\quad + s \mathcal{W}(q) e^{\pi\chi Pu} \int_0^1 \mathcal{T}(u, y) e^{\pi\gamma Py} \mathbb{E} \left[\partial_\tau [e^{\frac{\gamma}{2}Y_\tau(y)}] \left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)} \mathcal{T}(u, x) e^{\pi\gamma Px} dx \right)^{s-1} \right] dy. \end{aligned}$$

We now find that

$$\begin{aligned}
 & \mathbb{E} \left[\partial_\tau [e^{\frac{\gamma}{2} Y_\tau(y)}] \left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \mathcal{T}(u, x) e^{\pi\gamma P x} dx \right)^{s-1} \right] dy \\
 &= \lim_{N \rightarrow \infty} \mathbb{E} \left[\partial_\tau [e^{\frac{\gamma}{2} Y_{\tau, N}(y) - \frac{\gamma^2}{8} \mathbb{E}[Y_{\tau, N}(y)^2]}] \left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \mathcal{T}(u, x) e^{\pi\gamma P x} dx \right)^{s-1} \right] dy \\
 &= \lim_{N \rightarrow \infty} \mathbb{E} \left[\left(\frac{\gamma}{2} \partial_\tau Y_{\tau, N}(y) - \frac{\gamma^2}{4} \mathbb{E}[Y_{\tau, N}(y) \partial_\tau Y_{\tau, N}(y)] \right) e^{\frac{\gamma}{2} Y_{\tau, N}(y) - \frac{\gamma^2}{8} \mathbb{E}[Y_{\tau, N}(y)^2]} \left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \mathcal{T}(u, x) e^{\pi\gamma P x} dx \right)^{s-1} \right] dy \\
 &= \lim_{N \rightarrow \infty} \frac{\partial}{\partial \varepsilon} \Big|_{\varepsilon=0} \mathbb{E} \left[e^{\varepsilon \frac{\gamma}{2} \partial_\tau Y_{\tau, N}(y) - \varepsilon^2 \frac{\gamma^2}{4} \mathbb{E}[(\partial_\tau Y_{\tau, N}(y))^2] - \varepsilon \frac{\gamma^2}{4} \mathbb{E}[Y_{\tau, N}(y) \partial_\tau Y_{\tau, N}(y)]} e^{\frac{\gamma}{2} Y_{\tau, N}(y) - \frac{\gamma^2}{8} \mathbb{E}[Y_{\tau, N}(y)^2]} \right. \\
 &\quad \left. \times \left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \mathcal{T}(u, x) e^{\pi\gamma P x} dx \right)^{s-1} \right] dy \\
 &= \frac{\gamma^2}{4} (s-1) \int_0^1 \mathbb{E}[Y_\tau(z) \partial_\tau Y_\tau(y)] \mathcal{T}(u, z) e^{\pi\gamma P z} \mathbb{E} \left[e^{\frac{\gamma}{2} Y_\tau(y) + \frac{\gamma}{2} Y_\tau(z)} \left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \mathcal{T}(u, x) e^{\pi\gamma P x} dx \right)^{s-2} \right] dz \\
 &= \frac{\gamma^2 (s-1)}{4} \int_0^1 \left(\frac{i\pi}{6} - \frac{\partial_\tau \Theta_\tau(y-z)}{\Theta_\tau(y-z)} + \frac{1}{3} \frac{\partial_\tau \Theta'_\tau(0)}{\Theta'_\tau(0)} \right) \mathcal{T}(u, z) e^{\pi\gamma P z} \mathcal{V}_2(u, y, z) dz.
 \end{aligned}$$

We therefore obtain (3.12). \square

Proof of Theorem 3.4. To prove the BPZ equation we must combine the above expressions for the derivatives and check the equation. To see the cancellation we will need to perform an integration by parts on one of the terms in $\partial_{uu} \psi_\chi^\alpha(u, q)$. We will perform this at the level of the regularized field and introduce a smoothing by convolution that will be more regular than the truncation of the Fourier series defining Y_∞ .

For this, consider a small $\varepsilon > 0$. The field Y_∞ can be viewed as the restriction on the unit circle of a free boundary GFF X on the unit disk \mathbb{D} , with the identification $Y_\infty(x) = X(e^{2i\pi x})$ for $x \in [0, 1]$. Let $\rho : [0, +\infty) \mapsto [0, +\infty)$ be a C^∞ function with compact support in $[0, 1]$ and such that $\pi \int_0^\infty \rho(t) dt = 1$. For $z \in \mathbb{C}$, we write $\rho_\varepsilon(z) = \frac{1}{\varepsilon^2} \rho\left(\frac{z\bar{z}}{\varepsilon^2}\right)$ and introduce for $x \in [0, 1]$ the field

$$Y_{\infty, \varepsilon}(x) = X_{\infty, \varepsilon}(e^{2i\pi x}) = 2 \int_{\mathbb{D}} d^2 z X(z) \rho_\varepsilon(e^{2i\pi x} - z).$$

Furthermore set $Y_{\tau, \varepsilon}(x) = Y_{\infty, \varepsilon}(x) + F_\tau(x)$, and define the kernel $K_\varepsilon(x - y)$ via the equality

$$\mathbb{E}[Y_{\tau, \varepsilon}(x) Y_{\tau, \varepsilon}(y)] = K_\varepsilon(x - y) + 2 \log |q^{1/6} \eta(q)|.$$

It satisfies $K_\varepsilon(x - y) = K_\varepsilon(y - x)$ and $\lim_{\varepsilon \rightarrow 0} K_\varepsilon(x - y) = -2 \log |\Theta_\tau(x - y)|$. Let also $\mathcal{V}_{1, \varepsilon}(u, y)$ and $\mathcal{V}_{2, \varepsilon}(u, y, z)$ denote the exact same expressions as $\mathcal{V}_1(u, y)$ and $\mathcal{V}_2(u, y, z)$, but defined using the regularized field $Y_{\tau, \varepsilon}(x)$ instead of $Y_\tau(x)$. In the expression for $\partial_{uu} \psi_\chi^\alpha(u, q)$, by integration by parts at the level of the regularized field we have

$$\begin{aligned}
 2\chi P \pi s \mathcal{W}(q) e^{\pi\chi P u} \int_0^1 \partial_u \mathcal{T}(u, y) e^{\pi\gamma P y} \mathcal{V}_{1, \varepsilon}(u, y) dy &= \frac{2\chi s}{\gamma} \mathcal{W}(q) e^{\pi\chi P u} \int_0^1 \partial_u \mathcal{T}(u, y) [\partial_y e^{\pi\gamma P y}] \mathcal{V}_{1, \varepsilon}(u, y) dy \\
 &= -\frac{2\chi s}{\gamma} \mathcal{W}(q) e^{\pi\chi P u} \int_0^1 \partial_{uy} \mathcal{T}(u, y) e^{\pi\gamma P y} \mathcal{V}_{1, \varepsilon}(u, y) dy \\
 &\quad - \frac{2\chi s}{\gamma} \mathcal{W}(q) e^{\pi\chi P u} \int_0^1 \partial_u \mathcal{T}(u, y) e^{\pi\gamma P y} \partial_y \mathcal{V}_{1, \varepsilon}(u, y) dy.
 \end{aligned}$$

We now claim that the boundary contribution vanishes in the above integration by parts; for this, write

$$\partial_u \mathcal{T}(u, y) e^{\pi\gamma P y} \mathcal{V}_{1, \varepsilon}(u, y) = \frac{\gamma\chi}{2} \left(\frac{\Theta'_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta'_\tau(u)}{\Theta_\tau(u)} \right) \mathcal{T}(u, y) e^{\pi\gamma P y} \mathcal{V}_{1, \varepsilon}(u, y).$$

As y goes to 0, the above quantity is equivalent to

$$\frac{\gamma\chi}{2} \left(\frac{\Theta'_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta'_\tau(u)}{\Theta_\tau(u)} \right) \mathcal{T}(u, y) e^{\pi\gamma Py} \mathcal{V}_{1,\varepsilon}(u, y) \sim c \left(\frac{\Theta''_\tau(u)}{\Theta_\tau(u)} - \left(\frac{\Theta'_\tau(u)}{\Theta_\tau(u)} \right)^2 \right) \mathcal{V}_{1,\varepsilon}(u, y) y^{1-\frac{\alpha\gamma}{2}}$$

for some constant $c \in \mathbb{C}$ independent of u, τ and ε . Since $s-1 < 0$, $\mathcal{V}_{1,\varepsilon}(u, y)$ is always bounded as y goes to 0, uniformly in ε . If $\alpha < \frac{2}{\gamma}$, then $1 - \frac{\alpha\gamma}{2} > 0$ and thus the above quantity trivially converges to 0. If $\alpha \in [\frac{2}{\gamma}, Q)$, $y^{1-\frac{\alpha\gamma}{2}}$ no longer converges to 0, but the product $\mathcal{V}_{1,\varepsilon}(u, y) y^{1-\frac{\alpha\gamma}{2}}$ does, where we first let $\varepsilon \rightarrow 0$. This is because as y goes to 0 the quantity $\mathcal{V}_1(u, y)$ converges to a negative moment of GMC containing an insertion $\alpha + \gamma > Q$, which therefore vanishes; this is similar to the proof of [RZ18, Lemma A.4]. By symmetry, the exact same argument applies for y limiting to 1. We conclude that the boundary terms of the integration by parts performed above equal to 0.

By Girsanov's theorem (Theorem B.2), we find that

$$\partial_y \mathcal{V}_{1,\varepsilon}(u, y) dy = -\frac{\gamma^2}{2} (s-1) \int_0^1 \mathcal{T}(u, z) \partial_y K_\varepsilon(y-z) e^{\pi\gamma Pz} \mathcal{V}_{2,\varepsilon}(u, y, z) dy dz.$$

We may now write

$$\left(\partial_{uu} - l_\chi(l_\chi + 1)\wp(u) + 2i\pi\chi^2\partial_\tau \right) \psi_\chi^\alpha(u, q) = \Xi_0 + \Xi_1 + \Xi_2,$$

where Ξ_k contains all terms with a k -fold integral. We first consider Ξ_2 ; notice that

$$\Xi_2 = \lim_{\varepsilon \rightarrow 0} s(s-1) \mathcal{W}(q) e^{\pi\chi Pu} \int_0^1 \int_0^1 \Delta_{2,\varepsilon}(y, z) \mathcal{T}(u, y) \mathcal{T}(u, z) e^{\pi\gamma Py + \pi\gamma Pz} \mathcal{V}_2(u, y, z) dy dz$$

for

$$\begin{aligned} \Delta_{2,\varepsilon}(y, z) &= \chi\gamma \partial_y K_\varepsilon(y-z) \frac{\partial_u \mathcal{T}(u, y)}{\mathcal{T}(u, y)} + \frac{\partial_u \mathcal{T}(u, y)}{\mathcal{T}(u, y)} \frac{\partial_u \mathcal{T}(u, z)}{\mathcal{T}(u, z)} + i\pi \frac{\gamma^2}{2} \chi^2 \left(\frac{i\pi}{6} - \frac{\partial_\tau \Theta_\tau(y-z)}{\Theta_\tau(y-z)} + \frac{1}{3} \frac{\partial_\tau \Theta'_\tau(0)}{\Theta'_\tau(0)} \right) \\ &= \chi^2 \frac{\gamma^2}{2} \left[\partial_y K_\varepsilon(y-z) \left(\frac{\Theta'_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta'_\tau(u)}{\Theta_\tau(u)} \right) + \frac{1}{2} \left(\frac{\Theta'_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta'_\tau(u)}{\Theta_\tau(u)} \right) \left(\frac{\Theta'_\tau(u+z)}{\Theta_\tau(u+z)} - \frac{\Theta'_\tau(u)}{\Theta_\tau(u)} \right) \right. \\ &\quad \left. - \frac{1}{4} \frac{\Theta''_\tau(y-z)}{\Theta_\tau(y-z)} - \frac{\pi^2}{6} + \frac{1}{12} \frac{\Theta'''_\tau(0)}{\Theta'_\tau(0)} \right], \end{aligned}$$

where we use (A.2). Notice now that

$$\begin{aligned} \Delta_{2,\varepsilon}(y, z) + \Delta_{2,\varepsilon}(z, y) &= \frac{\chi^2 \gamma^2}{2} \left[\left(\frac{\Theta'_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta'_\tau(u)}{\Theta_\tau(u)} \right) \left(\frac{\Theta'_\tau(u+z)}{\Theta_\tau(u+z)} - \frac{\Theta'_\tau(u)}{\Theta_\tau(u)} \right) \right. \\ &\quad \left. + \partial_y K_\varepsilon(y-z) \left(\frac{\Theta'_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta'_\tau(u+z)}{\Theta_\tau(u+z)} \right) - \frac{1}{2} \frac{\Theta''_\tau(y-z)}{\Theta_\tau(y-z)} - \frac{\pi^2}{3} + \frac{1}{6} \frac{\Theta'''_\tau(0)}{\Theta'_\tau(0)} \right]. \end{aligned}$$

As ε goes to 0, the term $\partial_y K_\varepsilon(y-z) \left(\frac{\Theta'_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta'_\tau(u+z)}{\Theta_\tau(u+z)} \right)$ converges to $\frac{\Theta'_\tau(y-z)}{\Theta_\tau(y-z)} \left(\frac{\Theta'_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta'_\tau(u+z)}{\Theta_\tau(u+z)} \right)$. Notice also this limit is bounded as y tends to z , meaning we may apply the dominated convergence theorem in equation (3.13) below and exchange the limit in ε and the integration over y and z . Furthermore, adding a multiple of the identity (A.3) for $(a, b) = (u+y, u+z)$ gives the simplification

$$\lim_{\varepsilon \rightarrow 0} (\Delta_{2,\varepsilon}(y, z) + \Delta_{2,\varepsilon}(z, y)) = \frac{\chi^2 \gamma^2}{2} \left[\tilde{\Delta}(u, y) + \tilde{\Delta}(u, z) \right]$$

for

$$\tilde{\Delta}(u, x) = \frac{1}{2} \frac{\Theta''_\tau(u+x)}{\Theta_\tau(u+x)} - \frac{\Theta'_\tau(u+x)}{\Theta_\tau(u+x)} \frac{\Theta'_\tau(u)}{\Theta_\tau(u)} + \frac{1}{2} \frac{\Theta'_\tau(u)^2}{\Theta_\tau(u)^2} - \frac{\pi^2}{6} - \frac{1}{6} \frac{\Theta'''_\tau(0)}{\Theta'_\tau(0)}.$$

Notice now that the expression $\mathcal{T}(u, y)\mathcal{T}(u, z)e^{\pi\gamma Py+\pi\gamma Pz}\mathcal{V}_2(u, y, z)dydz$ in the integrand of Ξ_2 is symmetric under interchange of y and z , meaning that

(3.13)

$$\begin{aligned}\Xi_2 &= \lim_{\varepsilon \rightarrow 0} \frac{1}{2} s(s-1) \mathcal{W}(q) e^{\pi\chi Pu} \int_0^1 \int_0^1 \left(\Delta_{2,\varepsilon}(y, z) + \Delta_{2,\varepsilon}(z, y) \right) \mathcal{T}(u, y) \mathcal{T}(u, z) e^{\pi\gamma Py+\pi\gamma Pz} \mathcal{V}_2(u, y, z) dy dz \\ &= \frac{1}{2} s(s-1) \mathcal{W}(q) e^{\pi\chi Pu} \int_0^1 \int_0^1 \frac{\chi^2 \gamma^2}{2} \left(\tilde{\Delta}(u, y) + \tilde{\Delta}(u, z) \right) \mathcal{T}(u, y) \mathcal{T}(u, z) e^{\pi\gamma Py+\pi\gamma Pz} \mathcal{V}_2(u, y, z) dy dz \\ &= s(s-1) \mathcal{W}(q) e^{\pi\chi Pu} \int_0^1 \frac{\chi^2 \gamma^2}{2} \tilde{\Delta}(u, y) \mathcal{T}(u, y) e^{\pi\gamma Py} \mathcal{V}_1(u, y) dy.\end{aligned}$$

We now notice that

$$\Xi_1 + \Xi_2 = s \mathcal{W}(q) e^{\pi\chi Pu} \int_0^1 \Delta_1(u, y) \mathcal{T}(u, y) e^{\pi\gamma Py} \mathcal{V}_1(u, y) dy$$

for

$$\Delta_1(u, y) = -\frac{2\chi}{\gamma} \frac{\partial_{uy} \mathcal{T}(u, y)}{\mathcal{T}(u, y)} + \frac{\partial_{uu} \mathcal{T}(u, y)}{\mathcal{T}(u, y)} + 2i\pi\chi^2 \frac{\partial_\tau \mathcal{T}(u, y)}{\mathcal{T}(u, y)} + (s-1) \frac{\chi^2 \gamma^2}{2} \tilde{\Delta}(u, y).$$

We compute

$$\begin{aligned}\frac{\partial_{uu} \mathcal{T}(u, y)}{\mathcal{T}(u, y)} &= \frac{\gamma\chi}{2} \left(\frac{\Theta''_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta''_\tau(u)}{\Theta_\tau(u)} - \left(\frac{\Theta'_\tau(u+y)}{\Theta_\tau(u+y)} \right)^2 + \left(\frac{\Theta'_\tau(u)}{\Theta_\tau(u)} \right)^2 \right) + \frac{\gamma^2 \chi^2}{4} \left(\frac{\Theta'_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta'_\tau(u)}{\Theta_\tau(u)} \right)^2 \\ \frac{\partial_{uy} \mathcal{T}(u, y)}{\mathcal{T}(u, y)} &= \frac{\gamma\chi}{2} \left(\frac{\Theta''_\tau(u+y)}{\Theta_\tau(u+y)} - \left(\frac{\Theta'_\tau(u+y)}{\Theta_\tau(u+y)} \right)^2 \right) + \frac{\gamma\chi}{2} \left(\frac{\Theta'_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta'_\tau(u)}{\Theta_\tau(u)} \right) \left(\frac{\gamma\chi}{2} \frac{\Theta'_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\alpha\gamma}{2} \frac{\Theta'_\tau(y)}{\Theta_\tau(y)} \right) \\ \frac{\partial_\tau \mathcal{T}(u, y)}{\mathcal{T}(u, y)} &= \frac{1}{4\pi i} \left(-\frac{\alpha\gamma}{2} \frac{\Theta''_\tau(y)}{\Theta_\tau(y)} + \frac{\gamma\chi}{2} \frac{\Theta''_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\gamma\chi}{2} \frac{\Theta''_\tau(u)}{\Theta_\tau(u)} \right).\end{aligned}$$

The total prefactor of $\frac{\Theta'_\tau(u+y)^2}{\Theta_\tau(u+y)^2}$ in $\Delta_1(u, y)$ is therefore

$$-\frac{\gamma}{2}\chi + \left(1 + \frac{\gamma^2}{4}\right)\chi^2 - \frac{\gamma}{2}\chi^3 = -\frac{\gamma}{2}\chi\left(\chi - \frac{\gamma}{2}\right)\left(\chi - \frac{2}{\gamma}\right) = 0.$$

Similarly, the total prefactor of $\frac{\Theta'_\tau(u)^2}{\Theta_\tau(u)^2}$ in $\Delta_1(u, y)$ is $\frac{\gamma}{2}\chi - \frac{\alpha\gamma}{4}\chi^2 + \frac{\gamma}{4}\chi^3$. We may therefore write

$$\Delta_1(u, y) = \frac{\gamma}{2} \left(\chi - \frac{\alpha}{2}\chi^2 + \frac{1}{2}\chi^3 \right) \frac{\Theta'_\tau(u)^2}{\Theta_\tau(u)^2} + \chi \Delta_1^1(u, y) + \chi^2 \Delta_1^2(u, y) + \chi^3 \Delta_1^3(u, y)$$

for

$$\begin{aligned}\Delta_1^1(u, y) &= \frac{\gamma}{2} \left(\frac{\Theta''_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta''_\tau(u)}{\Theta_\tau(u)} \right) \\ \Delta_1^2(u, y) &= -(1 + \frac{\gamma^2}{4} + \frac{\alpha\gamma}{4}) \frac{\Theta''_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\alpha\gamma}{2} \frac{\Theta'_\tau(u)}{\Theta_\tau(u)} \frac{\Theta'_\tau(y)}{\Theta_\tau(y)} + \frac{\alpha\gamma}{2} \frac{\Theta'_\tau(y)}{\Theta_\tau(y)} \frac{\Theta'_\tau(u+y)}{\Theta_\tau(u+y)} + \frac{\alpha\gamma}{2} \frac{\Theta'_\tau(u+y)}{\Theta_\tau(u+y)} \frac{\Theta'_\tau(u)}{\Theta_\tau(u)} \\ &\quad - \frac{\alpha\gamma}{4} \frac{\Theta''_\tau(y)}{\Theta_\tau(y)} + \frac{\alpha\gamma\pi^2}{12} + \frac{\alpha\gamma}{12} \frac{\Theta'''_\tau(0)}{\Theta'_\tau(0)} + \frac{\pi^2\gamma^2}{12} + \frac{\gamma^2}{12} \frac{\Theta'''_\tau(0)}{\Theta'_\tau(0)} \\ &= -(1 + \frac{\gamma^2}{4}) \frac{\Theta''_\tau(u+y)}{\Theta_\tau(u+y)} + \frac{\alpha\gamma}{4} \frac{\Theta''_\tau(u)}{\Theta_\tau(u)} + \frac{\alpha\gamma\pi^2}{12} - \frac{\alpha\gamma}{6} \frac{\Theta'''_\tau(0)}{\Theta'_\tau(0)} + \frac{\pi^2\gamma^2}{12} + \frac{\gamma^2}{12} \frac{\Theta'''_\tau(0)}{\Theta'_\tau(0)} \\ \Delta_1^3(u, y) &= \frac{\gamma}{2} \frac{\Theta''_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\gamma}{4} \frac{\Theta''_\tau(u)}{\Theta_\tau(u)} - \frac{\pi^2\gamma}{12} - \frac{\gamma}{12} \frac{\Theta'''_\tau(0)}{\Theta'_\tau(0)},\end{aligned}$$

where we apply (A.3) for $(a, b) = (u + y, y)$. Adding $0 = (-\frac{\gamma}{2}\chi + (1 + \frac{\gamma^2}{4})\chi^2 - \frac{\gamma}{2}\chi^3) \frac{\Theta''(u+y)}{\Theta_\tau(u+y)}$, we obtain

$$\begin{aligned} \Delta_1(u, y) &= \left(\frac{\chi\gamma}{2} - \frac{\alpha\gamma}{4}\chi^2 + \frac{\gamma}{4}\chi^3\right) \frac{\Theta'_\tau(u)^2}{\Theta_\tau(u)^2} - \left(\frac{\chi\gamma}{2} - \frac{\alpha\gamma}{4}\chi^2 + \frac{\chi^3\gamma}{4}\right) \frac{\Theta''_\tau(u)}{\Theta_\tau(u)} \\ &\quad + \left(-\frac{\chi^2\alpha\gamma}{6} - \frac{\chi^3\gamma}{12} + \frac{\chi^2\gamma^2}{12}\right) \frac{\Theta'''_\tau(0)}{\Theta'_\tau(0)} + \left(\frac{\pi^2\alpha\gamma\chi^2}{12} - \frac{\pi^2\gamma\chi^3}{12} + \frac{\pi^2\chi^2\gamma^2}{12}\right). \end{aligned}$$

Finally, to conclude the proof, we compute that

$$\begin{aligned} \frac{\Xi_0 + \Xi_1 + \Xi_2}{\psi_\chi^\alpha(u, q)} &= \chi^2 P^2 \pi^2 - \left(\pi^2 \chi^2 P^2 + \frac{\pi^2 \chi \gamma l_\chi}{6} - \frac{\pi^2 l_\chi^2}{3}\right) + \left(-\frac{l_\chi^2}{3} + \frac{1}{6} l_\chi \chi^2 + \frac{\chi^2}{3} s\right) \frac{\Theta'''_\tau(0)}{\Theta'_\tau(0)} - l_\chi (l_\chi + 1) \wp(u) \\ &\quad + s \frac{\gamma}{2} \left(\chi - \frac{\alpha}{2} \chi^2 + \frac{1}{2} \chi^3\right) \frac{\Theta'_\tau(u)^2}{\Theta_\tau(u)^2} - s \left(\frac{\chi\gamma}{2} - \frac{\alpha\gamma}{4} \chi^2 + \frac{\chi^3\gamma}{4}\right) \frac{\Theta''_\tau(u)}{\Theta_\tau(u)} \\ &\quad + s \left(-\frac{\chi^2\alpha\gamma}{6} - \frac{\chi^3\gamma}{12} + \frac{\chi^2\gamma^2}{12}\right) \frac{\Theta'''_\tau(0)}{\Theta'_\tau(0)} + s \left(\frac{\pi^2\alpha\gamma\chi^2}{12} - \frac{\pi^2\gamma\chi^3}{12} + \frac{\pi^2\chi^2\gamma^2}{12}\right) \\ &= -\frac{l_\chi}{3} \left(\chi - \frac{\gamma}{2}\right) \left(\chi - \frac{2}{\gamma}\right) \frac{\Theta'''_\tau(0)}{\Theta'_\tau(0)} + l_\chi (l_\chi + 1) \left(\frac{\Theta'_\tau(u)^2}{\Theta_\tau(u)^2} - \frac{\Theta''_\tau(u)}{\Theta_\tau(u)} + \frac{1}{3} \frac{\Theta'''_\tau(0)}{\Theta'_\tau(0)}\right) - l_\chi (l_\chi + 1) \wp(u) \\ &= 0, \end{aligned}$$

where we use (A.8) in the last step. \square

3.3. Analyticity in α . We conclude this section by constructing an analytic extension of the deformed block in α .

Lemma 3.6. *(Analyticity in α)* Given $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$, there exists an open set in \mathbb{C}^3 containing $\{(\alpha, u, q) : \alpha \in (-\frac{4}{\gamma} + \chi, Q), u \in \mathfrak{B}, q = 0\}$ where $(\alpha, u, q) \mapsto \psi_\chi^\alpha(u, q)$ has an analytic continuation.

Proof. Following the notation of the proof of Proposition 3.2, analytically extending $\psi_\chi^\alpha(u, q)$ reduces to analytically extending $\mathbb{E} \left[(f_\nu(u))^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right]$. Thanks to Proposition 3.2, we have the desired analyticity with respect to u and q . For the analyticity in α , we repeat the argument given in Lemma 2.7. We again map the unit disk to the upper-half plane and the field Y_∞ to $X_{\mathbb{H}}$ using the same change of variable. By Girsanov's theorem (Theorem B.2) the analyticity in α of $\mathbb{E} \left[(f_\nu(u))^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right]$ reduces to the analyticity of

$$\begin{aligned} &\mathbb{E} \left[e^{\frac{\alpha}{2} F_\tau(0)} \mathcal{Q}(q) e^{\mathcal{X}(u, q) - \frac{1}{2} \mathbb{E}[\mathcal{X}(u, q)^2]} \left(\int_0^1 e^{\frac{\gamma}{2} Y_\infty(x)} (2 \sin(\pi x))^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x} dx \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] \\ &= \mathbb{E} \left[e^{\frac{\alpha}{2} F_\tau(0)} \mathcal{Q}(q) e^{\mathcal{X}(u, q) - \frac{1}{2} \mathbb{E}[\mathcal{X}(u, q)^2]} \left(\int_{\mathbb{R}} e^{\frac{\gamma}{2} X_{\mathbb{H}}(y)} |y|^{-\frac{\alpha\gamma}{2}} g_1(y) dy \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right]. \end{aligned}$$

In the last equality we have used the change of variable and g_1 is defined through the relation

$$(2 \sin(\pi x))^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x} dx = |y|^{-\frac{\alpha\gamma}{2}} g_1(y) dy.$$

The analyticity in α is now again a straightforward adaptation of the proof of [RZ20, Lemma 5.6]. \square

4. FROM BPZ TO GAUSS HYPERGEOMETRIC EQUATIONS

In this section, we apply separation of variables to the BPZ equation (3.5) for $\psi_\chi^\alpha(u, q)$ to convert it to the system (4.5)–(4.6) of inhomogeneous hypergeometric equations on normalized q -series coefficients of $\psi_\chi^\alpha(u, q)$. We then use this fact to prove that the resulting coefficients are analytic in α on a certain domain.

4.1. Separation of variables for the BPZ equation. Throughout this section we assume that $\tau \in i\mathbb{R}$ so that $q \in (0, 1)$. Recall the definition of l_χ from (3.1). Define $\psi_{\chi, n}^\alpha(u)$ as the coefficients of the series expansion

$$(4.1) \quad \psi_\chi^\alpha(u, q) = q^{\frac{P^2}{2} + \frac{1}{6\chi^2} l_\chi (l_\chi + 1)} \sum_{n=0}^{\infty} \psi_{\chi, n}^\alpha(u) q^n.$$

we introduce the normalization

$$(4.2) \quad \phi_\chi^\alpha(u, q) = \sin(\pi u)^{l_\chi} \psi_\chi^\alpha(u, q) \quad \text{and} \quad \phi_{\chi, n}^\alpha(u) = \sin(\pi u)^{l_\chi} \psi_{\chi, n}^\alpha(u) \text{ for } n \in \mathbb{N}_0$$

to remove the singularities at $u \in \{0, 1\}$ occurring due to the $\Theta(u)^{-l_\chi}$ factor in $\psi_\chi^\alpha(u, q)$. We introduce the hypergeometric differential operator

$$(4.3) \quad \mathcal{H}_\chi := w(1-w)\partial_w + (1/2 - l_\chi - (1-l_\chi)w)\partial_w.$$

In this section, we show that $\{\phi_{\chi, n}^\alpha(u)\}_{n \in \mathbb{N}_0}$ satisfy a system of hypergeometric ordinary differential equations governed by \mathcal{H}_χ after the change of variable $w = \sin^2(\pi u)$.

Let \mathbb{H}_+ and \mathbb{H}_- be the upper and lower half plane, respectively. We first clarify the nature of the change of variable by noting the following basic fact.

Lemma 4.1. *The map $u \mapsto w = \sin^2(\pi u)$ is a conformal (i.e. bi-holomorphic) map from $\{u : \operatorname{Re} u \in (0, \frac{1}{2}), \operatorname{Im} u > 0\}$ to \mathbb{H}_+ , and from $\{u : \operatorname{Re} u \in (\frac{1}{2}, 1), \operatorname{Im} u > 0\}$ to \mathbb{H}_- . In both cases, $\{u : \operatorname{Re} u = 1/2, \operatorname{Im} u > 0\}$ is mapped to $(1, \infty)$.*

For $i = 1, 2$, define the domains

$$(4.4) \quad D_i^u := \{u : \operatorname{Re} u \in (\frac{i-1}{2}, \frac{i}{2}), \operatorname{Im} u \in (0, \operatorname{Im} \tau)\} \quad \text{and} \quad D_i^w := \{w = \sin^2(\pi u) : u \in D_i^u\}.$$

Moreover, define the function $\phi_{\chi, n, i}^\alpha(w)$ on D_i^w by

$$\phi_{\chi, n, i}^\alpha(w) := \phi_{\chi, n}^\alpha(u) \text{ for } w = \sin^2(\pi u), \quad \text{where } u \in D_i^u.$$

Recalling the definition of $\wp_n(u)$ in (A.7), define $\tilde{\wp}_{n, i}(w)$ as a function on D_i^w by $\tilde{\wp}_{n, i}(\sin^2(\pi u)) = \wp_n(u)$ for $w = \sin^2(\pi u)$ with $u \in D_i^u$. By (A.6), the resulting function $\tilde{\wp}_{n, i}(w)$ is a polynomial in w for $n \geq 1$.

We now consider the set of equations

$$(4.5) \quad \left(\mathcal{H}_\chi - \left(\frac{1}{4}l_\chi^2 + \frac{1}{4}\chi^2 P^2 \right) \right) \phi_0(w) = 0,$$

$$(4.6) \quad \left(\mathcal{H}_\chi - \left(\frac{1}{4}l_\chi^2 + \frac{1}{4}\chi^2(P^2 + 2n) \right) \right) \phi_n(w) = \frac{l_\chi(l_\chi + 1)}{4\pi^2} \sum_{l=1}^n \tilde{\wp}_{l, i}(w) \phi_{n-l}(w), \quad \text{for } n \geq 1.$$

on sequences of functions $\{\phi_n(w)\}_{n \geq 0}$.

Proposition 4.2. *For $i = 1, 2$, equations (4.5)-(4.6) hold for $\{\phi_{\chi, n, i}^\alpha(w)\}_{n \geq 0}$ on D_i^w .*

Proof. The BPZ equation (3.5) implies that

$$\sum_{n \geq 0} \left[\partial_{uu} \psi_{\chi, n}^\alpha(u) - l_\chi(l_\chi + 1) \sum_{l=0}^n \wp_l(u) \psi_{\chi, n-l}^\alpha(u) - 2\pi^2 \chi^2 n \psi_{\chi, n}^\alpha(u) - 2\pi^2 \chi^2 \left(\frac{P^2}{2} + \frac{1}{6\chi^2} l_\chi(l_\chi + 1) \right) \psi_{\chi, n}^\alpha(u) \right] q^n = 0,$$

so we find for each n that

$$\left(\partial_{uu} - l_\chi(l_\chi + 1) \frac{\pi^2}{\sin^2(\pi u)} - \pi^2 \chi^2 (P^2 + 2n) \right) \psi_{\chi, n}^\alpha(u) = l_\chi(l_\chi + 1) \sum_{l=1}^n \wp_l(u) \psi_{\chi, n-l}^\alpha(u).$$

In terms of $\phi_{\chi, n}^\alpha(u)$, this yields

$$\begin{aligned} & \left(l_\chi(l_\chi + 1) \pi^2 \cos^2(\pi u) \sin(\pi u)^{-l_\chi - 2} + l_\chi \pi^2 \sin(\pi u)^{-l_\chi} - 2l_\chi \pi \cos(\pi u) \sin(\pi u)^{-l_\chi - 1} \partial_u \right. \\ & \quad \left. + \sin(\pi u)^{-l_\chi} \left(\partial_{uu} - l_\chi(l_\chi + 1) \frac{\pi^2}{\sin^2(\pi u)} - \pi^2 \chi^2 (P^2 + 2n) \right) \right) \phi_{\chi, n}^\alpha(u) \\ & \quad = l_\chi(l_\chi + 1) \sum_{l=1}^n \wp_l(u) \sin(\pi u)^{-l_\chi} \phi_{\chi, n-l}^\alpha(u). \end{aligned}$$

Multiplying by $\sin(\pi u)^{l_\chi}$ yields

$$(4.7) \quad \left(\partial_{uu} - 2\pi l_\chi \cot(\pi u) \partial_u - \pi^2 l_\chi^2 - \pi^2 \chi^2 (P^2 + 2n) \right) \phi_{\chi, n}^\alpha(u) = l_\chi(l_\chi + 1) \sum_{l=1}^n \wp_l(u) \phi_{\chi, n-l}^\alpha(u).$$

Notice that

$$2\pi\sqrt{w(1-w)}\partial_w = \partial_u,$$

hence we obtain

$$\begin{aligned} \partial_{uu} - 2\pi l_\chi \cot(\pi u)\partial_u - \pi^2 l_\chi^2 - \pi^2 \chi^2 (P^2 + 2n) \\ = 4\pi^2 w(1-w)\partial_{ww} + 2\pi^2(1-2w)\partial_w - 4\pi^2 l_\chi(1-w)\partial_w - \pi^2 l_\chi^2 - \pi^2 \chi^2 (P^2 + 2n) \\ = 4\pi^2 \left(w(1-w)\partial_{ww} + \left(\frac{1}{2} - l_\chi - (1-l_\chi)w \right) \partial_w - \left(\frac{l_\chi^2}{4} + \frac{\chi^2}{4} (P^2 + 2n) \right) \right). \end{aligned}$$

This implies that

$$\begin{aligned} \left(w(1-w)\partial_{ww} + (1/2 - l_\chi - (1-l_\chi)w)\partial_w - \left(\frac{l_\chi^2}{4} + \frac{\chi^2}{4} (P^2 + 2n) \right) \right) \phi_{\chi,n,1}^\alpha(w) \\ = \frac{l_\chi(l_\chi + 1)}{4\pi^2} \sum_{l=1}^n \tilde{\varphi}_{l,i}(u) \phi_{\chi,n-l,1}^\alpha(w), \end{aligned}$$

as desired. A similar argument shows the desired equation for $i = 2$. \square

We notice (4.5) and (4.6) are inhomogeneous Gauss hypergeometric equations with parameters $(A_{\chi,n}, B_{\chi,n}, C_\chi)$ defined by

$$(4.8) \quad C_\chi = \frac{1}{2} - l_\chi \quad A_{\chi,n} = -\frac{l_\chi}{2} + \mathbf{i} \frac{\chi}{2} \sqrt{P^2 + 2n} \quad B_{\chi,n} = -\frac{l_\chi}{2} - \mathbf{i} \frac{\chi}{2} \sqrt{P^2 + 2n}.$$

We summarize some well-known facts on the Gauss hypergeometric equation in Appendix C for the reader's convenience. We now use Proposition 4.2 to further extend the domain of definition for $\phi_{\chi,n,i}^\alpha(w)$.

Definition 4.3. Recalling that \mathbb{D} is the unit disk, we say that a function $f(w)$ satisfies Property (R) if $f(w)$ is analytic on \mathbb{D} and continuous on $\partial\mathbb{D}$.

Corollary 4.4. *Suppose C_χ is not an integer. For $i \in \{1, 2\}$, we may uniquely write*

$$\phi_{\chi,n,i}^\alpha(w) = \phi_{\chi,n,i}^{\alpha,1}(w) + w^{1-C_\chi} \phi_{\chi,n,i}^{\alpha,2}(w) \quad \text{on } D_i^w \cap \mathbb{D}$$

where w^{1-C_χ} has branch cut $(-\infty, 0)$ and $\phi_{\chi,n,i}^{\alpha,1}(w)$ and $w^{1-C_\chi} \phi_{\chi,n,i}^{\alpha,2}(w)$ are solutions to equations (4.5)–(4.6). Moreover, both $\phi_{\chi,n,i}^{\alpha,1}(w)$ and $\phi_{\chi,n,i}^{\alpha,2}(w)$ can be extended to functions on $\overline{\mathbb{D}}$ satisfying Property (R).

Proof. This follows from Corollary C.2 and an induction on n based on Proposition 4.2 using the fact that $\tilde{\varphi}_{l,i}(w)$ are polynomials in w . \square

By Corollary 4.4, we can extend $\phi_{\chi,n,1}^\alpha$ to a continuous function on $\overline{\mathbb{D} \cap \mathbb{H}_+}$ which is analytic on $\mathbb{D} \cap \mathbb{H}_+$. The same holds for $\phi_{\chi,n,2}^\alpha$ with \mathbb{H}_+ replaced by \mathbb{H}_- . In what follows, we will freely use the same notation to denote these extensions when applicable.

Lemma 4.5. *Under the extension of $\phi_{\chi,n,i}^\alpha$ to $\overline{\mathbb{D} \cap \mathbb{H}_+}$ or $\overline{\mathbb{D} \cap \mathbb{H}_-}$, for each $n \in \mathbb{N}_0$ we have*

$$(4.9) \quad \phi_{\chi,n,1}^\alpha(1) = \phi_{\chi,n,2}^\alpha(1) \quad \text{and} \quad \phi_{\chi,n,2}^\alpha(w) = e^{\pi\chi P - \pi\mathbf{i}l_\chi} \phi_{\chi,n,1}^\alpha(w) \quad \text{for } w \in [-1, 0] \cap \overline{D_1^w}.$$

Note that $[-1, 0] \cap \overline{D_1^w} = [-1, 0] \cap \overline{D_2^w}$.

Proof. After the continuous extension we must have $\phi_{\chi,n,1}^\alpha(w_t) = \phi_{\chi,n}^\alpha(\frac{1}{2} + \mathbf{i}t) = \phi_{\chi,n,2}^\alpha(w_t)$ for $w_t = \sin^2(\pi(\frac{1}{2} + \mathbf{i}t))$ with $t \in (0, \text{Im } \tau)$. Sending $t \rightarrow 0$ we obtain that $\phi_{\chi,n,1}^\alpha(1) = \phi_{\chi,n,2}^\alpha(1)$. Similarly, $\phi_{\chi,n,1}^\alpha(\sin^2(\pi\mathbf{i}t)) = \phi_{\chi,n}^\alpha(\mathbf{i}t)$ and $\phi_{\chi,n,2}^\alpha(\sin^2(\pi(1 + \mathbf{i}t))) = \phi_{\chi,n}^\alpha(1 + \mathbf{i}t)$ with $t \in (0, \text{Im } \tau)$. By Lemma 3.1 and the $e^{\pi\chi P u}$ factor in $\psi(u, \alpha)$, we have $\phi_{\chi,n}^\alpha(\mathbf{i}t + 1) = e^{\pi\chi P - \pi\mathbf{i}l_\chi} \phi_{\chi,n}^\alpha(\mathbf{i}t)$. Sending $t \rightarrow 0$, we have $\phi_{\chi,n,2}^\alpha(0) = e^{\pi\chi P - \pi\mathbf{i}l_\chi} \phi_{\chi,n,1}^\alpha(0)$. \square

Lemma 4.6. *We have that*

$$(4.10) \quad \phi_{\chi,n,2}^{\alpha,1}(w) = e^{\pi\chi P - \mathbf{i}\pi l_\chi} \phi_{\chi,n,1}^{\alpha,1}(w) \quad \text{and} \quad \phi_{\chi,n,2}^{\alpha,2}(w) = -e^{\pi\chi P + \mathbf{i}\pi l_\chi} \phi_{\chi,n,1}^{\alpha,2}(w).$$

Proof. By Lemma 4.5, we have $\phi_{\chi,n,2}^{\alpha,1}(0) = e^{\pi\chi P - i\pi l_\chi} \phi_{\chi,n,1}^{\alpha,1}(0)$. Set $\phi_n = \phi_{\chi,n,2}^{\alpha,1} - e^{\pi\chi P - i\pi l_\chi} \phi_{\chi,n,1}^{\alpha,1}$. Then ϕ_0 is a solution to (4.5) satisfying Property (R) with $\phi_0(0) = 0$. This yields that $\phi_0 \equiv 0$. Since ϕ_1 is a solution to (4.6) with $n = 1$, we similarly get $\phi_1 \equiv 0$. Continuing via induction on n , we get $\phi_n \equiv 0$, hence $\phi_{\chi,n,2}^{\alpha,1}(w) = e^{\pi\chi P - i\pi l_\chi} \phi_{\chi,n,1}^{\alpha,1}(w)$ for all n .

Under the extension of $\phi_{\chi,n,i}^\alpha(w)$ in Lemma 4.5, for $w \in [-c, 0]$ with small enough $c > 0$, we have

$$\phi_{\chi,n,2}^\alpha(w) - \phi_{\chi,n,2}^{\alpha,1}(w) = e^{\pi\chi P - i\pi l_\chi} (\phi_{\chi,n,1}^\alpha(w) - \phi_{\chi,n,1}^{\alpha,1}(w)).$$

On the other hand, since $D_1^w \subset \mathbb{H}_+$, $D_2^w \subset \mathbb{H}_-$, and w^{1-C_χ} has branch cut at $(-\infty, 0)$, on $(-c, 0)$ we have

$$\begin{aligned} \phi_{\chi,n,1}^\alpha(w) &= \phi_{\chi,n,1}^{\alpha,1}(w) + e^{\pi(1-C_\chi)\mathbf{i}} |w|^{1-C_\chi} \phi_{\chi,n,1}^{\alpha,2}(w) \\ \phi_{\chi,n,2}^\alpha(w) &= \phi_{\chi,n,2}^{\alpha,1}(w) + e^{-\pi(1-C_\chi)\mathbf{i}} |w|^{1-C_\chi} \phi_{\chi,n,2}^{\alpha,2}(w). \end{aligned}$$

Putting these together, we have $\phi_{\chi,n,2}^{\alpha,2}(w) = e^{\pi\chi P - i\pi l_\chi} e^{2(1-C_\chi)\pi\mathbf{i}} \phi_{\chi,n,1}^{\alpha,2}(w) = -e^{\pi\chi P + i\pi l_\chi} \phi_{\chi,n,1}^{\alpha,2}(w)$ on $(-c, 0)$. Therefore, $\phi_{\chi,n,2}^{\alpha,2}(w) = -e^{\pi\chi P + i\pi l_\chi} \phi_{\chi,n,1}^{\alpha,2}(w)$ everywhere by their analyticity. \square

4.2. Construction of a particular solution. The equations (4.5) and (4.6) have a 2-dimensional affine space of solutions given by adding to any particular solution the span of the Gauss hypergeometric functions $v_{1,\chi,n}(w)$ and $w^{1-C_\chi} v_{2,\chi,n}(w)$ defined by

$$\begin{aligned} v_{1,\chi,n}(w) &:= {}_2F_1(A_{\chi,n}, B_{\chi,n}, C_\chi, w) \\ v_{2,\chi,n}(w) &:= {}_2F_1(1 + A_{\chi,n} - C_\chi, 1 + B_{\chi,n} - C_\chi, 2 - C_\chi, w). \end{aligned}$$

In terms of C_χ , $A_{\chi,n}$ and $B_{\chi,n}$ from (4.8), define

$$(4.11) \quad \Gamma_{n,1} := \frac{\Gamma(C_\chi)\Gamma(C_\chi - A_{\chi,n} - B_{\chi,n})}{\Gamma(C_\chi - A_{\chi,n})\Gamma(C_\chi - B_{\chi,n})} \quad \text{and} \quad \Gamma_{n,2} := \frac{\Gamma(2 - C_\chi)\Gamma(C_\chi - A_{\chi,n} - B_{\chi,n})}{\Gamma(1 - A_{\chi,n})\Gamma(1 - B_{\chi,n})}.$$

We now construct a particular solution to (4.6) which will be used in the proof of Theorem 6.1. Because the equation (4.6) for each n depends on $\phi_{\chi,m,i}^\alpha(w)$ for $m < n$, our construction is inductive in nature.

Proposition 4.7. *The equation (4.6) has a particular solution $G_{\chi,n,i}^\alpha(w)$ of the form*

$$G_{\chi,n,i}^\alpha(w) = G_{\chi,n,i}^{\alpha,1}(w) + w^{1-C_\chi} G_{\chi,n,i}^{\alpha,2}(w)$$

for functions $G_{\chi,n,i}^{\alpha,j}(w)$ satisfying Property (R) for which

$$G_{\chi,n,1}^{\alpha,1}(1) = G_{\chi,n,2}^{\alpha,1}(1) = 0 \quad G_{\chi,n,2}^{\alpha,1}(0) = e^{\pi\chi P - i\pi l_\chi} G_{\chi,n,1}^{\alpha,1}(0) \quad G_{\chi,n,2}^{\alpha,2}(0) = -e^{\pi\chi P + i\pi l_\chi} G_{\chi,n,1}^{\alpha,2}(0)$$

and

$$G_{\chi,n,1}^{\alpha,1}(0) = V_{\chi,n}^{\alpha,1} \quad \text{and} \quad G_{\chi,n,1}^{\alpha,2}(0) = V_{\chi,n}^{\alpha,2},$$

where $V_{\chi,n}^{\alpha,j}$ are defined by

$$(4.12) \quad V_{\chi,n}^{\alpha,1} = \frac{1}{1 - C_\chi} \int_0^1 \frac{v_{2,\chi,n}(t) g_{\chi,n,1}^{\alpha,1}(t)}{(1-t)^{C_\chi - A_{\chi,n} - B_{\chi,n}}} dt - \frac{1}{1 - C_\chi} \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \int_0^1 \frac{v_{1,\chi,n}(t) g_{\chi,n,1}^{\alpha,1}(t)}{t^{1-C_\chi} (1-t)^{C_\chi - A_{\chi,n} - B_{\chi,n}}} dt$$

$$(4.13) \quad V_{\chi,n}^{\alpha,2} = \frac{1}{1 - C_\chi} \frac{\Gamma_{n,1}}{\Gamma_{n,2}} \int_0^1 \frac{v_{2,\chi,n}(t) t^{1-C_\chi} g_{\chi,n,1}^{\alpha,2}(t)}{(1-t)^{C_\chi - A_{\chi,n} - B_{\chi,n}}} dt - \frac{1}{1 - C_\chi} \int_0^1 \frac{v_{1,\chi,n}(t) g_{\chi,n,1}^{\alpha,2}(t)}{(1-t)^{C_\chi - A_{\chi,n} - B_{\chi,n}}} dt$$

with

$$g_{\chi,n,i}^{\alpha,j}(w) = \frac{l_\chi(l_\chi + 1)}{4\pi^2} \sum_{l=1}^n \tilde{\wp}_{l,i}(w) \phi_{\chi,n-l,i}^{\alpha,j}(w),$$

where we adopt the convention that the sum is empty for $n = 0$.

Proof. We proceed by induction on n . For $n = 0$, we take $G_{\chi,0,i}^\alpha(w) = 0$. For the inductive step, suppose the statement holds for all $m < n$. Notice that the inhomogeneous part of (4.6) is $g_{\chi,n,i}^{\alpha,1}(w) + w^{1-C_\chi} g_{\chi,n,i}^{\alpha,2}(w)$.

Further, $\tilde{\varphi}_{l,i}(w)$ is a polynomial in w for $l \geq 1$ by identity (A.6), so $g_{\chi,n,i}^{\alpha,j}(w)$ satisfies Property (R) by the inductive hypothesis. This means we may apply Lemma C.3 to the equations

$$\begin{aligned} & \left(w(1-w)\partial_{ww} + (C_\chi - (1 + A_{\chi,n} + B_{\chi,n})w)\partial_w - A_{\chi,n}B_{\chi,n} \right) G_{\chi,n,i}^{\alpha,1}(w) = g_{\chi,n,i}^{\alpha,1}(w) \\ & \left(w(1-w)\partial_{ww} + (C_\chi - (1 + A_{\chi,n} + B_{\chi,n})w)\partial_w - A_{\chi,n}B_{\chi,n} \right) w^{1-C_\chi} G_{\chi,n,i}^{\alpha,2}(w) = w^{1-C_\chi} g_{\chi,n,i}^{\alpha,2}(w), \end{aligned}$$

yielding particular solutions $G_{\chi,n,i}^{\alpha,1}(w)$ and $w^{1-C_\chi} G_{\chi,n,i}^{\alpha,2}(w)$ with $G_{\chi,n,i}^{\alpha,j}(w)$ satisfying Property (R), $G_{\chi,n,i}^{\alpha,j}(1) = 0$, and $G_{\chi,n,1}^{\alpha,j}(0) = V_{\chi,n}^{\alpha,j}$, where we make use of Gauss's identity (C.3) for the last equality. Furthermore, by the inductive hypothesis, we have $g_{\chi,n,2}^{\alpha,1}(w) = e^{\pi\chi P - i\pi l_\chi} g_{\chi,n,1}^{\alpha,1}(w)$ and $g_{\chi,n,2}^{\alpha,2}(w) = -e^{\pi\chi P + i\pi l_\chi} g_{\chi,n,1}^{\alpha,2}(w)$, which implies that $G_{\chi,n,2}^{\alpha,1}(0) = e^{\pi\chi P - i\pi l_\chi} G_{\chi,n,1}^{\alpha,1}(0)$ and $G_{\chi,n,2}^{\alpha,2}(0) = -e^{\pi\chi P + i\pi l_\chi} G_{\chi,n,1}^{\alpha,2}(0)$, completing the desired properties for $G_{\chi,n,i}^{\alpha,j}(w)$. \square

4.3. Analyticity of solutions in α . We now use this particular solution to show that $\phi_{\chi,n,i}^{\alpha,j}(w)$ are analytic in α on a specific domain. This will require the following analytic lemma.

Lemma 4.8. *Suppose that $f(\alpha, w)$ is continuous on $[0, 1]$ in w and analytic in α on an open set U . For $\operatorname{Re} a, \operatorname{Re} b > -1$ the functions*

$$\begin{aligned} g(\alpha) &:= \int_0^1 f(\alpha, w) w^a (1-w)^b dw \\ h(\alpha, w) &:= \int_0^w f(\alpha, t) t^a (1-t)^b dt \end{aligned}$$

are analytic in α in U . Further, $h(\alpha, w)$ is a continuous function of w on $[0, 1]$.

Proof. Let $\Delta \subset U$ be a cycle which is the boundary of a solid triangle. By compactness and continuity, we may find some C so that $|f(\alpha, w)| < C$ on $\Delta \times [0, 1]$, meaning that

$$\int_\Delta \int_0^1 |f(\alpha, w)| |w^a (1-w)^b| dw d\alpha \leq \int_\Delta \int_0^1 C w^{\operatorname{Re} a} (1-w)^{\operatorname{Re} b} dw d\alpha < \infty,$$

so we may apply Fubini's theorem to find

$$\int_\Delta g(\alpha) d\alpha = \int_0^1 \left[\int_\Delta f(\alpha, w) d\alpha \right] w^a (1-w)^b dw = 0,$$

which shows that $g(\alpha)$ is analytic on U by the (multivariate) Morera's theorem. The argument for $h(\alpha, w)$ is similar, where continuity in w on $[0, 1]$ follows by the fact that $\operatorname{Re} a, \operatorname{Re} b > -1$. \square

Lemma 4.9. *If C_χ is not an integer, for $i, j \in \{1, 2\}$ the quantities $V_{\chi,n}^{\alpha,j}$, $G_{\chi,n,i}^{\alpha,j}(w)$, and $\phi_{\chi,n,i}^{\alpha,j}(w)$ are analytic in α on an open complex neighborhood of $(-\frac{4}{\gamma} + \chi, Q)$ for $w \in \mathbb{D}$.*

Proof. We induct on n . Suppose the statement holds for all $m < n$. Because any solution to an inhomogenous hypergeometric equation is the sum of a particular solution and a solution to the homogeneous equation, by Proposition 4.2 we may write

$$(4.14) \quad \phi_{\chi,n,i}^\alpha(w) = G_{\chi,n,i}^\alpha(w) + X_{\chi,n,i}^1(\alpha) v_{1,\chi,n}(w) + X_{\chi,n,i}^2(\alpha) w^{1-C_\chi} v_{2,\chi,n}(w)$$

for $X_{\chi,n,i}^j(\alpha)$ independent of w , which implies that

$$(4.15) \quad \phi_{\chi,n,i}^{\alpha,j}(w) = G_{\chi,n,i}^{\alpha,j}(w) + X_{\chi,n,i}^j(\alpha) v_{j,\chi,n}(w).$$

For $V_{\chi,n}^{\alpha,j}$, notice that l_χ , $A_{\chi,n}$, $B_{\chi,n}$, C_χ , $v_{j,\chi,n}(w)$, $\Gamma_{n,1}$, and $\Gamma_{n,2}$ are analytic in α on a neighborhood of $(-\frac{4}{\gamma} + \chi, Q)$, so $V_{\chi,n}^{\alpha,j}$ is as well by the inductive hypothesis applied to $\phi_{\chi,n-l,i}^{\alpha,j}(w)$ and Lemma 4.8. For $G_{\chi,n,i}^\alpha(w)$, the conclusion follows from Lemma 4.8 and the explicit defining expression in Lemma C.3. Finally, for $\phi_{\chi,n,i}^{\alpha,j}(w)$, by Lemma 3.6, $\phi_{\chi,n,1}^\alpha(w)$ and $\phi_{\chi,n,2}^\alpha(w)$ are analytic in α on an open neighborhood of

$(-\frac{4}{\gamma} + \chi, Q)$ for $w \in \mathbb{D} \cap \mathbb{H}_+$ and $w \in \mathbb{D} \cap \mathbb{H}_-$, respectively. Choosing $w_1, w_2 \in \mathbb{D} \cap \mathbb{H}_+$ or $\mathbb{D} \cap \mathbb{H}_-$, we obtain the system of equations

$$\begin{aligned}\phi_{\chi,n,i}^\alpha(w_1) &= G_{\chi,n,i}^\alpha(w_1) + X_{\chi,n,i}^1(\alpha)v_{1,\chi,n}(w_1) + X_{\chi,n,i}^2(\alpha)w_1^{1-C_\chi}v_{2,\chi,n}(w_1) \\ \phi_{\chi,n,i}^\alpha(w_2) &= G_{\chi,n,i}^\alpha(w_2) + X_{\chi,n,i}^1(\alpha)v_{1,\chi,n}(w_2) + X_{\chi,n,i}^2(\alpha)w_2^{1-C_\chi}v_{2,\chi,n}(w_2),\end{aligned}$$

where for $k \in \{1, 2\}$, the expressions $\phi_{\chi,n,i}^\alpha(w_k)$, $G_{\chi,n,i}^\alpha(w_k)$, $w_k^{1-C_\chi}$, and $v_{j,\chi,n}(w_k)$ are analytic in α on a complex neighborhood of $(-\frac{4}{\gamma} + \chi, Q)$. Solving this system of linear equations yields expressions for $X_{\chi,n,i}^j(\alpha)$ which are meromorphic in α on this neighborhood. Finally, if $X_{\chi,n,i}^j(\alpha)$ had a pole at $\alpha = \alpha_0$, then by (4.14) we must have

$$\text{Res}_{\alpha=\alpha_0} [X_{\chi,n,i}^1(\alpha)v_{1,\chi,n}(w)] + \text{Res}_{\alpha=\alpha_0} [X_{\chi,n,i}^2(\alpha)w^{1-C_\chi}v_{2,\chi,n}(w)] = 0.$$

Taking w near 0 shows that this is impossible. We conclude that $X_{\chi,n,i}^j(\alpha)$ and hence $\phi_{\chi,n,i}^{\alpha,j}(w)$ is analytic in α in a complex neighborhood of $(-\frac{4}{\gamma} + \chi, Q)$. \square

5. OPERATOR PRODUCT EXPANSIONS FOR CONFORMAL BLOCKS

This section provides operator product expansions (OPEs) for the functions $\phi_\chi^\alpha(u, q)$ defined in (4.2). Mathematically, these OPEs characterize the behavior of $\phi_\chi^\alpha(u, q)$ as u tends to 0 in terms of the function $\mathcal{A}_{\gamma,P}^q(\alpha)$ from (2.9). Define the functions

$$(5.1) \quad W_\chi^-(\alpha, \gamma) = \pi^{l_\chi} (2\pi e^{i\pi})^{-\frac{1}{3}} \left(2 + \frac{2\gamma l_\chi}{\chi} + \frac{4l_\chi}{\chi\gamma} + \frac{6l_\chi^2}{\chi^2} \right)$$

$$(5.2) \quad W_\chi^+(\alpha, \gamma) = -e^{2i\pi l_\chi - 2i\pi\chi^2} (2\pi e^{i\pi})^{-\frac{1}{3}} \left(\frac{\gamma l_\chi}{\chi} + \frac{2l_\chi}{\chi^2} - 8l_\chi + \frac{6l_\chi^2}{\chi^2} \right) \pi^{-l_\chi - 1} \frac{1 - e^{2\pi\chi P - 2i\pi l_\chi}}{\chi(Q - \alpha)} \left(\frac{4}{\gamma^2} \right)^{\mathbf{1}_{\chi=\frac{2}{\gamma}}}$$

$$\frac{\Gamma(\frac{\alpha\chi}{2} - \frac{\chi^2}{2} + \frac{2\chi}{\gamma})\Gamma(1 - \alpha\chi)\Gamma(\alpha\chi - \chi^2)}{\Gamma(\frac{\alpha\chi}{2} - \frac{\chi^2}{2})\Gamma(1 - \frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}}.$$

We start with an easy result which corresponds to direct evaluation of $\phi_\chi^\alpha(u, q)$ at $u = 0$.

Lemma 5.1. *For $\alpha \in (-\frac{4}{\gamma} + \chi, Q)$, we have*

$$(5.3) \quad \phi_\chi^\alpha(0, q) = W_\chi^-(\alpha, \gamma) q^{\frac{P^2}{2} - \frac{1}{6} \frac{l_\chi(l_\chi+1)}{\chi^2} - \frac{1}{6} l_\chi - \frac{1}{6}} \Theta'(0)^{\frac{4}{3} \frac{l_\chi(l_\chi+1)}{\chi^2} + \frac{2}{3} l_\chi + \frac{2}{3}} \mathcal{A}_{\gamma,P}^q(\alpha - \chi).$$

Proof. By direct substitution, we have

$$\begin{aligned}\phi_\chi^\alpha(0, q) &= q^{\frac{P^2}{2} + \frac{\gamma l_\chi}{12\chi} - \frac{1}{6} \frac{l_\chi^2}{\chi^2}} \pi^{l_\chi} \Theta'_\tau(0)^{-\frac{2l_\chi^2}{3\chi^2} - \frac{2l_\chi}{3} + \frac{4l_\chi}{3\gamma\chi}} \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2} + \frac{\gamma\chi}{2}} e^{\pi\gamma P x} dx \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] \\ &= q^{\frac{P^2}{2} + \frac{\gamma l_\chi}{12\chi} - \frac{1}{6} \frac{l_\chi^2}{\chi^2}} \pi^{l_\chi} \Theta'_\tau(0)^{-\frac{2l_\chi^2}{3\chi^2} - \frac{2l_\chi}{3} + \frac{4l_\chi}{3\gamma\chi}} q^{-\frac{\gamma l_\chi}{6\chi} - \frac{l_\chi}{3\chi\gamma} - \frac{1}{6}} \eta(q)^{\frac{2\gamma l_\chi}{\chi} + \frac{4l_\chi}{\chi\gamma} + 2 + \frac{6l_\chi^2}{\chi^2}} \mathcal{A}_{\gamma,P}^q(\alpha - \chi) \\ &= q^{\frac{P^2}{2} - \frac{1}{6} \frac{l_\chi(l_\chi+1)}{\chi^2} - \frac{1}{6} l_\chi - \frac{1}{6}} \pi^{l_\chi} \Theta'_\tau(0)^{-\frac{2l_\chi^2}{3\chi^2} - \frac{2l_\chi}{3} + \frac{4l_\chi}{3\gamma\chi}} \Theta'_\tau(0)^{\frac{2\gamma l_\chi}{3\chi} + \frac{4l_\chi}{3\chi\gamma} + \frac{2}{3} + \frac{2l_\chi^2}{\chi^2}} (2\pi e^{i\pi})^{-\frac{2\gamma l_\chi}{3\chi} - \frac{4l_\chi}{3\chi\gamma} - \frac{2}{3} - \frac{2l_\chi^2}{\chi^2}} \mathcal{A}_{\gamma,P}^q(\alpha - \chi) \\ &= W_\chi^-(\alpha, \gamma) q^{\frac{P^2}{2} - \frac{1}{6} \frac{l_\chi(l_\chi+1)}{\chi^2} - \frac{1}{6} l_\chi - \frac{1}{6}} \Theta'(0)^{\frac{4}{3} \frac{l_\chi(l_\chi+1)}{\chi^2} + \frac{2}{3} l_\chi + \frac{2}{3}} \mathcal{A}_{\gamma,P}^q(\alpha - \chi). \quad \square\end{aligned}$$

We now characterize the next order asymptotics of $\phi_\chi^\alpha(u, q)$ as u goes to 0. For convenience, we use the notation

$$(5.4) \quad l_0 := l_{\frac{\gamma}{2}} \quad \text{and} \quad \tilde{l}_0 := l_{\frac{2}{\gamma}}.$$

The following asymptotic expansion is valid for $\chi = \frac{\gamma}{2}$, $\alpha \in (\frac{\gamma}{2}, \frac{2}{\gamma})$; its proof is by direct computation and is deferred to Appendix D. For the statement, recall the definition of \mathfrak{B} from Appendix A.

Lemma 5.2. *For $\alpha \in (\frac{\gamma}{2}, \frac{2}{\gamma})$ so that $0 < 1 + 2l_0 < 1$ and $\alpha + \frac{\gamma}{2} < Q$, $u \in \mathfrak{B}$, we have*

$$\lim_{u \rightarrow 0} \sin(\pi u)^{-2l_0 - 1} \left(\phi_{\frac{\gamma}{2}}^\alpha(u, q) - \phi_{\frac{\gamma}{2}}^\alpha(0, q) \right) = W_{\frac{\gamma}{2}}^+(\alpha, \gamma) q^{\frac{P^2}{2} + \frac{l_0}{6} - \frac{1}{6} \frac{l_0(1+l_0)}{\chi^2}} \Theta'_\tau(0)^{\frac{4}{3} \frac{l_0(1+l_0)}{\chi^2} - \frac{2}{3} l_0} \mathcal{A}_{\gamma,P}^q(\alpha + \frac{\gamma}{2}).$$

In the case $\chi = \frac{2}{\gamma}$ or in the case $\chi = \frac{\gamma}{2}$ with α close to Q , performing the asymptotic expansion is much more involved and requires an operation known as the OPE with reflection. This requires the following definitions, which will appear in the statement of Lemma 5.3. Let Z be a centered Gaussian process defined on the whole plane with covariance given for $x, y \in \mathbb{C}$ by

$$(5.5) \quad \mathbb{E}[Z(x)Z(y)] = 2 \log \frac{|x| \vee |y|}{|x - y|}.$$

For $\lambda > 0$ consider the process

$$(5.6) \quad \mathcal{B}_s^\lambda := \begin{cases} \hat{B}_s - \lambda s & s \geq 0 \\ \bar{B}_{-s} + \lambda s & s < 0, \end{cases}$$

where $(\hat{B}_s - \lambda s)_{s \geq 0}$ and $(\bar{B}_s - \lambda s)_{s \geq 0}$ are two independent Brownian motions with negative drift conditioned to stay negative. We also introduce the functions

$$(5.7) \quad \rho(\alpha, 1, e^{-i\pi \frac{\gamma\chi}{2} + \pi\gamma P}) := \frac{1}{2} \int_{-\infty}^{\infty} e^{\frac{\gamma}{2} \mathcal{B}_v^{\frac{Q-\alpha}{2}}} \left(e^{\frac{\gamma}{2} Z(-e^{-v/2})} + e^{-i\pi \frac{\gamma\chi}{2} + \pi\gamma P} e^{\frac{\gamma}{2} Z(e^{-v/2})} \right) dv,$$

$$(5.8) \quad \bar{R}(\alpha, 1, e^{-i\pi \frac{\gamma\chi}{2} + \pi\gamma P}) := \mathbb{E} \left[\left(\rho(\alpha, 1, e^{-i\pi \frac{\gamma\chi}{2} + \pi\gamma P}) \right)^{\frac{2}{\gamma}(Q-\alpha)} \right].$$

The function $\bar{R}(\alpha, 1, e^{-i\pi \frac{\gamma\chi}{2} + \pi\gamma P})$ is the reflection coefficient for boundary Liouville CFT, also known as the boundary two-point function. It was introduced in its most general form and computed in [RZ20]. An analogous function first appeared in the case of the Riemann sphere in [KRV19a] and a special case of \bar{R} was computed in [RZ18]. This reflection coefficient is important because it appears in the first order asymptotics of the probability for a one-dimensional GMC measure to be large. This is also why it is natural for this function to appear in the OPE expansions.

In [RZ20], the reflection coefficient was computed explicitly as

$$(5.9) \quad \bar{R}(\alpha, 1, e^{-i\pi \frac{\gamma\chi}{2} + \pi\gamma P}) = \frac{(2\pi)^{\frac{2}{\gamma}(Q-\alpha) - \frac{1}{2}} \left(\frac{2}{\gamma}\right)^{\frac{\gamma}{2}(Q-\alpha) - \frac{1}{2}}}{(Q-\alpha)\Gamma\left(1 - \frac{\gamma^2}{4}\right)^{\frac{2}{\gamma}(Q-\alpha)}} \frac{\Gamma_{\frac{\gamma}{2}}\left(\alpha - \frac{\gamma}{2}\right) e^{-i\pi\left(\frac{\chi}{2} + iP\right)(Q-\alpha)}}{\Gamma_{\frac{\gamma}{2}}\left(Q - \alpha\right) S_{\frac{\gamma}{2}}\left(\frac{\alpha}{2} + \frac{\chi}{2} + iP\right) S_{\frac{\gamma}{2}}\left(\frac{\alpha}{2} - \frac{\chi}{2} - iP\right)},$$

where we have used the notation $S_{\frac{\gamma}{2}}(x) := \frac{\Gamma_{\frac{\gamma}{2}}(x)}{\Gamma_{\frac{\gamma}{2}}(Q-x)}$. We now state the OPE with reflection, whose proof is also deferred to Appendix D.

Lemma 5.3. (*OPE with reflection*) Consider $u = it$ with $t \in (0, \frac{1}{2} \text{Im}(\tau))$. Let $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$. There exists small $\alpha_0 > 0$ such that for $\alpha \in (Q - \alpha_0, Q)$, we have the asymptotic expansion

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} \Theta_\tau(u+x)^{\frac{\chi\gamma}{2}} e^{\pi\gamma P x} dx \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] - \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2} + \frac{\chi\gamma}{2}} e^{\pi\gamma P x} dx \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] \\ &= -u^{1+2l_\chi} (2\pi)^{(Q-\alpha)\left(\frac{\gamma}{3} - \frac{\chi}{3} + \frac{2}{3\gamma}\right)} q^{\frac{1}{6}(Q-\alpha)(\chi + \frac{2}{\gamma} - 2Q)} \Theta'_\tau(0)^{(Q-\alpha)\left(\frac{2\chi}{3} - \frac{4}{3\gamma} - \frac{2}{3\chi}\right)} e^{i\pi(Q-\alpha)\left(\frac{4}{3\gamma} - \frac{2\chi}{3} - \frac{4}{3\chi}\right)} \\ &\times \frac{\Gamma\left(\frac{2\alpha}{\gamma} - \frac{4}{\gamma^2}\right) \Gamma\left(\frac{2Q-\alpha-\chi}{\gamma}\right)}{\Gamma\left(\frac{\alpha}{\gamma} - \frac{\chi}{\gamma}\right)} \bar{R}(\alpha, 1, e^{-i\pi \frac{\gamma\chi}{2} + \pi\gamma P}) \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\gamma}{2}(2Q-\alpha-\chi)} e^{\pi\gamma P x} dx \right)^{\frac{\alpha+\chi-2Q}{\gamma}} \right] + o(|u|^{1+2l_\chi}). \end{aligned}$$

We now propose an analytic continuation of $\alpha \mapsto \mathcal{A}_{\gamma, P}^q(\alpha)$ for $\alpha > Q$, which we will prove in Theorem 5.5 below. The key idea is that for $\chi = \frac{\gamma}{2}$ we have two ways to perform the OPE, one without reflection for $\alpha \in (\frac{\gamma}{2}, \frac{2}{\gamma})$ given by Lemma 5.2 and one with reflection for α close to Q given by Lemma 5.3. By correctly normalizing, we can restate the result of Lemma 5.2 as giving a first order expansion of a moment of GMC

similar to Lemma 5.3 via

$$\begin{aligned} & \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} \Theta_\tau(u+x)^{\frac{\gamma^2}{4}} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma} + \frac{1}{2}} \right] - \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2} + \frac{\gamma^2}{4}} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma} + \frac{1}{2}} \right] \\ &= \left(-\frac{\alpha}{\gamma} + \frac{1}{2}\right) e^{2i\pi l_0 - \frac{i\pi\gamma^2}{2}} q^{\frac{\alpha}{6\gamma} - \frac{1}{12}} \eta(q)^{-\frac{3\gamma^2}{8} - \frac{2\alpha}{\gamma} + 1 + \frac{3\alpha^2}{2}} \Theta'_\tau(0)^{\frac{\gamma^2}{4} - \frac{\alpha\gamma}{2}} \\ & \quad \times \frac{\Gamma(1 - \frac{\alpha\gamma}{2}) \Gamma(-1 + \frac{\alpha\gamma}{2} - \frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})} (1 - e^{\pi\gamma P - 2i\pi l_0}) \mathcal{A}_{\gamma,P}^q(\alpha + \frac{\gamma}{2}) + o(|u|^{1+2l_0}). \end{aligned}$$

In the equation above, $\alpha \mapsto \mathcal{A}_{\gamma,P}^q(\alpha + \frac{\gamma}{2})$ is a priori well-defined and analytic up to $\alpha = \frac{2}{\gamma}$ but beyond this point we expect to analytically continue the answer using Lemma 5.3 for $\chi = \frac{\gamma}{2}$. Therefore, for $\alpha \in (Q, 2Q + \frac{4}{\gamma})$, we define $\mathcal{A}_{\gamma,P}^q(\alpha)$ via

$$\begin{aligned} \mathcal{A}_{\gamma,P}^q(\alpha) &= q^{\frac{1}{12}(-\alpha\gamma - \frac{2\alpha}{\gamma} + 2)} \eta(q)^{\alpha\gamma + \frac{2\alpha}{\gamma} - 2 - \frac{3\alpha^2}{2}} e^{i\pi(\frac{\alpha\gamma}{2} - (\alpha - \frac{\gamma}{2} - Q)(\alpha - 2Q))} (2\pi)^{(\alpha - \frac{\gamma}{2} - Q)(Q - \alpha)} \\ & \quad \times \left(\left(-\frac{\alpha}{\gamma} + 1\right) (1 - e^{\pi\gamma P - i\pi\frac{\gamma^2}{2} + i\pi\frac{\alpha\gamma}{2}}) \right)^{-1} \frac{\Gamma(-\frac{\gamma^2}{4}) \Gamma(\frac{2\alpha}{\gamma} - 1 - \frac{4}{\gamma^2}) \Gamma(1 + \frac{4}{\gamma^2} - \frac{\alpha}{\gamma})}{\Gamma(\frac{\alpha\gamma}{2} - 1 - \frac{\gamma^2}{2}) \Gamma(1 + \frac{\gamma^2}{4} - \frac{\alpha\gamma}{2}) \Gamma(\frac{\alpha}{\gamma} - 1)} \\ & \quad \times \frac{(q^{\frac{1}{6}} \eta(q))^{(Q + \frac{\gamma}{2} - \alpha)(\alpha - 2Q)} \Theta'_\tau(0)^{(Q + \frac{\gamma}{2} - \alpha)(\gamma - \alpha)} (q^{-\frac{1}{12}} \eta(q))^{2(Q + \frac{\gamma}{2} - \alpha)(\alpha - Q)}}{q^{-\frac{\alpha\gamma}{12}} \eta(q)^{-\frac{\alpha\gamma}{2}} \Theta'_\tau(0)^{-\frac{\alpha\gamma}{2} + \frac{\gamma^2}{2}}} \\ & \quad \times \overline{R}(\alpha - \frac{\gamma}{2}, 1, e^{-i\pi\frac{\gamma^2}{4} + \pi\gamma P}) \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\gamma}{2}(2Q - \alpha)} e^{\pi\gamma Px} dx \right)^{\frac{\alpha}{\gamma} - \frac{4}{\gamma^2} - 1} \right] \\ &= -q^{\frac{1}{6}(1 - \frac{\alpha}{\gamma} - Q(Q + \frac{\gamma}{2} - \alpha))} \eta(q)^{\frac{3\alpha\gamma}{2} + \frac{2\alpha}{\gamma} - 2 - \frac{3\alpha^2}{2} + (Q + \frac{\gamma}{2} - \alpha)(3\alpha - 4Q)} \Theta'_\tau(0)^{(Q - \alpha)(\gamma - \alpha)} \\ & \quad \times \frac{e^{i\pi(\frac{\alpha\gamma}{2} - (\alpha - \frac{\gamma}{2} - Q)(\alpha - 2Q))} (2\pi)^{(\alpha - \frac{\gamma}{2} - Q)(Q - \alpha)}}{\left(-\frac{\alpha}{\gamma} + 1\right) (1 - e^{\pi\gamma P - i\pi\frac{\gamma^2}{2} + i\pi\frac{\alpha\gamma}{2}})} \frac{\Gamma(-\frac{\gamma^2}{4}) \Gamma(\frac{2\alpha}{\gamma} - 1 - \frac{4}{\gamma^2}) \Gamma(1 + \frac{4}{\gamma^2} - \frac{\alpha}{\gamma})}{\Gamma(\frac{\alpha\gamma}{2} - 1 - \frac{\gamma^2}{2}) \Gamma(1 + \frac{\gamma^2}{4} - \frac{\alpha\gamma}{2}) \Gamma(\frac{\alpha}{\gamma} - 1)} \\ & \quad \times \overline{R}(\alpha - \frac{\gamma}{2}, 1, e^{-i\pi\frac{\gamma^2}{4} + \pi\gamma P}) \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\gamma}{2}(2Q - \alpha)} e^{\pi\gamma Px} dx \right)^{\frac{\alpha}{\gamma} - \frac{4}{\gamma^2} - 1} \right]. \end{aligned}$$

In the last line, the GMC expectation is well-defined and analytic in α in a complex neighborhood of $(Q, 2Q)$ thanks to the moment bounds given by Lemma B.1 and the analyticity provided by Lemma 3.6. The prefactor in front of the GMC expectation is an explicit meromorphic function of α with known poles; the exact formula (5.9) shows that it is analytic in α in a complex neighborhood of $\alpha \in (Q, 2Q)$. We now use this definition to combine the OPE statements of Lemma 5.2 and Lemma 5.3 into a single expression.

Lemma 5.4. *Consider $u = it$ with $t \in (0, \frac{1}{2} \text{Im}(\tau))$. When $\chi = \frac{\gamma}{2}$ and $\alpha \in (\frac{\gamma}{2}, \frac{2}{\gamma}) \cup (Q - \alpha_0, Q)$, or $\chi = \frac{2}{\gamma}$ and $\alpha \in (Q - \alpha_0, Q)$, we have*

$$(5.10) \quad \lim_{u \rightarrow 0} \sin(\pi u)^{-2l_\chi - 1} \left(\phi_\chi^\alpha(u, q) - \phi_\chi^\alpha(0, q) \right) = W_\chi^+(\alpha, \gamma) q^{\frac{P^2}{2} + \frac{l_\chi}{6} - \frac{1}{6} \frac{l_\chi(1+l_\chi)}{x^2}} \Theta'_\tau(0)^{\frac{4}{3} \frac{l_\chi(l_\chi+1)}{x^2} - \frac{2}{3} l_\chi} \mathcal{A}_{\gamma,P}^q(\alpha + \chi).$$

Proof. For $\chi = \frac{\gamma}{2}$ and $\alpha \in (\frac{\gamma}{2}, \frac{2}{\gamma})$, the claim is given by Lemma 5.2. In the case $\chi = \frac{\gamma}{2}$ and $\alpha \in (Q - \alpha_0, Q)$, the claim is implied by Lemma 5.3 and the definition of $\mathcal{A}_{\gamma,P}^q(\alpha + \frac{\gamma}{2})$.

We now check the case $\chi = \frac{2}{\gamma}$. The claim of Lemma 5.3 for $\chi = \frac{2}{\gamma}$ means that we have

(5.11)

$$\begin{aligned} & \lim_{u \rightarrow 0} \sin(\pi u)^{-2\tilde{l}_0-1} \left(\phi_{\frac{2}{\gamma}}^{\alpha}(u, q) - \phi_{\frac{2}{\gamma}}^{\alpha}(0, q) \right) \\ &= -q^{\frac{P^2}{2} + \frac{\gamma^2}{24} \tilde{l}_0(1-\tilde{l}_0)} \Theta'_{\tau}(0)^{-\frac{\gamma^2}{6} \tilde{l}_0^2} \\ & \times \pi^{-1-\tilde{l}_0} (2\pi)^{\frac{\gamma}{3}(Q-\alpha)} q^{-\frac{\gamma}{6}(Q-\alpha)} \Theta'_{\tau}(0)^{-\frac{\gamma}{3}(Q-\alpha)} e^{-i\pi \frac{2\gamma}{3}(Q-\alpha)} \\ & \times \frac{\Gamma(\frac{2\alpha}{\gamma} - \frac{4}{\gamma^2}) \Gamma(\frac{2}{\gamma^2} + 1 - \frac{\alpha}{\gamma})}{\Gamma(\frac{\alpha}{\gamma} - \frac{2}{\gamma^2})} \bar{R}(\alpha, 1, e^{-i\pi + \pi\gamma P}) \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_{\tau}(x)} \Theta_{\tau}(x)^{-\frac{\gamma}{2}(2Q-\alpha-\frac{2}{\gamma})} e^{\pi\gamma P x} dx \right)^{\frac{\alpha}{\gamma} - \frac{2}{\gamma^2} - 1} \right]. \end{aligned}$$

By our definition of $\mathcal{A}_{\gamma, P}^q(\alpha + \frac{2}{\gamma})$, for $\alpha > \frac{\gamma}{2}$ we have

$$\begin{aligned} (5.12) \quad \mathcal{A}_{\gamma, P}^q(\alpha + \frac{2}{\gamma}) &= -q^{-\frac{1}{6}(1+\frac{\alpha}{\gamma}+Q(\gamma-\alpha))} \eta(q)^{\frac{13\alpha\gamma}{2}-1-\frac{2\alpha}{\gamma}-\frac{2}{\gamma^2}-\frac{9\alpha^2}{2}-2\gamma^2} \Theta'_{\tau}(0)^{(\frac{\gamma}{2}-\alpha)(\gamma-\frac{2}{\gamma}-\alpha)} \\ & \times \frac{e^{i\pi(\frac{\alpha\gamma}{2}+1-(\alpha-\gamma)(\alpha-\gamma-\frac{2}{\gamma}))} (2\pi)^{(\alpha-\gamma)(\frac{\gamma}{2}-\alpha)} \Gamma(-\frac{\gamma^2}{4}) \Gamma(\frac{2\alpha}{\gamma}-1) \Gamma(1+\frac{2}{\gamma^2}-\frac{\alpha}{\gamma})}{(1-\frac{\alpha}{\gamma}-\frac{2}{\gamma^2})(1+e^{\pi\gamma P-i\pi\frac{\gamma^2}{2}+i\pi\frac{\alpha\gamma}{2}}) \Gamma(\frac{\alpha\gamma}{2}-\frac{\gamma^2}{2}) \Gamma(\frac{\gamma^2}{4}-\frac{\alpha\gamma}{2}) \Gamma(\frac{\alpha}{\gamma}+\frac{2}{\gamma^2}-1)} \\ & \times \bar{R}(\alpha + \frac{2}{\gamma} - \frac{\gamma}{2}, 1, e^{-i\pi\frac{\gamma^2}{4}+\pi\gamma P}) \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_{\tau}(x)} \Theta_{\tau}(x)^{-\frac{\gamma}{2}(2Q-\alpha-\frac{2}{\gamma})} e^{\pi\gamma P x} dx \right)^{\frac{\alpha}{\gamma} - \frac{2}{\gamma^2} - 1} \right]. \end{aligned}$$

To land on the desired answer, by [RZ20, Theorem 1.7] we compute a ratio of reflection coefficients as

$$\begin{aligned} & \frac{\bar{R}(\alpha, 1, e^{-i\pi+\pi\gamma P})}{\bar{R}(\alpha + \frac{2}{\gamma} - \frac{\gamma}{2}, 1, e^{-i\pi\frac{\gamma^2}{4}+\pi\gamma P})} = \frac{\bar{R}(\alpha, 1, e^{-i\pi+\pi\gamma P})}{\bar{R}(\alpha + \frac{2}{\gamma}, 1, e^{\pi\gamma P})} \frac{\bar{R}(\alpha + \frac{2}{\gamma}, 1, e^{\pi\gamma P})}{\bar{R}(\alpha + \frac{2}{\gamma} - \frac{\gamma}{2}, 1, e^{-i\pi\frac{\gamma^2}{4}+\pi\gamma P})} \\ &= \frac{2}{\gamma(Q-\alpha)} (2\pi)^{\frac{4}{\gamma^2}-1} \frac{\Gamma(\frac{2\alpha}{\gamma}) \Gamma(1-\frac{2\alpha}{\gamma})}{\Gamma(1-\frac{\gamma^2}{4})^{\frac{4}{\gamma^2}-1} \Gamma(\frac{\gamma\alpha}{2}-\frac{\gamma^2}{2}) \Gamma(1-\frac{\gamma\alpha}{2}+\frac{\gamma^2}{4})} \frac{1 - e^{\frac{4\pi P}{\gamma} - \frac{4i\pi}{\gamma^2} + i\pi\frac{2\alpha}{\gamma}}}{1 + e^{\pi\gamma P - \frac{i\pi\gamma^2}{2} + i\pi\frac{\gamma\alpha}{2}}}. \end{aligned}$$

Substituting (5.12) into (5.11) and simplifying all the prefactors algebraically yields the desired claim. \square

We now define the quantities $\eta_{\chi, n}^{\pm}(\alpha)$ as coefficients of the q -series expansions

$$(5.13) \quad \Theta'_{\tau}(0)^{\frac{4}{3}\frac{l_{\chi}(l_{\chi}+1)}{x^2} + \frac{2}{3}l_{\chi} + \frac{2}{3}} = q^{\frac{1}{3}\frac{l_{\chi}(l_{\chi}+1)}{x^2} + \frac{1}{6}l_{\chi} + \frac{1}{6}} \sum_{n=0}^{\infty} \eta_{\chi, n}^{-}(\alpha) q^n$$

$$(5.14) \quad \Theta'_{\tau}(0)^{\frac{4}{3}\frac{l_{\chi}(l_{\chi}+1)}{x^2} - \frac{2}{3}l_{\chi}} = q^{\frac{1}{3}\frac{l_{\chi}(l_{\chi}+1)}{x^2} - \frac{1}{6}l_{\chi}} \sum_{n=0}^{\infty} \eta_{\chi, n}^{+}(\alpha) q^n.$$

Using this, we may translate the OPEs in Lemma 5.4 into their consequences on the q -series expansions. The following theorem also shows the q -series coefficients $\mathcal{A}_{\gamma, P, n}(\alpha)$ of $\mathcal{A}_{\gamma, P}^q(\alpha)$ are analytic in α in a complex neighborhood of $(-\frac{4}{\gamma}, 2Q)$. The non-trivial part of this claim is the analyticity at $\alpha = Q$, since we do not know a priori that our definition of $\mathcal{A}_{\gamma, P}^q(\alpha)$ for $\alpha > Q$ gives the correct analytic continuation.

Theorem 5.5. *Recall notations in Corollary 4.4 and the definition of $\mathcal{A}_{\gamma, P, n}(\alpha)$ from (2.11). For each $n \in \mathbb{N}$, the function $\mathcal{A}_{\gamma, P, n}(\alpha)$ can be analytically extended to a complex neighborhood of $(-\frac{4}{\gamma}, 2Q)$. Under this extension of $\mathcal{A}_{\gamma, P, n}(\alpha)$, for $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$ and $\alpha \in (\chi, Q)$, we have*

$$(5.15) \quad \phi_{\chi, n, 1}^{\alpha, 1}(0) = W_{\chi}^{-}(\alpha, \gamma) \left[\eta_{\chi, 0}^{-}(\alpha) \mathcal{A}_{\gamma, P, n}(\alpha - \chi) + \sum_{m=0}^{n-1} \eta_{\chi, n-m}^{-}(\alpha) \mathcal{A}_{\gamma, P, m}(\alpha - \chi) \right];$$

$$(5.16) \quad \phi_{\chi, n, 1}^{\alpha, 2}(0) = W_{\chi}^{+}(\alpha, \gamma) \left[\eta_{\chi, 0}^{+}(\alpha) \mathcal{A}_{\gamma, P, n}(\alpha + \chi) + \sum_{m=0}^{n-1} \eta_{\chi, n-m}^{+}(\alpha) \mathcal{A}_{\gamma, P, m}(\alpha + \chi) \right].$$

Proof. By Corollary 4.4, we have $\phi_{\chi,n,1}^\alpha(0) = \phi_{\chi,n,1}^{\alpha,1}(0)$ and

$$(5.17) \quad \phi_{\chi,n,1}^{\alpha,2}(0) = \lim_{w \rightarrow 0^-} w^{C_\chi - 1} (\phi_{\chi,n,1}^\alpha(w) - \phi_{\chi,n,1}^{\alpha,1}(w)) = \lim_{t \rightarrow 0^+} \sin(\pi it)^{-2l_\chi - 1} (\phi_{\chi,n}^\alpha(it) - \phi_{\chi,n}^{\alpha,1}(0)).$$

By (5.10) from Lemma 5.4 and (5.14), (5.16) holds for $\chi = \frac{\gamma}{2}$ and $\alpha \in (\frac{\gamma}{2}, \frac{2}{\gamma}) \cup (Q - \alpha_0, Q)$. Setting $n = 0$ in this equation, we find

$$\phi_{\chi,0,1}^{\alpha,2}(0) = W_\chi^+(\alpha, \gamma) \eta_{\chi,0}^+(\alpha) \mathcal{A}_{\gamma,P,0}(\alpha + \chi).$$

By Lemma 4.9, when $\chi = \frac{\gamma}{2}$, $\phi_{\chi,n,1}^{\alpha,2}(0)$ is analytic in α on a complex neighborhood of $(-\frac{4}{\gamma} + \frac{\gamma}{2}, Q)$. Combined with the explicit expression for $W_\chi^+(\alpha, \gamma)$ from (5.2) and the fact that $\eta_{\chi,0}^+(\alpha) = (2\pi e^{i\pi})^{\frac{4}{3} \frac{l_\chi(l_\chi+1)}{x^2} - \frac{2}{3}l_\chi}$, this shows that

$$[W_\chi^+(\alpha, \gamma) \eta_{\chi,0}^+(\alpha)]^{-1} \phi_{\chi,0,1}^{\alpha,2}(0)$$

provides an analytic extension of $\mathcal{A}_{\gamma,P,0}(\alpha)$ to a complex neighborhood of Q . Combined with Lemma 2.7 and the definition of $\mathcal{A}_{\gamma,P,0}^q(\alpha)$ on a complex neighborhood of $(Q, 2Q)$, this allows us to glue the two definitions together to analytically extend $\mathcal{A}_{\gamma,P,0}(\alpha)$ to a complex neighborhood of $(-\frac{4}{\gamma}, 2Q)$. An induction on n yields a similar analytic extension for $\mathcal{A}_{\gamma,P,n}(\alpha)$ with $n \geq 1$.

Finally, to show the desired OPEs (5.15) and (5.16), we notice that (5.15) follows from (5.3) and (5.13), and (5.16) follows from (5.10) and (5.14). \square

Corollary 5.6. *For $w \in \mathbb{D}$, recall the definitions of $V_{\chi,n}^{\alpha,j}$ in (4.12) and (4.13) and $G_{\chi,n,i}^{\alpha,j}(w)$ and $\phi_{\chi,n,i}^{\alpha,j}(w)$ in Proposition 4.7 and Corollary 4.4, respectively. The quantities $V_{\chi,n}^{\alpha,j}$, $G_{\chi,n,i}^{\alpha,j}(w)$, and $\phi_{\chi,n,i}^{\alpha,j}(w)$ may be analytically extended as functions of α to a complex neighborhood of $(-\frac{4}{\gamma} + \chi, 2Q - \chi)$.*

Proof. We induct on n . Suppose such analytic extensions exist for all $m < n$. For $V_{\chi,n}^{\alpha,j}$ and $G_{\chi,n,i}^{\alpha,j}(w)$, the conclusion follows from Lemma 4.8 in the same way as in the proof of Lemma 4.9. For $\phi_{\chi,n,i}^{\alpha,j}(w)$, for α in an open neighborhood of (χ, Q) , setting $w = 0$ in (4.15) yields

$$X_{\chi,n,i}^j(\alpha) = \phi_{\chi,n,i}^{\alpha,j}(0) - G_{\chi,n,i}^{\alpha,j}(0).$$

Substituting this back in yields

$$\phi_{\chi,n,i}^{\alpha,j}(w) = G_{\chi,n,i}^{\alpha,j}(w) + \left(\phi_{\chi,n,i}^{\alpha,j}(0) - G_{\chi,n,i}^{\alpha,j}(0) \right) v_{j,\chi,n}(w),$$

which upon substituting the expressions (5.15) and (5.16) from Theorem 5.5 for $\phi_{\chi,n,i}^{\alpha,j}(0)$ yields an expression for $\phi_{\chi,n,i}^{\alpha,j}(w)$ providing the desired analytic continuation. \square

6. EQUIVALENCE OF THE PROBABILISTIC CONFORMAL BLOCK AND NEKRASOV PARTITION FUNCTION

In this section, we prove our main result Theorem 2.11 by showing that the q -series coefficients of $\tilde{\mathcal{A}}_{\gamma,P}^q(\alpha)$ and $\mathcal{Z}_{\gamma,P}^q(\alpha)$ both satisfy a system (6.2) of shift equations. We present the proofs of these shift equations in Sections 6.1 and 6.2 while making use of Proposition 6.2 and Theorem 6.4, which will be proved in Sections 6.4–6.6. We put these together to prove Theorem 2.11 in Section 6.3.

6.1. Shift equations for series coefficients of the conformal block. The goal of this section will be to prove Theorem 6.1, which gives the *shift equations* (6.1) and (6.2) relating values of $\mathcal{A}_{\gamma,P,0}(\alpha)$ and $\tilde{\mathcal{A}}_{\gamma,P,n}(\alpha)$ at different values of α . The proof will proceed by combining the Gauss hypergeometric equations from Proposition 4.2 and the operator product expansions from Theorem 5.4. In particular, we will compare the two functions $\phi_{\chi,0,i}^\alpha(w)$ for $i \in \{1, 2\}$ within the solution spaces of (4.5) and (4.6) and apply (2.7).

Theorem 6.1. *Recall the analytic extensions given in Theorem 5.5 for $\{\mathcal{A}_{\gamma,P,n}(\alpha)\}$ and $\tilde{\mathcal{A}}_{\gamma,P,n}(\alpha)$ defined in (2.11) and (2.17), respectively. For $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$ and α in a complex neighborhood of $(-\frac{4}{\gamma} + \chi, 2Q - \chi)$, we have*

$$(6.1) \quad \mathcal{A}_{\gamma,P,0}(\alpha - \chi) = - \frac{W_\chi^+(\alpha, \gamma) \Gamma_{0,2}}{W_\chi^-(\alpha, \gamma) \Gamma_{0,1}} \frac{1 + e^{\pi\chi P + i\pi l_\chi} \eta_{\chi,0}^+(\alpha)}{1 - e^{\pi\chi P - i\pi l_\chi} \eta_{\chi,0}^-(\alpha)} \mathcal{A}_{\gamma,P,0}(\alpha + \chi)$$

and

(6.2)

$$\begin{aligned} \tilde{\mathcal{A}}_{\gamma,P,n}(\alpha-\chi) + \sum_{m=0}^{n-1} \frac{\eta_{\chi,n-m}^-(\alpha)}{\eta_{\chi,0}^-(\alpha)} \tilde{\mathcal{A}}_{\gamma,P,m}(\alpha-\chi) &= \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \frac{\Gamma_{0,1}}{\Gamma_{0,2}} \tilde{\mathcal{A}}_{\gamma,P,n}(\alpha+\chi) + \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \frac{\Gamma_{0,1}}{\Gamma_{0,2}} \sum_{m=0}^{n-1} \frac{\eta_{\chi,n-m}^+(\alpha)}{\eta_{\chi,0}^+(\alpha)} \tilde{\mathcal{A}}_{\gamma,P,m}(\alpha+\chi) \\ &+ \left(W_{\chi}^-(\alpha, \gamma)^{-1} \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \frac{1 + e^{\pi\chi P + i\pi l_{\chi}}}{1 - e^{\pi\chi P - i\pi l_{\chi}}} V_{\chi,n}^{\alpha,2} + W_{\chi}^-(\alpha, \gamma)^{-1} V_{\chi,n}^{\alpha,1} \right) \eta_{\chi,0}^-(\alpha)^{-1} \mathcal{A}_{\gamma,P,0}(\alpha-\chi)^{-1}, \end{aligned}$$

where we interpret $V_{\chi,n}^{\alpha,1}$ and $V_{\chi,n}^{\alpha,2}$ as defined in (4.12) and (4.13) via their analytic continuation given in Corollary 5.6.

Proof. By Corollary 5.6 and the uniqueness of meromorphic extension, it suffices to check this for $\alpha \in (\chi, Q)$, where Theorem 5.5 applies. For $i \in \{1, 2\}$, let $G_{\chi,n,i}^{\alpha}(w)$ be the solutions to (4.6) given by Proposition 4.7. Expressing the decomposition of (C.4) in two different bases, for some $X_{\chi,n,i}^j(\alpha), Y_{\chi,n,i}^j(\alpha)$ for $i, j \in \{1, 2\}$, we have

$$\begin{aligned} \phi_{\chi,n,i}^{\alpha}(w) &= G_{\chi,n,i}^{\alpha}(w) + X_{\chi,n,i}^1(\alpha) {}_2F_1(A_{\chi,n}, B_{\chi,n}, C_{\chi}; w) \\ &\quad + X_{\chi,n,i}^2(\alpha) w^{1-C_{\chi}} {}_2F_1(1 + A_{\chi,n} - C_{\chi}, 1 + B_{\chi,n} - C_{\chi}, 2 - C_{\chi}; w) \\ \phi_{\chi,n,i}^{\alpha}(w) &= G_{\chi,n,i}^{\alpha}(w) + Y_{\chi,n,i}^1(\alpha) {}_2F_1(A_{\chi,n}, B_{\chi,n}, 1 + A_{\chi,n} + B_{\chi,n} - C_{\chi}; 1 - w) \\ &\quad + Y_{\chi,n,i}^2(\alpha) (1 - w)^{C_{\chi} - A_{\chi,n} - B_{\chi,n}} {}_2F_1(C_{\chi} - A_{\chi,n}, C_{\chi} - B_{\chi,n}, 1 + C_{\chi} - A_{\chi,n} - B_{\chi,n}; 1 - w). \end{aligned}$$

Together, these equations imply for $i \in \{1, 2\}$ that

$$\begin{aligned} \phi_{\chi,n,i}^{\alpha,1}(w) &= G_{\chi,n,i}^{\alpha,1}(w) + X_{\chi,n,i}^1(\alpha) {}_2F_1(A_{\chi,n}, B_{\chi,n}, C_{\chi}; w) \\ \phi_{\chi,n,i}^{\alpha,2}(w) &= G_{\chi,n,i}^{\alpha,2}(w) + X_{\chi,n,i}^2(\alpha) {}_2F_1(1 + A_{\chi,n} - C_{\chi}, 1 + B_{\chi,n} - C_{\chi}, 2 - C_{\chi}; w). \end{aligned}$$

In terms of the connection coefficients defined in (4.11), we have for $i \in \{1, 2\}$ by the connection equation (C.2) that

$$Y_{\chi,n,i}^1(\alpha) = \Gamma_{n,1} X_{\chi,n,i}^1(\alpha) + \Gamma_{n,2} X_{\chi,n,i}^2(\alpha).$$

Because $\phi_{\chi,n,1}^{\alpha}(1) = \phi_{\chi,n,2}^{\alpha}(1)$, $G_{\chi,n,1}^{\alpha}(1) = G_{\chi,n,2}^{\alpha}(1) = 0$, and $C_{\chi} - A_{\chi,n} - B_{\chi,n} = \frac{1}{2}$, this implies that

$$(6.3) \quad X_{\chi,n,1}^1(\alpha) - X_{\chi,n,2}^1(\alpha) = -\frac{\Gamma_{n,2}}{\Gamma_{n,1}} (X_{\chi,n,1}^2(\alpha) - X_{\chi,n,2}^2(\alpha)).$$

In addition, by Lemma 4.6 and Theorem 5.5, we have

$$\begin{aligned} X_{\chi,n,1}^1(\alpha) + G_{\chi,n,1}^{\alpha,1}(0) = \phi_{\chi,n,1}^{\alpha,1}(0) &= W_{\chi}^-(\alpha, \gamma) \left[\eta_{\chi,0}^-(\alpha) \mathcal{A}_{\gamma,P,n}(\alpha-\chi) + \sum_{m=0}^{n-1} \eta_{\chi,n-m}^-(\alpha) \mathcal{A}_{\gamma,P,m}(\alpha-\chi) \right] \\ X_{\chi,n,1}^2(\alpha) + G_{\chi,n,1}^{\alpha,2}(0) = \phi_{\chi,n,1}^{\alpha,2}(0) &= W_{\chi}^+(\alpha, \gamma) \left[\eta_{\chi,0}^+(\alpha) \mathcal{A}_{\gamma,P,n}(\alpha+\chi) + \sum_{m=0}^{n-1} \eta_{\chi,n-m}^+(\alpha) \mathcal{A}_{\gamma,P,m}(\alpha+\chi) \right] \\ X_{\chi,n,2}^1(\alpha) + G_{\chi,n,2}^{\alpha,1}(0) = \phi_{\chi,n,2}^{\alpha,1}(0) &= e^{\pi\chi P - i\pi l_{\chi}} (X_{\chi,n,1}^1(\alpha) + G_{\chi,n,1}^{\alpha,1}(0)) \\ X_{\chi,n,2}^2(\alpha) + G_{\chi,n,2}^{\alpha,2}(0) = \phi_{\chi,n,2}^{\alpha,2}(0) &= -e^{\pi\chi P + i\pi l_{\chi}} (X_{\chi,n,1}^2(\alpha) + G_{\chi,n,1}^{\alpha,2}(0)) \end{aligned}$$

for $W_{\chi}^{\pm}(\alpha, \gamma)$ defined in (5.1) and (5.2). Combining (6.3), the last two equalities, and Proposition 4.7, we find that

$$(1 - e^{\pi\chi P - i\pi l_{\chi}}) X_{\chi,n,1}^1(\alpha) = -\frac{\Gamma_{n,2}}{\Gamma_{n,1}} (1 + e^{\pi\chi P + i\pi l_{\chi}}) X_{\chi,n,1}^2(\alpha).$$

Finally, substituting Theorem 5.5 into the first equality, we find that

$$\begin{aligned}
 \eta_{\chi,0}^-(\alpha)\mathcal{A}_{\gamma,P,n}(\alpha-\chi) &= W_{\chi}^-(\alpha,\gamma)^{-1}(X_{\chi,n,1}^1(\alpha) + G_{\chi,n,1}^{\alpha,1}(0)) - \sum_{m=0}^{n-1} \eta_{\chi,n-m}^-(\alpha)\mathcal{A}_{\gamma,P,m}(\alpha-\chi) \\
 &= -W_{\chi}^-(\alpha,\gamma)^{-1} \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \frac{1 + e^{\pi\chi P + i\pi l_{\chi}}}{1 - e^{\pi\chi P - i\pi l_{\chi}}} X_{\chi,n,1}^2(\alpha) + W_{\chi}^-(\alpha,\gamma)^{-1} G_{\chi,n,1}^{\alpha,1}(0) - \sum_{m=0}^{n-1} \eta_{\chi,n-m}^-(\alpha)\mathcal{A}_{\gamma,P,m}(\alpha-\chi) \\
 &= -W_{\chi}^+(\alpha,\gamma)W_{\chi}^-(\alpha,\gamma)^{-1} \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \frac{1 + e^{\pi\chi P + i\pi l_{\chi}}}{1 - e^{\pi\chi P - i\pi l_{\chi}}} \eta_{\chi,0}^+(\alpha)\mathcal{A}_{\gamma,P,n}(\alpha+\chi) \\
 &\quad - W_{\chi}^+(\alpha,\gamma)W_{\chi}^-(\alpha,\gamma)^{-1} \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \frac{1 + e^{\pi\chi P + i\pi l_{\chi}}}{1 - e^{\pi\chi P - i\pi l_{\chi}}} \sum_{m=0}^{n-1} \eta_{\chi,n-m}^+(\alpha)\mathcal{A}_{\gamma,P,m}(\alpha+\chi) \\
 &\quad + W_{\chi}^-(\alpha,\gamma)^{-1} \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \frac{1 + e^{\pi\chi P + i\pi l_{\chi}}}{1 - e^{\pi\chi P - i\pi l_{\chi}}} G_{\chi,n,1}^{\alpha,2}(0) + W_{\chi}^-(\alpha,\gamma)^{-1} G_{\chi,n,1}^{\alpha,1}(0) - \sum_{m=0}^{n-1} \eta_{\chi,n-m}^-(\alpha)\mathcal{A}_{\gamma,P,m}(\alpha-\chi).
 \end{aligned}$$

Specializing to the case $n = 0$, we find that

$$\mathcal{A}_{\gamma,P,0}(\alpha-\chi) = -\frac{W_{\chi}^+(\alpha,\gamma)}{W_{\chi}^-(\alpha,\gamma)} \frac{\Gamma_{0,2}}{\Gamma_{0,1}} \frac{1 + e^{\pi\chi P + i\pi l_{\chi}}}{1 - e^{\pi\chi P - i\pi l_{\chi}}} \frac{\eta_{\chi,0}^+(\alpha)}{\eta_{\chi,0}^-(\alpha)} \mathcal{A}_{\gamma,P,0}(\alpha+\chi),$$

which yields (6.1). For (6.2), divide both sides of the equation by $\eta_{\chi,0}^-(\alpha)\mathcal{A}_{\gamma,P,0}(\alpha-\chi)$ and apply (6.1) to find that

$$\begin{aligned}
 \tilde{\mathcal{A}}_n(\alpha-\chi) + \sum_{m=0}^{n-1} \frac{\eta_{\chi,n-m}^-(\alpha)}{\eta_{\chi,0}^-(\alpha)} \tilde{\mathcal{A}}_m(\alpha-\chi) &= \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \frac{\Gamma_{0,1}}{\Gamma_{0,2}} \tilde{\mathcal{A}}_n(\alpha+\chi) + \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \frac{\Gamma_{0,1}}{\Gamma_{0,2}} \sum_{m=0}^{n-1} \frac{\eta_{\chi,n-m}^+(\alpha)}{\eta_{\chi,0}^+(\alpha)} \tilde{\mathcal{A}}_m(\alpha+\chi) \\
 &\quad + \left(W_{\chi}^-(\alpha,\gamma)^{-1} \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \frac{1 + e^{\pi\chi P + i\pi l_{\chi}}}{1 - e^{\pi\chi P - i\pi l_{\chi}}} G_{\chi,n,1}^{\alpha,2}(0) + W_{\chi}^-(\alpha,\gamma)^{-1} G_{\chi,n,1}^{\alpha,1}(0) \right) \eta_{\chi,0}^-(\alpha)^{-1} \mathcal{A}_{\gamma,P,0}(\alpha-\chi)^{-1},
 \end{aligned}$$

which implies (6.2) by Proposition 4.7. \square

Remark. For $n = 2$, the shift equation (6.2) for $\chi = \frac{\gamma}{2}$ becomes

$$\tilde{\mathcal{A}}_{\gamma,P,2}(\alpha - \frac{\gamma}{2}) + \frac{\eta_{\frac{\gamma}{2},2}^-(\alpha)}{\eta_{\frac{\gamma}{2},0}^-(\alpha)} = \frac{\Gamma_{2,2}\Gamma_{0,1}}{\Gamma_{2,1}\Gamma_{0,2}} \tilde{\mathcal{A}}_{\gamma,P,2}(\alpha + \frac{\gamma}{2}) + \frac{\Gamma_{2,2}\Gamma_{0,1}}{\Gamma_{2,1}\Gamma_{0,2}} \frac{\eta_{\frac{\gamma}{2},2}^+(\alpha)}{\eta_{\frac{\gamma}{2},0}^+(\alpha)} + X$$

for

$$X := \eta_{\frac{\gamma}{2},0}^-(\alpha)^{-1} \mathcal{A}_{\gamma,P,0}(\alpha - \frac{\gamma}{2})^{-1} \left(W_{\frac{\gamma}{2}}^-(\alpha,\gamma)^{-1} \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \frac{1 + e^{\pi\frac{\gamma}{2}P + i\pi l_0}}{1 - e^{\pi\frac{\gamma}{2}P - i\pi l_0}} V_{\frac{\gamma}{2},n}^{\alpha,2} + W_{\frac{\gamma}{2}}^-(\alpha,\gamma)^{-1} V_{\frac{\gamma}{2},n}^{\alpha,1} \right).$$

By Proposition 4.2 and Theorem 5.5, we find that

$$\begin{aligned}
 \phi_{\frac{\gamma}{2},0}^{\alpha}(w) &= W_{\frac{\gamma}{2}}^-(\alpha,\gamma) \eta_{\frac{\gamma}{2},0}^-(\alpha) \mathcal{A}_{\gamma,P,0}(\alpha - \frac{\gamma}{2}) {}_2F_1(A_{\frac{\gamma}{2},0}, B_{\frac{\gamma}{2},0}, C_{\frac{\gamma}{2}}; w) \\
 &\quad + W_{\frac{\gamma}{2}}^+(\alpha,\gamma) \eta_0^+(\alpha) \mathcal{A}_{\gamma,P,0}(\alpha + \frac{\gamma}{2}) w^{1-C_{\frac{\gamma}{2}}} {}_2F_1(1 + A_{\frac{\gamma}{2},0} - C_{\frac{\gamma}{2}}, 1 + B_{\frac{\gamma}{2},0} - C_{\frac{\gamma}{2}}, 2 - C_{\frac{\gamma}{2}}; w).
 \end{aligned}$$

Noting from (A.6) that $\wp_2(w) = 16\pi^2 w$, we find that

$$\begin{aligned}
 g_{\frac{\gamma}{2},2,1}^{\alpha,1}(w) &= 4wl_0(l_0 + 1) W_{\frac{\gamma}{2}}^-(\alpha,\gamma) \eta_{\frac{\gamma}{2},0}^-(\alpha) \mathcal{A}_{\gamma,P,0}(\alpha - \frac{\gamma}{2}) {}_2F_1(A_{\frac{\gamma}{2},0}, B_{\frac{\gamma}{2},0}, C_{\frac{\gamma}{2}}; w) \\
 g_{\frac{\gamma}{2},2,1}^{\alpha,2}(w) &= 4wl_0(l_0 + 1) W_{\frac{\gamma}{2}}^+(\alpha,\gamma) \eta_0^+(\alpha) \mathcal{A}_{\gamma,P,0}(\alpha + \frac{\gamma}{2}) w^{1-C_{\frac{\gamma}{2}}} {}_2F_1(1 + A_{\frac{\gamma}{2},0} - C_{\frac{\gamma}{2}}, 1 + B_{\frac{\gamma}{2},0} - C_{\frac{\gamma}{2}}, 2 - C_{\frac{\gamma}{2}}; w).
 \end{aligned}$$

Together these imply after some computation that

$$(6.4) \quad X = \frac{4l_0(l_0+1)}{1-C_{\frac{\gamma}{2}}} \int_0^1 \frac{t}{t^{1-C_{\frac{\gamma}{2}}}(1-t)^{C_{\frac{\gamma}{2}}-A_{\frac{\gamma}{2},2}-B_{\frac{\gamma}{2},2}}} \\ \left(t^{1-C_{\frac{\gamma}{2}}} {}_2F_1(1+A_{\frac{\gamma}{2},2}-C_{\frac{\gamma}{2}}, 1+B_{\frac{\gamma}{2},2}-C_{\frac{\gamma}{2}}, 2-C_{\frac{\gamma}{2}}, t) - \frac{\Gamma_{2,2}}{\Gamma_{2,1}} {}_2F_1(A_{\frac{\gamma}{2},2}, B_{\frac{\gamma}{2},2}, C_{\frac{\gamma}{2}}, t) \right) \\ \left({}_2F_1(A_{\frac{\gamma}{2},0}, B_{\frac{\gamma}{2},0}, C_{\frac{\gamma}{2}}; t) - \frac{\Gamma_{0,1}}{\Gamma_{0,2}} t^{1-C_{\frac{\gamma}{2}}} {}_2F_1(1+A_{\frac{\gamma}{2},0}-C_{\frac{\gamma}{2}}, 1+B_{\frac{\gamma}{2},0}-C_{\frac{\gamma}{2}}, 2-C_{\frac{\gamma}{2}}; t) \right) dt.$$

By our main result Theorem 2.11, we compute

$$\tilde{\mathcal{A}}_2(\alpha) = \mathcal{Z}_2(P, \alpha, \gamma) = -\alpha(Q - \frac{\alpha}{2}) + 2 + 4 \frac{\frac{\alpha^2}{4}(Q - \frac{\alpha}{2})^2 - \frac{\alpha}{2}(Q - \frac{\alpha}{2})}{2Q^2 + 2P^2},$$

and by (5.13) and (5.14), we have

$$\frac{\eta_{\frac{\gamma}{2},2}^-(\alpha)}{\eta_{\frac{\gamma}{2},0}^-(\alpha)} = -\frac{16l_0(l_0+1)}{\gamma^2} - 2l_0 - 2 \quad \text{and} \quad \frac{\eta_{\frac{\gamma}{2},2}^+(\alpha)}{\eta_{\frac{\gamma}{2},0}^+(\alpha)} = -\frac{16l_0(l_0+1)}{\gamma^2} + 2l_0.$$

Putting these together, we find that (6.2) for $n=2$ and $\chi = \frac{\gamma}{2}$ implies that

$$X = \frac{8l_0(l_0+1)}{\gamma^2} \left[-1 + \frac{(4l_0 + \gamma^2)(4l_0 + \gamma^2 + 4)}{4\gamma^2(Q^2 + P^2)} + \frac{\Gamma_{2,2}\Gamma_{0,1}}{\Gamma_{2,1}\Gamma_{0,2}} \left(1 - \frac{(4l_0 - \gamma^2)(4l_0 + 4 - \gamma^2)}{4\gamma^2(Q^2 + P^2)} \right) \right],$$

which we verified numerically in Mathematica for a few generic values of α, γ, P . We do not know a direct method to evaluate the defining integral of X from (6.4).

6.2. Shift equations for $\mathcal{Z}_{\gamma,P}^q(\alpha)$. Define the ratio $N := -\frac{\alpha}{\gamma}$. If $N \in \mathbb{N}$, for $\gamma \in (0, 2)$, $\alpha \in (-\frac{4}{\gamma}, Q)$, and $q \in (0, 1)$, the function $\mathcal{A}_{\gamma,P}^q(\alpha)$ is given by the Dotsenko-Fateev integral

$$(6.5) \quad \mathcal{A}_{\gamma,P}^q(\alpha) := q^{\frac{\alpha^2}{24} - \frac{\alpha}{12}Q + \frac{1}{6}\eta(q)^{\frac{5}{4}\alpha\gamma + \frac{2\alpha}{\gamma} - \frac{5}{4}\alpha^2 - 2}} \\ \left(\int_0^1 \right)^N \prod_{1 \leq i < j \leq N} |\Theta_\tau(x_i - x_j)|^{-\frac{\gamma^2}{2}} \prod_{i=1}^N \Theta_\tau(x_i)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x_i} \prod_{i=1}^N dx_i.$$

Our proof of Theorem 2.11 is based on two properties of the above Dotsenko-Fateev integral.

Proposition 6.2. *For $\gamma \in (0, 2)$, $\alpha \in (-\frac{4}{\gamma}, Q)$, and $P \in \mathbb{R}$, we have*

$$(6.6) \quad \mathcal{A}_{\gamma,P,0}(\alpha) = e^{\frac{i\pi\alpha^2}{2}} \left(\frac{\gamma}{2} \right)^{\frac{\gamma\alpha}{4}} e^{-\frac{\pi\alpha P}{2}} \Gamma\left(1 - \frac{\gamma^2}{4}\right)^{\frac{\alpha}{\gamma}} \frac{\Gamma_{\frac{\gamma}{2}}(Q - \frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(\frac{\gamma}{2} + \frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(Q - \frac{\alpha}{2} - iP)\Gamma_{\frac{\gamma}{2}}(Q - \frac{\alpha}{2} + iP)}{\Gamma_{\frac{\gamma}{2}}(\frac{\gamma}{2})\Gamma_{\frac{\gamma}{2}}(Q - iP)\Gamma_{\frac{\gamma}{2}}(Q + iP)\Gamma_{\frac{\gamma}{2}}(Q - \alpha)}.$$

Corollary 6.3. *If $N \in \mathbb{N}$ and $N < \frac{4}{\gamma^2}$, we have*

$$\mathcal{A}_{\gamma,P,0}(\alpha) = e^{\frac{i\pi\gamma^2 N^2}{2}} \frac{e^{\frac{\pi\gamma P N}{2}}}{\Gamma(1 - \frac{\gamma^2}{4})^N} \prod_{j=1}^N \frac{\Gamma(1 - \frac{j\gamma^2}{4})\Gamma(1 + \frac{(2N-j+1)\gamma^2}{4})}{\Gamma(1 + \frac{j\gamma^2}{4} + \frac{i\gamma P}{2})\Gamma(1 + \frac{j\gamma^2}{4} - \frac{i\gamma P}{2})}.$$

Proof. This follows by specializing Proposition 6.2. □

Theorem 6.4 (Zamolodchikov's recursion for integer parameters). *If $N \in \mathbb{N}$ and $N < \frac{4}{\gamma^2}$, we have*

$$(6.7) \quad \tilde{\mathcal{A}}_{\gamma,P}^q(\alpha) = \sum_{n=1}^{\infty} \sum_{m=1}^N q^{2nm} \frac{R_{\gamma,m,n}^q(\alpha)}{P^2 - P_{m,n}^2} \tilde{\mathcal{A}}_{\gamma,P-m,n}^q(\alpha) + [q^{-\frac{1}{12}}\eta(q)]^{\alpha(Q - \frac{\alpha}{2}) - 2},$$

where $R_{\gamma,m,n}^q(\alpha)$ and $P_{m,n}$ are defined in (2.25) and (2.24).

We defer the proofs of Proposition 6.2 and Theorem 6.4 to Sections 6.4–6.6. First, we show in Theorem 6.5 that Corollary 6.3 and Theorem 6.4 imply the desired result when $N \in \mathbb{N}$.

Theorem 6.5. *For $\gamma \in (0, 2)$, $\alpha \in (-\frac{4}{\gamma}, Q)$, $P \in \mathbb{R}$, and $q \in (0, 1)$, if $N \in \mathbb{N}$ with $0 < N < \frac{4}{\gamma^2}$, then $\tilde{\mathcal{A}}_{\gamma,P}^q(\alpha)$ admits a meromorphic continuation to $P \in \mathbb{C}$ for which $\mathcal{Z}_{\gamma,P}^q(\alpha) = \tilde{\mathcal{A}}_{\gamma,P}^q(\alpha)$ as formal q -series.*

Proof. The meromorphic continuation of $\tilde{\mathcal{A}}_{\gamma,P}^q(\alpha)$ is given by the Dotsenko-Fateev integral (6.5) and the explicit meromorphic expression for $\mathcal{A}_{\gamma,P,0}(\alpha)$ in Corollary 6.3. By Theorem 6.4, (2.23), and (2.26), when $N \in \mathbb{N}$ and $N < \frac{4}{\gamma^2}$, the formal q -series expansions for both $\mathcal{Z}_{\gamma,P}^q(\alpha)$ and the meromorphic continuation of $\tilde{\mathcal{A}}_{\gamma,P}^q(\alpha)$ solve the recursion (6.7). Denoting their difference by

$$\Delta_{\gamma,P}^q(\alpha) = \sum_{n=0}^{\infty} \Delta_{\gamma,P,n}(\alpha) q^n,$$

we find by subtraction that

$$\sum_{n=0}^{\infty} \Delta_{\gamma,P,n}(\alpha) q^n = \sum_{n=1}^{\infty} \sum_{m=1}^N q^{2nm} \frac{R_{\gamma,m,n}^q(\alpha)}{P^2 - P_{m,n}^2} \tilde{\mathcal{A}}_{\gamma,P-m,n}^q(\alpha) \sum_{k=0}^{\infty} \Delta_{\gamma,P,k}(\alpha) q^k.$$

Equating q -series coefficients of both sides expresses $\Delta_{\gamma,P,n}(\alpha)$ as a linear combination of $\Delta_{\gamma,P,m}(\alpha)$ with $m < n$. By the form of the right hand side, we find that $\Delta_{\gamma,P,0}(\alpha) = 0$, hence an induction shows that $\Delta_{\gamma,P,n}(\alpha) = 0$ as needed. \square

6.3. Proof of Theorem 2.11. The final ingredient in our proof is Proposition 6.6 showing that the shift equations (6.2) have a unique solution up to constant factor. For this result and the following proof of Theorem 2.11, we notice the shift equations (6.2) may be put in the form

$$(6.8) \quad X_n(\alpha - \chi) = Y_n(\chi, \alpha) X_n(\alpha + \chi) + Z_n(\chi, \alpha)$$

for unknown functions $X_n(\alpha)$ and functions $Y_n(\chi, \alpha) = \frac{\Gamma_{n,2}\Gamma_{0,1}}{\Gamma_{n,1}\Gamma_{0,2}}$ and

$$(6.9) \quad Z_n(\chi, \alpha) = - \sum_{m=0}^{n-1} \frac{\eta_{\chi,n-m}^-(\alpha)}{\eta_{\chi,0}^-(\alpha)} \tilde{\mathcal{A}}_{\gamma,P,m}(\alpha - \chi) + \frac{\Gamma_{n,2}\Gamma_{0,1}}{\Gamma_{n,1}\Gamma_{0,2}} \sum_{m=0}^{n-1} \frac{\eta_{\chi,n-m}^+(\alpha)}{\eta_{\chi,0}^+(\alpha)} \tilde{\mathcal{A}}_{\gamma,P,m}(\alpha + \chi) \\ + \left(W_{\chi}^-(\alpha, \gamma)^{-1} \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \frac{1 + e^{\pi\chi P + i\pi l_{\chi}}}{1 - e^{\pi\chi P - i\pi l_{\chi}}} V_{\chi,n}^{\alpha,2} + W_{\chi}^-(\alpha, \gamma)^{-1} V_{\chi,n}^{\alpha,1} \right) \eta_{\chi,0}^-(\alpha)^{-1} \mathcal{A}_{\gamma,P,0}(\alpha - \chi)^{-1}$$

with the quantities $W_{\chi}^-(\alpha, \gamma)$ from (5.1), $V_{\chi,n}^{\alpha,i}$ from (4.12) and (4.13), and $\eta_{\chi,m}^{\pm}(\alpha)$ from (5.13) and (5.14).

Proposition 6.6. *For $n > 0$, if the shift equations (6.2) with $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$ hold for $\tilde{\mathcal{A}}_{\gamma,P,n}(\alpha)$ replaced by continuous functions $X_n^1(\alpha), X_n^2(\alpha)$ on a complex neighborhood of $(-\frac{4}{\gamma}, Q)$, and $X_n^1(\alpha_0) = X_n^2(\alpha_0)$ for some $\alpha_0 \in (-\frac{4}{\gamma}, Q)$, then $X_n^1(\alpha) = X_n^2(\alpha)$ on a complex neighborhood of $(-\frac{4}{\gamma}, Q)$.*

Proof. Define $\Delta_n(\alpha) := X_n^1(\alpha) - X_n^2(\alpha)$ so that $\Delta_n(\alpha_0) = 0$. Subtracting the given equations for $i = 1, 2$, we obtain that

$$(6.10) \quad \Delta_n(\alpha - \chi) = Y_n(\chi, \alpha) \Delta_n(\alpha + \chi).$$

Noting that $n > 0$, by the explicit expression

$$Y_n(\chi, \alpha) = \frac{\Gamma(\frac{1}{2} - \frac{1}{2}l_{\chi} + \mathbf{i}\frac{\chi}{2}\sqrt{P^2 + 2n})\Gamma(\frac{1}{2} - \frac{1}{2}l_{\chi} - \mathbf{i}\frac{\chi}{2}\sqrt{P^2 + 2n})}{\Gamma(1 + \frac{1}{2}l_{\chi} + \mathbf{i}\frac{\chi}{2}\sqrt{P^2 + 2n})\Gamma(1 + \frac{1}{2}l_{\chi} - \mathbf{i}\frac{\chi}{2}\sqrt{P^2 + 2n})} \frac{\Gamma(1 + \frac{1}{2}l_{\chi} + \mathbf{i}\frac{\chi}{2}P)\Gamma(1 + \frac{1}{2}l_{\chi} - \mathbf{i}\frac{\chi}{2}P)}{\Gamma(\frac{1}{2} - \frac{1}{2}l_{\chi} + \mathbf{i}\frac{\chi}{2}P)\Gamma(\frac{1}{2} - \frac{1}{2}l_{\chi} - \mathbf{i}\frac{\chi}{2}P)},$$

we find that $Y_n(\chi, \alpha)$ is meromorphic with no real zeroes or poles in α for real P . As a consequence, (6.10) implies that $\Delta_n(\alpha) = 0$ for any $\alpha \in (-\frac{4}{\gamma}, Q)$ which can be reached from α_0 by a sequence of additions or subtractions of γ or $\frac{4}{\gamma}$ without leaving $(-\frac{4}{\gamma}, Q)$. Because $(-\frac{4}{\gamma}, Q)$ has length $\frac{\gamma}{2} + \frac{6}{\gamma} > \gamma + \frac{4}{\gamma}$, we may use (6.10) to extend $\Delta_n(\alpha)$ from $(-\frac{4}{\gamma}, Q)$ to a function on \mathbb{R} which we still denote $\Delta_n(\alpha)$ and which still satisfies (6.10). For this function, for $\gamma^2 \notin \mathbb{Q}$, since $\alpha_0 + \mathbb{Z}\gamma + \mathbb{Z}\frac{4}{\gamma}$ is dense in \mathbb{R} , continuity in α implies that $\Delta_n(\alpha) = 0$ for $\alpha \in (-\frac{4}{\gamma}, Q)$. Continuity in γ then implies that $\Delta_n(\alpha) = 0$ for all $\gamma \in (0, 2)$ and hence that $X_n^1(\alpha) = X_n^2(\alpha)$, as desired. \square

We now prove Theorem 2.11 by combining Proposition 6.6 with a detailed analytic analysis of the shift equations. We will need the following analogue of Lemma 4.8.

Lemma 6.7. *Suppose that $f(\alpha, \chi, w)$ is continuous on $[0, 1]$ in w and analytic in (α, χ) on an open set U . For $\operatorname{Re} a, \operatorname{Re} b > -1$ the functions*

$$g(\alpha, \chi) := \int_0^1 f(\alpha, \chi, w) w^a (1-w)^b dw$$

$$h(\alpha, \chi, w) := \int_0^w f(\alpha, \chi, t) t^a (1-t)^b dt$$

are analytic on (α, χ) in U . Further, $h(\alpha, \chi, w)$ is a continuous function of w on $[0, 1]$.

Proof. The proof is analogous to that of Lemma 4.8, but choosing $\Delta \subset U$ to be a cycle which is a product of the boundary of a solid triangle in either the α - or χ -coordinate and a segment in the other. \square

Proof of Theorem 2.11. It suffices to check that $\mathcal{Z}_{\gamma, P, n}(\alpha) = \tilde{\mathcal{A}}_{\gamma, P, n}(\alpha)$ for all $n > 0$, where we note that $\mathcal{Z}_{\gamma, P, 0}(\alpha) = \tilde{\mathcal{A}}_{\gamma, P, 0}(\alpha) = 1$. We say that a function of α, χ , and w is good if it has meromorphic extension in (α, χ) to a neighborhood of

$$D := \{(\alpha, \chi) \in \mathbb{R}^2 \mid -\chi < \alpha < \chi^{-1}, \chi \in [0, \infty)\}.$$

For $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$, define the normalized expressions

$$\tilde{V}_{\chi, n}^{\alpha, j} := \frac{V_{\chi, n}^{\alpha, j}}{\mathcal{A}_{\gamma, P, 0}(\alpha - \chi)} \quad \tilde{G}_{\chi, n, i}^{\alpha, j}(w) := \frac{G_{\chi, n, i}^{\alpha, j}(w)}{\mathcal{A}_{\gamma, P, 0}(\alpha - \chi)} \quad \tilde{\phi}_{\chi, n, i}^{\alpha, j}(w) := \frac{\phi_{\chi, n, i}^{\alpha, j}(w)}{\mathcal{A}_{\gamma, P, 0}(\alpha - \chi)}.$$

We induct on n to prove the strengthened claim that the following statements hold:

- (a) $\tilde{V}_{\chi, n}^{\alpha, j}$ may be extended to a good function of α and χ independent of γ ;
- (b) $\tilde{G}_{\chi, n, i}^{\alpha, j}(w)$ may be extended to a good function of α, χ , and w independent of γ which is continuous on $[0, 1]$ in w ;
- (c) $\tilde{\phi}_{\chi, n, i}^{\alpha, j}(w)$ may be extended to a good function of α, χ , and w independent of γ which is continuous on $[0, 1]$ in w ;
- (d) $\mathcal{Z}_{\gamma, P, n}(\alpha) = \tilde{\mathcal{A}}_{\gamma, P, n}(\alpha)$.

In what follows, we use the same notation for the extensions of $\tilde{V}_{\chi, n}^{\alpha, j}$, $\tilde{G}_{\chi, n, i}^{\alpha, j}(w)$, and $\tilde{\phi}_{\chi, n, i}^{\alpha, j}(w)$.

Suppose that the claim holds for $m < n$. For (a), notice by definition that $l_\chi, A_{\chi, n}, B_{\chi, n}, C_\chi, v_{j, \chi, n}(w), \Gamma_{n, 1}, \Gamma_{n, 2}, \eta_{\chi, n}^\pm(\alpha)$, and $W_\chi^-(\alpha, \gamma)$ are functions of α, χ , and w independent of γ . We conclude from the defining expressions (4.12) and (4.13) and the inductive hypothesis for $\tilde{\phi}_{\chi, m, i}^{\alpha, j}(w)$ that the same holds for $\tilde{V}_{\chi, n}^{\alpha, j}$. The fact that it is good follows from Lemma 6.7 and the fact that $\phi_{\chi, m, i}^{\alpha, j}(w)$ satisfies Property (R).

For (b), the explicit defining expression for $\tilde{G}_{\chi, n, i}^{\alpha, j}(w)$ in Lemma C.3 shows that it may be extended to a function of α, χ , and w independent of γ . This function is good and continuous on $[0, 1]$ by Lemma 6.7.

For (d), we now show that for $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$, the shift equations (6.2) hold if $\tilde{\mathcal{A}}_{\gamma, P, n}(\alpha)$ is replaced by $\mathcal{Z}_{\gamma, P, n}(\alpha)$. By the explicit expression (6.9), (a) for n , and (d) for $m < n$, we find that $Z_n(\chi, \alpha)$ is a good function of χ and α alone independent of γ . Furthermore, $\mathcal{Z}_{\gamma, P, n}(\alpha)$ and $Y_n(\chi, \alpha)$ are both good functions of α and χ alone. Combining these facts, we conclude that the function

$$F_n(\alpha, \chi) := \mathcal{Z}_{2\chi, P, n}(\alpha - \chi) - Y_n(\chi, \alpha) \mathcal{Z}_{2\chi, P, n}(\alpha + \chi) - Z_n(\chi, \alpha)$$

is a good function of α and χ alone. Now, for $\alpha \in (-\frac{4}{\gamma} + \frac{\gamma}{2}, 0)$, choose $\chi_k := -\frac{\alpha}{2k}$ so that

$$-\frac{\alpha}{2\chi_k} = k \quad 0 < k < \frac{4k^2}{\alpha^2} \text{ for large } k.$$

This implies that for large k we have

$$F_n(\alpha, \chi_k) = \mathcal{Z}_{2\chi_k, P, n}(\alpha - \chi_k) - Y_n(\chi_k, \alpha) \mathcal{Z}_{2\chi_k, P, n}(\alpha + \chi_k) - Z_n(\chi_k, \alpha) = 0$$

by Theorem 6.5 for $\gamma = 2\chi_k$ and Theorem 6.1. Since the sequence χ_k has an accumulation point at 0, this implies by meromorphicity that $F_n(\alpha, \chi) = 0$ for (α, χ) in a neighborhood of D . For $\gamma \in (0, 2)$, setting $\chi = \frac{\gamma}{2}$ and $\chi = \frac{2}{\gamma}$ then yields the desired (6.2) for $\mathcal{Z}_{\gamma, P, n}(\alpha)$.

To complete (d), Theorem 6.1 and what we just showed imply that both $\tilde{\mathcal{A}}_{\gamma,P,n}(\alpha)$ and $\mathcal{Z}_{\gamma,P,n}(\alpha)$ satisfy the shift equations (6.2) for $\chi = \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$. Because both $\tilde{\mathcal{A}}_{\gamma,P,n}(\alpha)$ and $\mathcal{Z}_{\gamma,P,n}(\alpha)$ are continuous in γ and α and equal at $\alpha = 0$, Proposition 6.6 implies they are equal for $\alpha \in (-\frac{4}{\gamma}, Q)$, as needed.

Finally, for (c), substituting $w = 0$ into (4.15) implies that

$$X_{\chi,n,i}^j(\alpha) = \phi_{\chi,n,i}^{\alpha,j}(0) - V_{\chi,n}^{\alpha,j},$$

where by Theorem 5.5 and Proposition 6.2 we have

$$\begin{aligned} \tilde{\phi}_{\chi,n,1}^{\alpha,1}(0) &= W_{\chi}^{-}(\alpha, \gamma) \left[\eta_{\chi,0}^{-}(\alpha) \tilde{\mathcal{A}}_{\gamma,P,n}(\alpha - \chi) + \sum_{m=0}^{n-1} \eta_{\chi,n-m}^{-}(\alpha) \tilde{\mathcal{A}}_{\gamma,P,m}(\alpha - \chi) \right] \\ \tilde{\phi}_{\chi,n,1}^{\alpha,2}(0) &= -W_{\chi}^{-}(\alpha, \gamma) \frac{\Gamma_{0,1}}{\Gamma_{0,2}} \frac{1 - e^{\pi\chi P - i\pi l_{\chi}}}{1 + e^{\pi\chi P + i\pi l_{\chi}}} \frac{\eta_{\chi,0}^{-}(\alpha)}{\eta_{\chi,0}^{+}(\alpha)} \left[\eta_{\chi,0}^{+}(\alpha) \tilde{\mathcal{A}}_{\gamma,P,n}(\alpha + \chi) + \sum_{m=0}^{n-1} \eta_{\chi,n-m}^{+}(\alpha) \tilde{\mathcal{A}}_{\gamma,P,m}(\alpha + \chi) \right]. \end{aligned}$$

Substituting this back into (4.15) and applying (d) shows that

$$\tilde{\phi}_{\chi,n,i}^{\alpha,j}(w) = \tilde{G}_{\chi,n,i}^{\alpha,j}(w) + \frac{X_{\chi,n,i}^j(\alpha)}{\mathcal{A}_{\gamma,P,0}(\alpha - \chi)} v_{j,\chi,n}(w)$$

is a sum of good functions of α , χ , and w which are continuous on $[0, 1]$ in w . This completes the induction and the proof. \square

6.4. Proof of Proposition 6.2. By Theorem 6.1, we find that

$$\mathcal{A}_{\gamma,P,0}(\alpha - \chi) = -\frac{W_{\chi}^{+}(\alpha, \gamma)}{W_{\chi}^{-}(\alpha, \gamma)} \frac{\Gamma_{0,2}}{\Gamma_{0,1}} \frac{1 + e^{\pi\chi P + i\pi l_{\chi}}}{1 - e^{\pi\chi P - i\pi l_{\chi}}} \frac{\eta_{\chi,0}^{+}(\alpha)}{\eta_{\chi,0}^{-}(\alpha)} \mathcal{A}_{\gamma,P,0}(\alpha + \chi).$$

In this expression, by (5.1) and (5.2) we have

$$-\frac{W_{\chi}^{+}(\alpha, \gamma)}{W_{\chi}^{-}(\alpha, \gamma)} = \frac{e^{2i\pi l_{\chi} - 2i\pi\chi^2} (2\pi e^{i\pi})^{-\frac{1}{3}} \left(\frac{\gamma l_{\chi}}{\chi} + \frac{2l_{\chi}}{\chi^2} - 8l_{\chi} + \frac{6l_{\chi}^2}{\chi^2} \right) \pi^{-2l_{\chi}-1} \frac{1 - e^{2\pi\chi P - 2i\pi l_{\chi}}}{\chi(Q-\alpha)} \frac{\Gamma(\frac{\alpha\chi}{2} - \frac{\chi^2}{2} + \frac{2\chi}{\gamma}) \Gamma(1-\alpha\chi) \Gamma(\alpha\chi - \chi^2)}{\Gamma(\frac{\alpha\chi}{2} - \frac{\chi^2}{2}) \Gamma(1 - \frac{\gamma^2}{4}) \frac{2\chi}{\gamma}}}{(2\pi e^{i\pi})^{-\frac{1}{3}} \left(2 + \frac{2\gamma l_{\chi}}{\chi} + \frac{4l_{\chi}}{\chi\gamma} + \frac{6l_{\chi}^2}{\chi^2} \right)} \left(\frac{4}{\gamma^2} \right) \mathbf{1}_{\chi=\frac{2}{\gamma}}$$

and by the reflection and duplication formulas we have

$$\begin{aligned} \frac{\Gamma_{0,2}}{\Gamma_{0,1}} &= \frac{\Gamma(2 - C_{\chi}) \Gamma(C_{\chi} - A_{\chi,0}) \Gamma(C_{\chi} - B_{\chi,0})}{\Gamma(C_{\chi}) \Gamma(1 - A_{\chi,0}) \Gamma(1 - B_{\chi,0})} \\ &= \frac{\Gamma(\frac{3}{2} + l_{\chi}) \Gamma(\frac{1}{2} - \frac{1}{2}l_{\chi} - \mathbf{i}\frac{\chi P}{2}) \Gamma(\frac{1}{2} - \frac{1}{2}l_{\chi} + \mathbf{i}\frac{\chi P}{2})}{\Gamma(\frac{1}{2} - l_{\chi}) \Gamma(1 + \frac{1}{2}l_{\chi} - \mathbf{i}\frac{\chi P}{2}) \Gamma(1 + \frac{1}{2}l_{\chi} + \mathbf{i}\frac{\chi P}{2})} = \frac{2^{2l_{\chi}} \Gamma(\frac{3}{2} + l_{\chi}) \cos(\frac{\pi}{2}l_{\chi} - \mathbf{i}\pi\frac{\chi P}{2}) \cos(\frac{\pi}{2}l_{\chi} + \mathbf{i}\pi\frac{\chi P}{2})}{\pi^3 \Gamma(\frac{1}{2} - l_{\chi}) \Gamma(1 + l_{\chi} - \mathbf{i}\chi P) \Gamma(1 + l_{\chi} + \mathbf{i}\chi P)}, \end{aligned}$$

and by (5.13) and (5.14) we have $\frac{\eta_{\chi,0}^{+}(\alpha)}{\eta_{\chi,0}^{-}(\alpha)} = (2\pi e^{i\pi})^{-\frac{4}{3}l_{\chi} - \frac{2}{3}}$. Putting these together, we find that

$$\mathcal{A}_{\gamma,P,0}(\alpha - \chi) = X_{\chi,\gamma,P}^1(\alpha) X_{\chi,\gamma,P}^2(\alpha) \mathcal{A}_{\gamma,P,0}(\alpha + \chi)$$

for

$$\begin{aligned} X_{\chi,\gamma,P}^1(\alpha) &= e^{2i\pi l_{\chi} - 2i\pi\chi^2} (2\pi e^{i\pi})^{-\frac{1}{3}} \left(\frac{\gamma l_{\chi}}{\chi} + \frac{2l_{\chi}}{\chi^2} - 8l_{\chi} + \frac{6l_{\chi}^2}{\chi^2} \right) \pi^{-2l_{\chi}-1} (2\pi e^{i\pi})^{\frac{1}{3}} \left(2 + \frac{2\gamma l_{\chi}}{\chi} + \frac{4l_{\chi}}{\chi\gamma} + \frac{6l_{\chi}^2}{\chi^2} \right) (2\pi e^{i\pi})^{-\frac{4}{3}l_{\chi} - \frac{2}{3}} \\ &= \pi^{-1} 2^{2l_{\chi}} e^{4i\pi l_{\chi} - 2i\pi\chi^2} \\ X_{\chi,\gamma,P}^2(\alpha) &= \frac{(1 + e^{\pi\chi P - i\pi l_{\chi}})(1 + e^{\pi\chi P + i\pi l_{\chi}})}{\cos(\frac{\pi}{2}l_{\chi} - \mathbf{i}\pi\frac{\chi P}{2}) \cos(\frac{\pi}{2}l_{\chi} + \mathbf{i}\pi\frac{\chi P}{2})} \frac{\Gamma(\frac{2\chi}{\gamma} - l_{\chi}) \Gamma(1 + 2l_{\chi} - \chi^2) \Gamma(-2l_{\chi})}{(1 + 2l_{\chi}) \Gamma(-l_{\chi}) \Gamma(1 - \frac{\gamma^2}{4}) \frac{2\chi}{\gamma}} \\ &\quad \frac{2^{2l_{\chi}} \pi \Gamma(\frac{3}{2} + l_{\chi})}{\Gamma(\frac{1}{2} - l_{\chi}) \Gamma(1 + l_{\chi} - \mathbf{i}\chi P) \Gamma(1 + l_{\chi} + \mathbf{i}\chi P)} \left(\frac{4}{\gamma^2} \right) \mathbf{1}_{\chi=\frac{2}{\gamma}} \\ &= e^{\pi\chi P} 2^{-2l_{\chi}} \pi \Gamma(1 - \frac{\gamma^2}{4})^{-\frac{2\chi}{\gamma}} \frac{\Gamma(\frac{2\chi}{\gamma} - l_{\chi}) \Gamma(1 + 2l_{\chi} - \chi^2) \Gamma(1 + 2l_{\chi})}{\Gamma(1 + l_{\chi}) \Gamma(1 + l_{\chi} - \mathbf{i}\chi P) \Gamma(1 + l_{\chi} + \mathbf{i}\chi P)} \left(\frac{4}{\gamma^2} \right) \mathbf{1}_{\chi=\frac{2}{\gamma}} \end{aligned}$$

We conclude that

$$\frac{\mathcal{A}_{\gamma,P,0}(\alpha - \chi)}{\mathcal{A}_{\gamma,P,0}(\alpha + \chi)} = e^{4i\pi l_\chi - 2i\pi\chi^2} e^{\pi\chi P} \Gamma(1 - \frac{\gamma^2}{4})^{-\frac{2\chi}{\gamma}} \frac{\Gamma(\frac{2\chi}{\gamma} - l_\chi) \Gamma(1 + 2l_\chi - \chi^2) \Gamma(1 + 2l_\chi)}{\Gamma(1 + l_\chi) \Gamma(1 + l_\chi - i\chi P) \Gamma(1 + l_\chi + i\chi P)} (\frac{4}{\gamma^2})^{1_{\chi=\frac{\gamma}{2}}}.$$

Let $A(\alpha)$ be the claimed expression for $\mathcal{A}_{\gamma,P,0}(\alpha)$; by explicit computation, we find that

$$\frac{\mathcal{A}_{\gamma,P,0}(\alpha - \chi)}{\mathcal{A}_{\gamma,P,0}(\alpha + \chi)} = \frac{A(\alpha - \chi)}{A(\alpha + \chi)}.$$

This implies that $\frac{\mathcal{A}_{\gamma,P,0}(\alpha)}{A(\alpha)}$ is doubly periodic with periods $\frac{\gamma}{2}$ and $\frac{\gamma}{\gamma}$. If $\gamma^2 \notin \mathbb{Q}$, this implies by continuity in α that $\mathcal{A}_{\gamma,P,0}(\alpha) = A(\alpha)$, which implies by continuity in γ that $\mathcal{A}_{\gamma,P,0}(\alpha) = A(\alpha)$ for all α, γ .

6.5. Preliminaries for Zamolodchikov's recursion. We now present a key preliminary result for our proof. Proposition 6.8 and Corollary 6.9 give a key identity relating values of $\mathcal{A}_{\gamma,P}^q(\alpha)$ at $P_{m,n}$ and $P_{-m,n}$.

Proposition 6.8. *If $N \in \mathbb{N}$ and $N < \frac{4}{\gamma^2}$, we have*

$$(6.11) \quad \mathcal{A}_{\gamma,P_{m,n}}^q(\alpha) = q^{2n+(m-1)\frac{\gamma^2}{2}} e^{-\frac{i\pi\alpha\gamma}{2}} \mathcal{A}_{\gamma,P_{m-2,n}}^q(\alpha).$$

Proof. Define the functions

$$g_{m,n}(u) := q^{\frac{\alpha^2}{24} - \frac{\alpha}{12}Q + \frac{1}{6}\eta(q)} \frac{5}{4}\alpha\gamma + \frac{2\alpha}{\gamma} - \frac{5}{4}\alpha^2 - 2\Theta_\tau(u) - \frac{\alpha\gamma}{2} + \frac{m\gamma^2}{4} + \frac{\alpha\gamma}{4} \Theta_\tau(1-u) - \frac{m\gamma^2}{4} - \frac{\alpha\gamma}{4} e^{\pi\gamma P_{m,n}u}$$

$$f(P, u) := \left(\int_0^1 \right)^{N-1} \prod_{1 \leq i < j \leq N-1} |\Theta_\tau(x_i - x_j)|^{-\frac{\gamma^2}{2}}$$

$$\prod_{i=1}^{N-1} \Theta_\tau(x_i - u) - \frac{\gamma^2}{4} \Theta_\tau(u - x_i) - \frac{\gamma^2}{4} \Theta_\tau(x_i) - \frac{\alpha\gamma}{2} e^{\pi\gamma P x_i} \prod_{i=1}^{N-1} dx_i,$$

which by the Dotsenko-Fateev integral expression (6.5) satisfy

$$(6.12) \quad \mathcal{A}_{\gamma,P_{m,n}}^q(\alpha) = e^{i\pi(N-1)\frac{\gamma^2}{4}} \int_0^1 g_{m,n}(u) f(P_{m,n}, u) du$$

for $u \in (0, 1)$. In addition, notice that $f(P, u)$ is 1-periodic in u , and, noting that $\pi\gamma P_{m,n} - \frac{i\pi m\gamma^2}{2} = 2\pi i n$, we find

$$g_{m,n}(u+1) = e^{\pi\gamma P_{m,n} + i\pi(\frac{\alpha\gamma}{2} - \frac{m\gamma^2}{4} - \frac{\alpha\gamma}{4})} g_{m,n}(u) = g_{m,n}(u).$$

Define the fundamental domain \mathbb{T}_0 to be the region bounded by $0, 1, \tau, 1 + \tau$. We see that $f(P, u)$ is holomorphic in u on the interior of \mathbb{T}_0 , so integrating along a contour limiting to the boundary of \mathbb{T}_0 , we conclude that

$$(6.13) \quad \int_0^1 g_{m,n}(u) f(P, u) du + \int_1^{1+\tau} g_{m,n}(u) f(P, u) du - \int_0^\tau g_{m,n}(u) f(P, u) du - \int_\tau^{1+\tau} g_{m,n}(u) f(P, u) du = 0.$$

Because both $g_{m,n}(u)$ and $f(P, u)$ are 1-periodic in u , we find that

$$\int_0^\tau g_{m,n}(u) f(P, u) du = \int_1^{1+\tau} g_{m,n}(u) f(P, u) du$$

and thus (6.13) implies that

$$\int_0^1 g_{m,n}(u) f(P, u) du = \int_\tau^{1+\tau} g_{m,n}(u) f(P, u) du = \int_0^1 g_{m,n}(u + \tau) f(P, u + \tau) du.$$

By direct computation we find that

$$g_{m,n}(u + \tau) = e^{\pi P_{m,n}\gamma\tau} e^{-2\pi i(-\frac{\alpha\gamma}{2} + \frac{m\gamma^2}{4} + \frac{\alpha\gamma}{4})(u - \frac{1}{2} + \frac{\tau}{2})} e^{-2\pi i(-\frac{m\gamma^2}{4} - \frac{\alpha\gamma}{4})(u - \frac{1}{2} + \frac{\tau}{2})} g_{m,n}(u)$$

$$= e^{\pi P_{m,n}\gamma\tau} e^{i\pi\alpha\gamma(u - \frac{1}{2} + \frac{\tau}{2})} g_{m,n}(u)$$

$$f(P, u + \tau) = e^{(N-1)(i\pi\gamma^2 u + \frac{i\pi\gamma^2\tau}{2})} f(P - i\gamma, u).$$

Combining these, we find that

$$\begin{aligned} g_{m,n}(u+\tau)f(P_{m,n},u+\tau) &= e^{\pi P_{m,n}\gamma\tau} e^{i\pi\alpha\gamma(u-\frac{1}{2}+\frac{\tau}{2})} e^{(N-1)(i\pi\gamma^2u+\frac{i\pi\gamma^2\tau}{2})} g_{m,n}(u)f(P_{m,n}-i\gamma,u) \\ &= q^{2n+(m-1)\frac{\gamma^2}{2}} e^{-\frac{i\pi\alpha\gamma}{2}} g_{m-2,n}(u)f(P_{m-2,n},u). \end{aligned}$$

Integrating both sides and recalling (6.12), we find as desired that

$$\mathcal{A}_{\gamma,P_{m,n}}^q(\alpha) = q^{2n+(m-1)\frac{\gamma^2}{2}} e^{-\frac{i\pi\alpha\gamma}{2}} \mathcal{A}_{\gamma,P_{m-2,n}}^q(\alpha). \quad \square$$

Corollary 6.9. *If $N \in \mathbb{N}$ and $N < \frac{4}{\gamma^2}$, we have*

$$(6.14) \quad \mathcal{A}_{\gamma,P_{m,n}}^q(\alpha) = q^{2nm} e^{-\frac{i\pi\alpha\gamma m}{2}} \mathcal{A}_{\gamma,P_{-m,n}}^q(\alpha).$$

Proof. This follows by an m -fold application of Proposition 6.8. \square

6.6. Proof of Theorem 6.4. We are now ready to prove Theorem 6.4. Our proof proceeds by studying the $P \rightarrow \infty$ limit and poles of $\tilde{\mathcal{A}}_{\gamma,P}^q(\alpha)$. The $P \rightarrow \infty$ limit is computed in the following Lemma 6.10.

Lemma 6.10. *We have*

$$(6.15) \quad \lim_{P \rightarrow \infty} \tilde{\mathcal{A}}_{\gamma,P}^q(\alpha) = [q^{-\frac{1}{12}}\eta(q)]^{\alpha(Q-\frac{\alpha}{2})-2}.$$

Proof. Recall from (2.2 that $Y_\tau(x) = Y_\infty(x) + F_\tau(x)$, where $F_\tau(x)$ is an almost surely continuous Gaussian random field independent of $Y_\infty(x)$ with $\mathbb{E}[F_\tau(1)^2] = 4 \log(q^{\frac{1}{12}}/\eta(\tau))$. As a result, we have the identity of GMC measures

$$e^{\frac{\gamma}{2}Y_\tau(x)} dx = e^{\frac{\gamma}{2}F_\tau(x) - \frac{\gamma^2}{8}\mathbb{E}[F_\tau(x)^2]} e^{\frac{\gamma}{2}Y_\infty(x)} dx.$$

Applying this identity in the definition of $\tilde{\mathcal{A}}_{\gamma,P}^q(\alpha)$ and multiplying the numerator and denominator of the defining expression for $\tilde{\mathcal{A}}_{\gamma,P}^q(\alpha)$ by $e^{\alpha\pi P}$ yields

$$(6.16) \quad \tilde{\mathcal{A}}_{\gamma,P}^q(\alpha) = [q^{-\frac{1}{12}}\eta(q)]^{\alpha\gamma + \frac{2\alpha}{\gamma} - \alpha^2 - 2} \frac{\mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2}F_\tau(x) - \frac{\gamma^2}{8}\mathbb{E}[F_\tau(x)^2]} e^{\frac{\gamma}{2}Y_\infty(x)} [2 \sin(\pi x)]^{-\frac{\alpha\gamma}{2}} e^{\frac{\alpha\gamma}{4}\mathbb{E}[F_\tau(x)F_\tau(0)]} e^{\gamma\pi P(x-1)} dx \right)^{-\frac{\alpha}{\gamma}} \right]}{\mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\infty(x)} [2 \sin(\pi x)]^{-\frac{\alpha\gamma}{2}} e^{\gamma\pi P(x-1)} dx \right)^{-\frac{\alpha}{\gamma}} \right]}.$$

Fix a small $\varepsilon > 0$. Because $\lim_{P \rightarrow \infty} e^{\gamma P(x-1)} = 0$ for $x \in (0, 1 - \varepsilon)$, we have

$$(6.17) \quad \lim_{P \rightarrow \infty} \int_0^{1-\varepsilon} e^{\frac{\gamma}{2}F_\tau(x) - \frac{\gamma^2}{8}\mathbb{E}[F_\tau(x)^2]} e^{\frac{\gamma}{2}Y_\infty(x)} [2 \sin(\pi x)]^{-\frac{\alpha\gamma}{2}} e^{\frac{\alpha\gamma}{4}\mathbb{E}[F_\tau(x)F_\tau(0)]} e^{\gamma\pi P(x-1)} dx = 0$$

$$(6.18) \quad \lim_{P \rightarrow \infty} \int_0^{1-\varepsilon} e^{\frac{\gamma}{2}Y_\infty(x)} [2 \sin(\pi x)]^{-\frac{\alpha\gamma}{2}} e^{\gamma\pi P(x-1)} dx = 0.$$

Combining (6.17) and (6.18) yields

$$(6.19) \quad \lim_{P \rightarrow \infty} \tilde{\mathcal{A}}_{\gamma,P}^q(\alpha) = \lim_{P \rightarrow \infty} [q^{-\frac{1}{12}}\eta(q)]^{\alpha\gamma + \frac{2\alpha}{\gamma} - \alpha^2 - 2} \frac{\mathbb{E} \left[\left(\int_{1-\varepsilon}^1 e^{\frac{\gamma}{2}F_\tau(x) - \frac{\gamma^2}{8}\mathbb{E}[F_\tau(x)^2]} e^{\frac{\gamma}{2}Y_\infty(x)} [2 \sin(\pi x)]^{-\frac{\alpha\gamma}{2}} e^{\frac{\alpha\gamma}{4}\mathbb{E}[F_\tau(x)F_\tau(0)]} e^{\gamma\pi P(x-1)} dx \right)^{-\frac{\alpha}{\gamma}} \right]}{\mathbb{E} \left[\left(\int_{1-\varepsilon}^1 e^{\frac{\gamma}{2}Y_\infty(x)} [2 \sin(\pi x)]^{-\frac{\alpha\gamma}{2}} e^{\gamma\pi P(x-1)} dx \right)^{-\frac{\alpha}{\gamma}} \right]}.$$

In the rest of the proof, we use the following notations

$$\begin{aligned} \mathfrak{F}_{\gamma,P,\varepsilon}^{q,1}(\alpha) &:= \mathbb{E} \left[\left(\int_{1-\varepsilon}^1 e^{\frac{\gamma}{2}F_\tau(x) - \frac{\gamma^2}{8}\mathbb{E}[F_\tau(x)^2]} e^{\frac{\gamma}{2}Y_\infty(x)} [2 \sin(2\pi x)]^{-\frac{\alpha\gamma}{2}} e^{\frac{\alpha\gamma}{4}\mathbb{E}[F_\tau(x)F_\tau(0)] + \gamma\pi P(x-1)} dx \right)^{-\frac{\alpha}{\gamma}} \right], \\ \mathfrak{F}_{\gamma,P,\varepsilon}^{q,2}(\alpha) &:= \mathbb{E} \left[\left(\int_{1-\varepsilon}^1 e^{\frac{\gamma}{2}F_\tau(x) - \frac{\gamma^2}{8}\mathbb{E}[F_\tau(x)^2]} e^{\frac{\gamma}{2}Y_\infty(x)} [2 \sin(\pi x)]^{-\frac{\alpha\gamma}{2}} e^{\frac{\alpha\gamma}{4}\mathbb{E}[F_\tau(x)F_\tau(0)] + \gamma\pi P(x-1)} dx \right)^{-\frac{\alpha}{\gamma}} \right]. \end{aligned}$$

In $\mathfrak{F}_{\gamma,P,\varepsilon}^{q,2}(\alpha)$, we may bound $F_\tau(x)$ from above and below by $F_\tau(1) \pm \sup_{x \in [1-\varepsilon,1]} |F_\tau(x) - F_\tau(1)|$ for all $x \in [1-\varepsilon,1]$. Owing to this and the independence between Y_∞ and F_τ , we get for some $C = C(\alpha, \gamma) > 0$ that

$$(6.20) \quad \left| \mathfrak{F}_{\gamma,P,\varepsilon}^{q,2}(\alpha) - \mathfrak{F}_{\gamma,P,\varepsilon}^{q,1}(\alpha) \right| \leq \mathbb{E} \left[\exp \left(\frac{\alpha}{2\gamma} \sup_{x \in [1-\varepsilon,1]} |F_\tau(x) - F_\tau(1)| + C \mathbb{E} \left[\sup_{x \in [1-\varepsilon,1]} |F_\tau(x) - F_\tau(1)| \right] \right) - 1 \right] \mathfrak{F}_{\gamma,P,\varepsilon}^{q,1}(\alpha).$$

Since F_τ is almost surely continuous at 1, $\sup_{x \in [1-\varepsilon,1]} |F_\tau(x) - F_\tau(1)|$ converges in probability to 0 as ε goes to 0. Thus, for any $\delta > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, the right hand side of (6.20) is less than δ . Owing to this, for all $\varepsilon < \varepsilon_0$ and some $C_1 = C_1(\alpha, \gamma) > 0$ we have

$$(6.21) \quad \left| \text{r.h.s. of (6.19)} - \lim_{P \rightarrow \infty} [q^{-\frac{1}{12}} \eta(q)]^{\alpha\gamma + \frac{2\alpha}{\gamma} - \alpha^2 - 2} \frac{\mathfrak{F}_{\gamma,P,\varepsilon}^{q,1}(\alpha)}{\mathbb{E} \left[\left(\int_{1-\varepsilon}^1 e^{\frac{\gamma}{2} Y_\infty(x)} [2 \sin(\pi x)]^{-\frac{\alpha\gamma}{2}} e^{\gamma\pi P(x-1)} dx \right)^{-\frac{\alpha}{\gamma}} \right]} \right| \leq C_1 \delta \lim_{P \rightarrow \infty} \frac{\mathfrak{F}_{\gamma,P,\varepsilon}^{q,1}(\alpha)}{\mathbb{E} \left[\left(\int_{1-\varepsilon}^1 e^{\frac{\gamma}{2} Y_\infty(x)} [2 \sin(\pi x)]^{-\frac{\alpha\gamma}{2}} e^{\gamma\pi P(x-1)} dx \right)^{-\frac{\alpha}{\gamma}} \right]}.$$

Because $\mathbb{E}[F_\tau(1)F_\tau(0)] = -4 \log[q^{-\frac{1}{12}} \eta(q)]$ and Y_∞ and $F_\tau(1)$ are independent, we have

$$\begin{aligned} \mathfrak{F}_{\gamma,P,\varepsilon}^{q,1}(\alpha) &= \mathbb{E} \left[\left(\int_{1-\varepsilon}^1 e^{\frac{\gamma}{2} F_\tau(1) - \frac{\gamma^2}{8} \mathbb{E}[F_\tau(1)^2]} e^{\frac{\gamma}{2} Y_\infty(x)} [2 \sin(\pi x)]^{-\frac{\alpha\gamma}{2}} [q^{-\frac{1}{12}} \eta(q)]^{-\alpha\gamma} e^{\gamma P(x-1)} dx \right)^{-\frac{\alpha}{\gamma}} \right] \\ &= \mathbb{E} \left[e^{-\frac{\alpha}{2} F_\tau(1) + \frac{\alpha\gamma}{8} \mathbb{E}[F_\tau(1)^2]} [q^{-\frac{1}{12}} \eta(q)]^{\alpha^2} \right] \mathbb{E} \left[\left(\int_{1-\varepsilon}^1 e^{\frac{\gamma}{2} Y_\infty(x)} [2 \sin(\pi x)]^{-\frac{\alpha\gamma}{2}} e^{\gamma P(x-1)} dx \right)^{-\frac{\alpha}{\gamma}} \right] \\ &= [q^{-\frac{1}{12}} \eta(q)]^{-\frac{\alpha\gamma}{2} + \frac{\alpha^2}{2}} \mathbb{E} \left[\left(\int_{1-\varepsilon}^1 e^{\frac{\gamma}{2} Y_\infty(x)} [2 \sin(\pi x)]^{-\frac{\alpha\gamma}{2}} e^{\gamma P(x-1)} dx \right)^{-\frac{\alpha}{\gamma}} \right], \end{aligned}$$

where the last line follows by noting that

$$\mathbb{E}[e^{-\frac{\alpha}{2} F_\tau(1)}] = [q^{-\frac{1}{12}} \eta(q)]^{-\frac{\alpha^2}{2}}$$

since $\mathbb{E}[F_\tau(1)^2] = -4 \log[q^{-\frac{1}{12}} \eta(q)]$. Substituting the relation in the equation above into both sides of (6.21) shows that for any $\delta > 0$ there exists ε_0 such that for all $\varepsilon < \varepsilon_0$ we have

$$(6.22) \quad \left| \text{r.h.s. of (6.19)} - [q^{-\frac{1}{12}} \eta(q)]^{\alpha(Q - \frac{\alpha}{2}) - 2} \right| \leq C_1 \delta.$$

Since $\tilde{\mathcal{A}}_{\gamma,P}^q(\alpha)$ does not depend on ε , using (6.19), we get

$$\lim_{P \rightarrow \infty} \tilde{\mathcal{A}}_{\gamma,P}^q(\alpha) = \lim_{\varepsilon \rightarrow 0} \text{r.h.s. of (6.19)} = [q^{-\frac{1}{12}} \eta(q)]^{\alpha(Q - \frac{\alpha}{2}) - 2},$$

where the last equality follows by taking the limit $\varepsilon \rightarrow 0$ on both sides of (6.22). \square

Proof of Theorem 6.4. First, notice that $\mathcal{A}_{\gamma,P}^q(\alpha)$ is analytic in P and $\mathcal{A}_{\gamma,P,0}(\alpha)$ has simple zeros at $P = \pm P_{m,n}$ for $n \in \mathbb{N}$ and $1 \leq m \leq N$. We now compute the residue of $\mathcal{A}_{\gamma,P,0}(\alpha)^{-1}$ at each of its poles. Define the function

$$f(P) := \prod_{j=1}^N \Gamma\left(1 + \frac{j\gamma^2}{4} + \frac{\mathbf{i}\gamma P}{2}\right) \Gamma\left(1 + \frac{j\gamma^2}{4} - \frac{\mathbf{i}\gamma P}{2}\right).$$

We find that

$$\begin{aligned} \operatorname{Res}_{P=P_{m,n}} f(P) &= \prod_{j=1}^N \Gamma\left(1 + \frac{j\gamma^2}{4} + n + m\frac{\gamma^2}{4}\right) \prod_{j=1, j \neq m}^N \Gamma\left(1 + \frac{j\gamma^2}{4} - n - m\frac{\gamma^2}{4}\right) \operatorname{Res}_{P=P_{m,n}} \Gamma\left(1 + \frac{m\gamma^2}{4} + \frac{\mathbf{i}\gamma P}{2}\right) \\ &= \frac{2}{\mathbf{i}\gamma} \frac{(-1)^{n-1}}{(n-1)!} \prod_{j=1}^N \Gamma\left(1 + \frac{j\gamma^2}{4} + n + m\frac{\gamma^2}{4}\right) \prod_{j=1, j \neq m}^N \Gamma\left(1 + \frac{j\gamma^2}{4} - n - m\frac{\gamma^2}{4}\right), \end{aligned}$$

where we note that

$$\operatorname{Res}_{P=P_{m,n}} \Gamma\left(1 + \frac{m\gamma^2}{4} + \frac{\mathbf{i}\gamma P}{2}\right) = \frac{2}{\mathbf{i}\gamma} \operatorname{Res}_{x=1-n} \Gamma(x) = \frac{2}{\mathbf{i}\gamma} \frac{(-1)^{n-1}}{(n-1)!}.$$

We now compute

$$\begin{aligned} \frac{\operatorname{Res}_{P=P_{m,n}} f(P)}{f(P_{-m,n})} &= \frac{2}{\mathbf{i}\gamma} \frac{(-1)^{n-1}}{(n-1)!} \frac{\prod_{j=1}^N \prod_{l=-n}^{n-1} \left(1 + \frac{j\gamma^2}{4} + \frac{m\gamma^2}{4} + l\right)}{n! \prod_{j=1, j \neq m}^N \prod_{l=-n+1}^n \left(\frac{j\gamma^2}{4} - \frac{m\gamma^2}{4} + l\right)} \\ &= \frac{2}{\mathbf{i}\gamma} \frac{(-1)^{n-1}}{(n-1)!} \frac{\prod_{j=m+1}^{N+m} \prod_{l=-n+1}^n \left(\frac{j\gamma^2}{4} + l\right)}{n! \prod_{j=1-m, j \neq 0}^{N-m} \prod_{l=-n+1}^n \left(\frac{j\gamma^2}{4} + l\right)} \\ &= \frac{2}{\mathbf{i}\gamma} \frac{(-1)^{n-1}}{(n-1)!} \frac{\prod_{j=N-m+1}^{N+m} \prod_{l=-n+1}^n \left(\frac{j\gamma^2}{4} + l\right)}{n! \prod_{j=1-m, j \neq 0}^m \prod_{l=-n+1}^n \left(\frac{j\gamma^2}{4} + l\right)} \\ &= \frac{2}{\mathbf{i}\gamma} \frac{(-1)^{n-1}}{(n-1)!} \frac{\prod_{j=-m}^{m-1} \prod_{l=-n}^{n-1} \left(1 + \frac{\gamma^2}{4} + \frac{N\gamma^2}{4} + \frac{j\gamma^2}{4} + l\right)}{n! \prod_{j=1-m, j \neq 0}^m \prod_{l=-n+1}^n \left(\frac{j\gamma^2}{4} + l\right)} \\ &= \frac{1}{2P_{m,n}} R_{\gamma, m, n}^q(\alpha). \end{aligned}$$

By the P -dependence of $\mathcal{A}_{\gamma, P, 0}(\alpha)$ from Corollary 6.3, we find that

$$\operatorname{Res}_{P=P_{m,n}} \mathcal{A}_{\gamma, P, 0}(\alpha)^{-1} = e^{\frac{\mathbf{i}\pi\alpha\gamma m}{2}} \frac{1}{2P_{m,n}} R_{\gamma, m, n}^q(\alpha) \mathcal{A}_{\gamma, P_{-m, n}, 0}(\alpha)^{-1}.$$

Combining this with Corollary 6.9, we obtain

$$\operatorname{Res}_{P=P_{m,n}} \tilde{\mathcal{A}}_{\gamma, P}^q(\alpha) = q^{2nm} \frac{1}{2P_{m,n}} R_{\gamma, m, n}^q(\alpha) \tilde{\mathcal{A}}_{\gamma, P_{-m, n}}^q(\alpha).$$

By the definition of $\tilde{\mathcal{A}}_{\gamma, P}^q(\alpha)$ and by Lemma 6.10, we have

$$\lim_{q \rightarrow 0} \tilde{\mathcal{A}}_{\gamma, P}^q(\alpha) = 1 \quad \text{and} \quad \lim_{P \rightarrow \infty} \tilde{\mathcal{A}}_{\gamma, P}^q(\alpha) = [q^{-\frac{1}{12}} \eta(q)]^{\alpha(Q - \frac{\alpha}{2}) - 2}.$$

By the residue expansion and the symmetry

$$\operatorname{Res}_{P=-P_{m,n}} \tilde{\mathcal{A}}_{\gamma, P}^q(\alpha) = - \operatorname{Res}_{P=P_{m,n}} \tilde{\mathcal{A}}_{\gamma, P}^q(\alpha),$$

we obtain as desired that

$$\begin{aligned} \tilde{\mathcal{A}}_{\gamma, P}^q(\alpha) &= \sum_{n=1}^{\infty} \sum_{m=1}^N \frac{2P_{m,n}}{P^2 - P_{m,n}^2} \operatorname{Res}_{P=P_{m,n}} [\tilde{\mathcal{A}}_{\gamma, P}^q(\alpha)] + [q^{-\frac{1}{12}} \eta(q)]^{\alpha(Q - \frac{\alpha}{2}) - 2} \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^N q^{2nm} \frac{R_{\gamma, m, n}^q(\alpha)}{P^2 - P_{m,n}^2} \tilde{\mathcal{A}}_{\gamma, P_{-m, n}}^q(\alpha) + [q^{-\frac{1}{12}} \eta(q)]^{\alpha(Q - \frac{\alpha}{2}) - 2}. \quad \square \end{aligned}$$

APPENDIX A. CONVENTIONS ON THETA AND ELLIPTIC FUNCTIONS

This appendix collects our conventions on theta and elliptic functions and presents a few identities between them which are used in the main text. We fix $q = e^{\mathbf{i}\pi\tau}$ and define the Jacobi theta function by

$$(A.1) \quad \Theta_{\tau}(u) := \vartheta_{11}(e^{\mathbf{i}\pi u}, e^{\mathbf{i}\pi\tau}) = -2q^{1/4} \sin(\pi u) \prod_{k=1}^{\infty} (1 - q^{2k})(1 - 2\cos(2\pi u)q^{2k} + q^{4k}).$$

It satisfies the heat equation

$$(A.2) \quad \mathbf{i}\pi\partial_\tau\Theta_\tau(x) = \frac{1}{4}\Theta_\tau''(x)$$

and the identity

$$(A.3) \quad \frac{\Theta_\tau''(a-b)}{\Theta_\tau(a-b)} + \frac{\Theta_\tau''(a)}{\Theta_\tau(a)} + \frac{\Theta_\tau''(b)}{\Theta_\tau(b)} - 2\frac{\Theta_\tau'(a-b)}{\Theta_\tau(a-b)}\left(\frac{\Theta_\tau'(a)}{\Theta_\tau(a)} - \frac{\Theta_\tau'(b)}{\Theta_\tau(b)}\right) - 2\frac{\Theta_\tau'(a)\Theta_\tau'(b)}{\Theta_\tau(a)\Theta_\tau(b)} - \frac{\Theta_\tau'''(0)}{\Theta_\tau'(0)} = 0.$$

When considering powers of the Jacobi theta function, we choose a branch cut so that we have the identities

$$\begin{aligned} \Theta_\tau(-z)^\alpha &= e^{\mathbf{i}\pi\alpha}\Theta_\tau(z)^\alpha & \text{and} & \quad \Theta_\tau(z+1)^\alpha = e^{-\mathbf{i}\pi\alpha}\Theta_\tau(z). \\ \Theta_\tau(z+\tau)^\alpha &= e^{-2\pi\mathbf{i}\alpha(z-\frac{1}{2}+\frac{\tau}{2})}\Theta_\tau(z)^\alpha & \text{and} & \quad \Theta_\tau(-z-\tau)^\alpha = e^{-2\pi\mathbf{i}\alpha(z+\frac{1}{2}+\frac{\tau}{2})}\Theta_\tau(-z)^\alpha. \end{aligned}$$

We define also the Dedekind eta function by

$$(A.4) \quad \eta(q) = q^{\frac{1}{12}} \prod_{k=1}^{\infty} (1 - q^{2k})$$

so that

$$\Theta_\tau'(0) = -2\pi q^{1/4} \prod_{k=1}^{\infty} (1 - q^{2k})^3 = -2\pi\eta(q)^3.$$

The following identity on $\Theta_\tau(\frac{\tau}{2} + \cdot)$ is used in Section 3.

$$(A.5) \quad \Theta_\tau\left(\frac{\tau}{2} + z\right) = -\mathbf{i}e^{-\mathbf{i}\pi z}q^{-\frac{1}{3}}\eta(q) \prod_{n=1}^{\infty} (1 - q^{2n-1}e^{2\pi iz})(1 - q^{2n-1}e^{-2\pi iz}).$$

We recall the expansion from [DLMF, Equation (23.8.1)] for the Weierstrass \wp function given by

$$(A.6) \quad \wp(u) = \frac{\pi^2}{\sin^2(\pi u)} - 8\pi^2 \sum_{n=1}^{\infty} \frac{nq^{2n}}{1 - q^{2n}} \cos(2\pi nu) - \frac{\pi^2}{3} + 8\pi^2 \sum_{n=1}^{\infty} \frac{q^{2n}}{(1 - q^{2n})^2},$$

which implies that $\wp(u)$ admits a q -expansion

$$(A.7) \quad \wp(u) := \sum_{n=0}^{\infty} \wp_n(u)q^n, \quad \text{where } \wp_n(u) \equiv 0 \text{ for odd } n.$$

We also have the identity

$$(A.8) \quad \wp(u) = \frac{\Theta_\tau'(u)^2}{\Theta_\tau(u)^2} - \frac{\Theta_\tau''(u)}{\Theta_\tau(u)} + \frac{1}{3} \frac{\Theta_\tau'''(0)}{\Theta_\tau'(0)}.$$

Finally, we define the double gamma function $\Gamma_{\frac{\gamma}{2}}(x)$ by

$$(A.9) \quad \log \Gamma_{\frac{\gamma}{2}}(x) := \int_0^\infty \frac{dt}{t} \left[\frac{e^{-xt} - e^{-\frac{Q}{2}t}}{(1 - e^{-\frac{\gamma}{2}t})(1 - e^{-\frac{2t}{\gamma}})} - \frac{(\frac{Q}{2} - x)^2}{2} e^{-t} + \frac{x - \frac{Q}{2}}{t} \right]$$

so that for $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$, we have

$$\Gamma_{\frac{\gamma}{2}}(z + \chi) = \sqrt{2\pi} \frac{\chi^{\chi z - \frac{1}{2}}}{\Gamma(\chi z)} \Gamma_{\frac{\gamma}{2}}(z).$$

To define the u -deformed block in (3.4), we need to define fractional powers of $\Theta_\tau(z)$, for which we recall the following fact.

Lemma A.1. *Suppose f is analytic on a simply connected domain D such that $f(z) \neq 0$ for each $z \in D$. Then, there exists an analytic function g on D such that $f = e^g$.*

We will focus on $\mathfrak{B} := \{z : 0 < \text{Im}(z) < \frac{3}{4} \text{Im}(\tau)\}$. The number $\frac{3}{4}$ is only for convenience, which can be replaced by any number between $(\frac{1}{2}, 1)$. Note that $\Theta_\tau(z) \neq 0$ if $z/2 \in \mathfrak{B}$. Let g be the function on \mathfrak{B} such that $\Theta_\tau(z) = e^g$ and $\text{Im}g(1) = 0$.

Lemma A.2. *For $u \in \mathfrak{B}$, we have $g(u+1) - g(u) = -\pi\mathbf{i}$. Moreover, there exists $q_0 > 0$ such that if $q \in (0, q_0)$, we have $\text{Im}g' < 0$ on \mathfrak{B} .*

Proof. Note that $\overline{g'} = \Theta'_\tau / \Theta_\tau$. Recall now that the log-derivative of $\Theta_\tau(z)$ is given by

$$(A.10) \quad \frac{\Theta'_\tau(z)}{\Theta_\tau(z)} = \pi \frac{\cos(\pi z)}{\sin(\pi z)} + 4\pi \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} \sin(2\pi n z).$$

For $u \in \mathfrak{B}$, we have $\int_u^{u+1} \sin(2\pi n z) dz = 0$ for each positive integer n , and $\int_u^{u+1} \frac{\cos(\pi z)}{\sin(\pi z)} dz = -\mathbf{i}$. This gives $g(u+1) - g(u) = -\pi \mathbf{i}$.

Since $\operatorname{Re}(z) = \frac{e^{-4\pi \operatorname{Im}(z)} - 1}{|e^{2i\pi z} - 1|^2}$ and $\operatorname{Im}(\sin z) = \cos(\operatorname{Re}(z))(e^{\operatorname{Im}(z)} - e^{-\operatorname{Im}(z)})$, by (A.10) we have

$$\begin{aligned} \operatorname{Im} \left(\frac{\Theta'_\tau(z)}{\Theta_\tau(z)} \right) &= \pi \operatorname{Re} \left(\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right) + 2\pi \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} (e^{2\pi n \operatorname{Im}(z)} - e^{-2\pi n \operatorname{Im}(z)}) \cos(2\pi n \operatorname{Re}(z)) \\ &= -\pi \frac{1 - e^{-4\pi \operatorname{Im}(z)}}{|e^{2i\pi z} - 1|^2} + 2\pi \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} (e^{2\pi n \operatorname{Im}(z)} - e^{-2\pi n \operatorname{Im}(z)}) \cos(2\pi n \operatorname{Re}(z)). \end{aligned}$$

Since $\operatorname{Im}(z) > 0$, we have $\pi \frac{1 - e^{-4\pi \operatorname{Im}(z)}}{|e^{2i\pi z} - 1|^2} > \frac{\pi}{4} (1 - e^{-4\pi \operatorname{Im}(z)})$. Note that

$$2\pi \sum_{n=1}^{\infty} \frac{q^{2n}}{1-q^{2n}} (e^{2\pi n \operatorname{Im}(z)} - e^{-2\pi n \operatorname{Im}(z)}) \cos(2\pi n \operatorname{Re}(z)) < \frac{2\pi}{1-q^2} \left(\frac{q^2 e^{2\pi \operatorname{Im}(z)}}{1-q^2 e^{2\pi \operatorname{Im}(z)}} - \frac{q^2 e^{-2\pi \operatorname{Im}(z)}}{1-q^2 e^{-2\pi \operatorname{Im}(z)}} \right).$$

Set $h(x) = \frac{q^2 x}{1-q^2 x}$. Since $h'(x) = \frac{q^2}{(1-q^2 x)^2} \leq \frac{q^2}{(1-q^2 e^{2\pi \operatorname{Im}(z)})^2}$ for $x \in [e^{-2\pi \operatorname{Im}(z)}, e^{2\pi \operatorname{Im}(z)}]$, we have

$$\begin{aligned} \frac{2\pi}{1-q^2} \left(\frac{q^2 e^{2\pi \operatorname{Im}(z)}}{1-q^2 e^{2\pi \operatorname{Im}(z)}} - \frac{q^2 e^{-2\pi \operatorname{Im}(z)}}{1-q^2 e^{-2\pi \operatorname{Im}(z)}} \right) &< \frac{2\pi}{1-q^2} \frac{q^2}{(1-q^2 e^{2\pi \operatorname{Im}(z)})^2} (e^{2\pi \operatorname{Im}(z)} - e^{-2\pi \operatorname{Im}(z)}) \\ &= \frac{2\pi}{1-q^2} \frac{q^2 e^{2\pi \operatorname{Im}(z)}}{(1-q^2 e^{2\pi \operatorname{Im}(z)})^2} (1 - e^{-4\pi \operatorname{Im}(z)}) \\ &< \frac{2\pi}{1-q_0^2} \frac{q_0}{(1-q_0)^2} (1 - e^{-4\pi \operatorname{Im}(z)}). \end{aligned}$$

When $\operatorname{Im} z < \frac{3}{4} \operatorname{Im} \tau$, we have $q^2 e^{2\pi \operatorname{Im} z} < q^{\frac{1}{2}}$. By the monotonicity of $\frac{x}{(1-x)^2}$ on $(0, 1)$, we have

$$\frac{2\pi}{1-q^2} \frac{q^2 e^{2\pi \operatorname{Im}(z)}}{(1-q^2 e^{2\pi \operatorname{Im}(z)})^2} (1 - e^{-4\pi \operatorname{Im}(z)}) < \frac{2\pi}{1-q^2} \frac{q^{\frac{1}{2}}}{(1-q^{\frac{1}{2}})^2} (1 - e^{-4\pi \operatorname{Im}(z)}).$$

Since $\lim_{q \rightarrow 0} \frac{2\pi}{1-q^2} \frac{q^{\frac{1}{2}}}{(1-q^{\frac{1}{2}})^2} = 0$, we have the existence of q_0 with the desired property. \square

Lemma A.3. Fix $q \in (0, q_0)$ and $u \in \mathfrak{B}$. Let the straight line between $\Theta_\tau(u)$ and $\Theta_\tau(u+1) = -\Theta_\tau(u)$ divide the complex plane into two open half planes \mathbb{H}_u^- and \mathbb{H}_u^+ , where \mathbb{H}_u^- contains a small clockwise rotation of $\Theta_\tau(u)$ viewed as a vector. Then, for $x \in (0, 1)$, we have $\Theta_\tau(u+x) \in \mathbb{H}_u^-$.

Proof. Let $f(x) = \operatorname{Im} g(u+x)$ for $x \in \mathbb{R}$. It suffices to show that if $\operatorname{Im}(u) \in (0, \frac{1}{2} \operatorname{Im}(\tau))$, we have

$$(A.11) \quad f(1) - f(0) = -\pi \quad \text{and} \quad f'(x) < 0 \quad \text{for all } x \in \mathbb{R}.$$

By Lemma A.2, $g(u+1) - g(u) = -\pi \mathbf{i}$ and $\operatorname{Im} g' < 0$ on \mathfrak{B} . Therefore $f(1) - f(0) = -\pi$ and $f'(x) < 0$ for $x \in \mathbb{R}$. \square

The function g extends to $\partial \mathfrak{B}$ piecewise continuously.

Lemma A.4. Fix $q \in (0, q_0)$ and $c \in (0, 1]$. Given a finite measure ν whose support equals $[0, 1]$, let $f_\nu(u) := \int_0^1 e^{cg(u+x)} \nu(dx)$ for $u \in \overline{\mathfrak{B}} := \mathfrak{B} \cup \partial \mathfrak{B}$. Then $f_\nu(u)$ is analytic on \mathfrak{B} and continuous on $\overline{\mathfrak{B}}$. Moreover, $f_\nu(u+1) = e^{-c\pi \mathbf{i}} f_\nu(u)$, f_ν is nonzero on $[0, 1]$, and $f_\nu(1) > 0$.

Proof. Since g is piecewise continuous on $\partial \mathfrak{B}$, f_ν is continuous on $\overline{\mathfrak{B}}$. Because

$$e^{-cg(u)} f_\nu(u) = \int_0^1 e^{cg(u+x) - cg(u)} \nu(dx),$$

by Lemma A.3, we have $\operatorname{Im}(e^{cg(u+x) - cg(u)}) < 0$ for $x \in (0, 1)$. Therefore $\operatorname{Im}(e^{-cg(u)} f_\nu(u)) < 0$, hence $f_\nu(u) \neq 0$. Since $g(z+1) - g(z) = -\pi \mathbf{i}$ for $z \in \mathfrak{B}$, we have $f_\nu(u+1) = e^{-c\pi \mathbf{i}} f_\nu(u)$.

If $c \in (0, 1)$, for $u \in (0, 1)$, we have $\text{Im}(e^{cg(u+x)}) > 0$ for $x \in (0, 1-u)$. Since the support of m is $[0, 1]$, we have $f_\nu(u) \neq 0$. On the other hand, since $\Theta_\tau(1+x) > 0$ for $x \in (0, 1)$, we have $f(1) > 0$. \square

Definition A.5. In Lemma A.4, let h be the function on \mathfrak{B} such that $f = e^h$ on \mathfrak{B} and $\lim_{z \rightarrow 1} \text{Im } h(z) = 0$. For each $\beta \in \mathbb{R}$, define $f_\nu^\beta := e^{\beta h}$.

APPENDIX B. SOME USEFUL FACTS IN PROBABILITY

In this appendix, we present a few probabilistic facts used throughout the paper.

Lemma B.1. (*Moments of GMC*) For $\gamma \in (0, 2)$, and $\alpha \in (-\frac{4}{\gamma}, Q)$, we have

$$0 < \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\infty(x)} \sin(\pi x)^{-\frac{\alpha\gamma}{2}} dx \right)^{-\frac{\alpha}{\gamma}} \right] < \infty.$$

Similarly for $\gamma \in (0, 2)$, $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$, $u \in \mathfrak{B}$, $P \in \mathbb{R}$, and $\alpha \in (-\frac{4}{\gamma} + \chi, Q)$, we have

$$0 < \mathbb{E} \left[\left| \int_0^1 e^{\frac{\gamma}{2} Y_\infty(x)} (2 \sin(\pi x))^{-\frac{\alpha\gamma}{2}} \Theta_\tau(x+u)^{\frac{\chi\gamma}{2}} e^{\pi\gamma Px} dx \right|^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] < \infty.$$

Proof. For the first claim, since the function integrated against the GMC measure is positive, we are in the classical case of the existence of a moment of GMC with an insertion of weight α . Following [DKRV16, Lemma 3.10], adapted here to the case of one-dimensional GMC, the condition is thus $\alpha < Q$ and $-\frac{\alpha}{\gamma} < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q-\alpha)$ which is equivalent simply to $\alpha \in (-\frac{4}{\gamma}, Q)$. The second claim is more difficult since the integrand $\Theta_\tau(x+u)^{\frac{\chi\gamma}{2}}$ is a complex valued quantity. For the case of positive moments where $\alpha \in (-\frac{4}{\gamma} + \chi, \chi)$, one can simply use the bound

$$\mathbb{E} \left[\left| \int_0^1 e^{\frac{\gamma}{2} Y_\infty(x)} (2 \sin(\pi x))^{-\frac{\alpha\gamma}{2}} \Theta_\tau(x+u)^{\frac{\chi\gamma}{2}} e^{\pi\gamma Px} dx \right|^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] \leq M \mathbb{E} \left[\left| \int_0^1 e^{\frac{\gamma}{2} Y_\infty(x)} \sin(\pi x)^{-\frac{\alpha\gamma}{2}} dx \right|^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right],$$

which is valid for some constant $M > 0$. The claim then reduces to the positive case. For negative moments corresponding to $\alpha \in (\chi, Q)$, we need to lower bound our expectation by the positive case. We know thanks to Lemma A.2 that for all $x \in (0, 1)$, $\Theta_\tau(x+u)^{\frac{\chi\gamma}{2}}$ remains strictly contained in a half-space, touching the boundary of the half-space only at $x = 0$ and $x = 1$. The GMC measure is thus a strictly positive sum of vectors strictly contained in this half space, which implies the claim

$$\mathbb{E} \left[\left| \int_0^1 e^{\frac{\gamma}{2} Y_\infty(x)} (2 \sin(\pi x))^{-\frac{\alpha\gamma}{2}} \Theta_\tau(x+u)^{\frac{\chi\gamma}{2}} e^{\pi\gamma Px} dx \right|^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] \geq M' \mathbb{E} \left[\left| \int_0^1 e^{\frac{\gamma}{2} Y_\infty(x)} \sin(\pi x)^{-\frac{\alpha\gamma}{2}} dx \right|^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right]$$

for some $M' > 0$. Therefore this case also reduces to the positive case, providing the desired bound. \square

Next, we state a version of Girsanov's theorem for GMC used frequently in the main text, a version of Kahane's inequality, and the Williams decomposition theorem from [Wil74].

Theorem B.2. Let $Y(x)$ be either of the Gaussian fields $Y_\infty(x)$ or $Y_\tau(x)$ on $[0, 1]$ defined in Section 2.1. Let \mathcal{X} be a Gaussian variable measurable with respect to Y , and let F be a bounded continuous function. Then we have

$$(B.1) \quad \mathbb{E} \left[e^{\mathcal{X} - \frac{1}{2} \mathbb{E}[\mathcal{X}^2]} \int_0^1 e^{\frac{\gamma}{2} Y(x)} dx \right] = \mathbb{E} \left[\int_0^1 e^{\frac{\gamma}{2} \mathbb{E}[\mathcal{X} Y(x)]} e^{\frac{\gamma}{2} Y(x)} dx \right].$$

Theorem B.3 (Kahane's inequality). Let $(Z_0(x))_{x \in D}$, $(Z_1(x))_{x \in D}$ be two continuous centered Gaussian processes such that for all $x, y \in D$ we have

$$|\mathbb{E}[Z_0(x)Z_0(y)] - \mathbb{E}[Z_1(x)Z_1(y)]| \leq C.$$

For $u \in [0, 1]$, define

$$Z_u = \sqrt{1-u}Z_0 + \sqrt{u}Z_1, \quad W_u = \int_D e^{Z_u(x) - \frac{1}{2} \mathbb{E}[Z_u(x)^2]} \sigma(dx).$$

For all smooth functions F with at most polynomial growth at infinity and σ a complex Radon measure over D , we have

$$\left| \mathbb{E} \left[F \left(\int_D e^{Z_0(x) - \frac{1}{2} \mathbb{E}[Z_0(x)^2]} \sigma(dx) \right) \right] - \mathbb{E} \left[F \left(\int_D e^{Z_1(x) - \frac{1}{2} \mathbb{E}[Z_1(x)^2]} \sigma(dx) \right) \right] \right| \leq \sup_{u \in [0,1]} \frac{C}{2} \mathbb{E}[|W_u|^2 |F''(W_u)|].$$

Theorem B.4. Let $(B_s - vs)_{s \geq 0}$ be a Brownian motion with negative drift, i.e. $v > 0$ and let $M = \sup_{s \geq 0} (B_s - vs)$. Then conditionally on M the law of the path $(B_s - vs)_{s \geq 0}$ is given by the joining of the following two independent paths:

1. A Brownian motion $(B_s^1 + vs)_{0 \leq s \leq \tau_M}$ with positive drift v run until its hitting time τ_M of M .
2. $(M + B_t^2 - vt)_{t \geq 0}$ where $(B_t^2 - vt)_{t \geq 0}$ is a Brownian motion with negative drift conditioned to stay negative.

Moreover, for all $C > 0$ we have

$$(B.2) \quad (B_{\tau_C}^1 + v(\tau_C - s) - C)_{0 \leq s \leq \tau_C} \stackrel{d}{=} (\tilde{B}_s - vs)_{0 \leq s \leq L_{-C}},$$

where τ_C denotes the hitting time of C , $(\tilde{B}_s - vs)_{s \geq 0}$ is a Brownian motion with drift $-v$ conditioned to stay negative, and L_{-C} is the last time $(\tilde{B}_s - vs)_{s \geq 0}$ hits $-C$.

APPENDIX C. INHOMOGENEOUS GAUSS HYPERGEOMETRIC EQUATIONS

This appendix presents background on the Gauss hypergeometric equation and a construction of a particular solution to the inhomogeneous version. For parameters A, B, C , the (inhomogeneous) Gauss hypergeometric equation with inhomogeneous part $g(w)$ is the second order ODE

$$(C.1) \quad (w(1-w)\partial_w + (C - (1+A+B)w)\partial_w - AB)f(w) = g(w)$$

for an unknown function $f(w)$. If $g(w) = 0$, the equation (C.1) is homogeneous and has 2-dimensional solution space spanned by $v_1(w)$ and $w^{1-C}v_2(w)$ for the functions

$$v_1(w) = {}_2F_1(A, B, C; w) \quad \text{and} \quad v_2(w) = {}_2F_1(1+A-C, 1+B-C, 2-C; w)$$

holomorphic near $w = 0$. Both $v_1(w)$ and $v_2(w)$ satisfy the Property (R) defined in Definition 4.3.

A separate basis of solutions to (C.1) with similar good behavior at $w = 1$ is given by

$${}_2F_1(A, B, 1+A+B-C, 1-w) \quad \text{and} \quad (1-w)^{C-A-B} {}_2F_1(C-A, C-B, 1+C-A-B, 1-w).$$

These two bases of solutions are related by *connection equations*, one of which is

$$(C.2) \quad {}_2F_1(A, B, 1+A+B-C, 1-w) = \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} v_1(w) + \frac{\Gamma(2-C)\Gamma(C-A-B)}{\Gamma(1-A)\Gamma(1-B)} w^{1-C} v_2(w).$$

If $\text{Re}[C] > \text{Re}[A+B]$, they satisfy Gauss's identity

$$(C.3) \quad {}_2F_1(A, B, C, 1) = \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)},$$

and the function $\frac{{}_2F_1(A, B, C, w)}{\Gamma(C)}$ is holomorphic as a function of A, B, C . In particular, this means that ${}_2F_1(A, B, C, w)$ is holomorphic for $C \notin \{0, -1, -2, \dots\}$.

If $g(w)$ is not identically zero, then we may write

$$(C.4) \quad f(w) = f_{\text{homog}}(w) + f_{\text{part}}(w),$$

where $f_{\text{part}}(w)$ is a particular solution solving (C.1) and $f_{\text{homog}}(w)$ solves the homogeneous version of (C.1).

Applying the variation of parameters method with homogeneous solutions $\{v_1(w), w^{1-C}v_2(w)\}$ yields the particular solution

$$(C.5) \quad f_{\text{part}}(w) := -\frac{v_1(w)}{1-C} \int_0^w \frac{v_2(t)g(t)}{(1-t)^{C-A-B}} dt + \frac{w^{1-C}v_2(w)}{1-C} \int_0^w \frac{v_1(t)g(t)}{t^{1-C}(1-t)^{C-A-B}} dt.$$

Lemma C.1. *Suppose $\operatorname{Re}(C - A - B) \in (0, 1)$. Fix $X \in \{0, 1 - C\}$. Suppose $g(w) = w^X \tilde{g}(w)$, and $\tilde{g}(w)$ is a function satisfying Property (R). Then the solution $f_{\text{part}}(w)$ defined in (C.5) satisfies Property (R) is of the form $w^X \tilde{f}(w)$ with \tilde{f} satisfying Property (R) if $X = 1 - C$.*

Proof. We note that $w^{1-C} \int_0^w \frac{v_1(t)g(t)}{t^{1-C}(1-t)^{C-A-B}} dt = w \int_0^1 \frac{v_1(wt)g(wt)}{t^{1-C}(1-wt)^{C-A-B}} dt = w \int_0^1 \frac{v_1(wt)(wt)^X \tilde{g}(wt)}{t^{1-C}(1-wt)^{C-A-B}} dt$. Since $\int_0^1 \frac{v_1(wt)(wt)^X \tilde{g}(wt)}{t^{1-C}(1-wt)^{C-A-B}} dt$ satisfies Property (R), we are done. \square

Corollary C.2. *Suppose $g(w)$ is of the form $g_1(w) + w^{1-C}g_2(w)$, where g_1 and g_2 satisfy Property (R). Then any solution $f(w)$ to (C.1) can be written in the same form.*

We give in Lemma C.3 one construction of a family of particular solutions.

Lemma C.3. *Suppose $g(w) = w^X \tilde{g}(w)$, where $X \in \{0, 1 - C\}$ and $\tilde{g}(w)$ satisfies Property (R). The equation (C.1) has particular solution*

$$f(w) = f_{\text{part}}(w) + \begin{cases} Zv_1(w) & X = 0 \\ Z'w^{1-C}v_2(w) & X = 1 - C, \end{cases}$$

where

$$Z = \frac{1}{1-C} \int_0^1 \frac{v_2(t)g(t)}{(1-t)^{C-A-B}} dt - \frac{1}{1-C} \frac{v_2(1)}{v_1(1)} \int_0^1 \frac{v_1(t)g(t)}{t^{1-C}(1-t)^{C-A-B}} dt$$

$$Z' = \frac{1}{1-C} \frac{v_1(1)}{v_2(1)} \int_0^1 \frac{v_2(t)g(t)}{(1-t)^{C-A-B}} dt - \frac{1}{1-C} \int_0^1 \frac{v_1(t)g(t)}{t^{1-C}(1-t)^{C-A-B}} dt.$$

This solution satisfies $f(1) = 0$ and satisfies Property (R) if $X = 0$ and is of the form $w^{1-C} \tilde{f}(w)$ with \tilde{f} satisfying Property (R) if $X = 1 - C$.

Proof. Direct computation shows that the claimed function $f(w)$ is a particular solution to (C.1) and that $f(1) = 0$. The analytic properties follow from Lemma C.1 and the fact that $v_1(w)$ and $v_2(w)$ both satisfy Property (R). \square

APPENDIX D. PROOF OF OPE LEMMAS

In this appendix, we provide the proofs of Lemmas 5.2 and 5.3, which were used in the proof of the OPE in Section 5.

Proof of Lemma 5.2. Recall the notation $l_0 = l_{\frac{\gamma}{2}}$ from (5.4). We start with

(D.1)

$$\begin{aligned} & \sin(\pi u)^{-2l_0-1} \left(\phi_{\frac{\gamma}{2}}^\alpha(u, q) - \phi_{\frac{\gamma}{2}}^\alpha(0, q) \right) \\ &= \sin(\pi u)^{-2l_0-1} q^{\frac{P^2}{2} + \frac{l_0}{6}(1 - \frac{4l_0}{\gamma^2})} \Theta'_\tau(0)^{-\frac{8}{3} \frac{l_0^2 - l_0}{\gamma^2} + \frac{1}{3} l_0} \left(e^{\frac{\gamma P u \pi}{2}} \sin(\pi u)^{l_0} \Theta_\tau(u)^{-l_0} - \pi^{l_0} \Theta'_\tau(0)^{-l_0} \right) \\ & \quad \times \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta(x)^{-\frac{\alpha\gamma}{2}} \Theta_\tau(u+x)^{\frac{\gamma^2}{4}} e^{\pi\gamma P x} dx \right)^{-\frac{\alpha}{\gamma} + \frac{1}{2}} \right] \\ &+ \sin(\pi u)^{-2l_0-1} q^{\frac{P^2}{2} + \frac{l_0}{6}(1 - \frac{4l_0}{\gamma^2})} \Theta'_\tau(0)^{-\frac{8}{3} \frac{l_0^2 - l_0}{\gamma^2} + \frac{1}{3} l_0} \pi^{l_0} \Theta'_\tau(0)^{-l_0} \\ & \quad \times \left(\mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} \Theta_\tau(u+x)^{\frac{\gamma^2}{4}} e^{\pi\gamma P x} dx \right)^{-\frac{\alpha}{\gamma} + \frac{1}{2}} \right] - \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2} + \frac{\gamma^2}{4}} e^{\pi\gamma P x} dx \right)^{-\frac{\alpha}{\gamma} + \frac{1}{2}} \right] \right). \end{aligned}$$

By our bound on α , we see that $-2l_0 > 0$, and thus we have

$$\lim_{u \rightarrow 0} \sin(\pi u)^{-2l_0-1} \left(e^{\frac{\gamma P u \pi}{2}} \sin(\pi u)^{l_0} \Theta_\tau(u)^{-l_0} - \pi^{l_0} \Theta'_\tau(0)^{-l_0} \right) = \lim_{u \rightarrow 0} O(u^{-2l_0}) = 0.$$

For the other piece, define the functions

$$g(u) := \int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} \Theta_\tau(u+x)^{\frac{\gamma^2}{4}} e^{\pi\gamma P x} dx \quad \text{and} \quad g(t, u) := (1-t)g(0) + tg(u).$$

For $f(u) = \mathbb{E}[g(u)^{-\frac{\alpha}{\gamma} + \frac{1}{2}}]$, we obtain

$$f(u) - f(0) = \int_0^1 \partial_t \mathbb{E}[g(t, u)^{-\frac{\alpha}{\gamma} + \frac{1}{2}}] dt = \left(-\frac{\alpha}{\gamma} + \frac{1}{2}\right) \int_0^1 \mathbb{E}[(g(u) - g(0))g(t, u)^{-\frac{\alpha}{\gamma} - \frac{1}{2}}] dt.$$

Now, for all $t \in [0, 1]$, by Girsanov's theorem (Theorem B.2) we have

$$\begin{aligned} & \mathbb{E}[(g(u) - g(0))g(t, u)^{-\frac{\alpha}{\gamma} - \frac{1}{2}}] \\ &= \int_0^1 \Theta_\tau(y)^{-\frac{\alpha\gamma}{2}} \left(\Theta_\tau(u+y)^{\frac{\gamma^2}{4}} - \Theta_\tau(y)^{\frac{\gamma^2}{4}} \right) e^{\pi\gamma Py} \\ & \quad \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} q^{\frac{\gamma^2}{12}} \eta(q)^{\frac{\gamma^2}{2}} \frac{\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}}}{|\Theta_\tau(x-y)|^{\frac{\gamma^2}{2}}} \left((1-t)\Theta_\tau(x)^{\frac{\gamma^2}{4}} + t\Theta_\tau(u+x)^{\frac{\gamma^2}{4}} \right) e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma} - \frac{1}{2}} \right] dy. \end{aligned}$$

For $\delta \in (\frac{1}{1-2l_0}, 1)$, if $y \in [u^{1-\delta}, 1 - u^{1-\delta}]$, then

$$\sin(\pi u)^{-2l_0-1} \Theta_\tau(y)^{-\frac{\alpha\gamma}{2}} \left(\Theta_\tau(u+y)^{\frac{\gamma^2}{4}} - \Theta_\tau(y)^{\frac{\gamma^2}{4}} \right) e^{\pi\gamma Py} = O(u^{\delta(1-2l_0)-1}) = o(1),$$

which by the dominated convergence theorem implies that

$$\begin{aligned} & \lim_{u \rightarrow 0} \sin(\pi u)^{-2l_0-1} \int_{u^{1-\delta}}^{1-u^{1-\delta}} \Theta_\tau(y)^{-\frac{\alpha\gamma}{2}} \left(\Theta_\tau(u+y)^{\frac{\gamma^2}{4}} - \Theta_\tau(y)^{\frac{\gamma^2}{4}} \right) e^{\pi\gamma Py} \\ & \quad \times \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} q^{\frac{\gamma^2}{12}} \eta(q)^{\frac{\gamma^2}{2}} \frac{\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}}}{|\Theta_\tau(x-y)|^{\frac{\gamma^2}{2}}} \left((1-t)\Theta_\tau(x)^{\frac{\gamma^2}{4}} + t\Theta_\tau(u+x)^{\frac{\gamma^2}{4}} \right) e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma} - \frac{1}{2}} \right] dy = 0. \end{aligned}$$

As a result, we only need to study the limit when the integration variable y is contained in the two intervals $[0, u^{1-\delta}]$ and $[1 - u^{1-\delta}, 1]$. Let us first focus on $[0, u^{1-\delta}]$. We have that

$$\begin{aligned} & \lim_{u \rightarrow 0} \sin(\pi u)^{-2l_0-1} \int_0^{u^{1-\delta}} \Theta_\tau(y)^{-\frac{\alpha\gamma}{2}} \left(\Theta_\tau(u+y)^{\frac{\gamma^2}{4}} - \Theta_\tau(y)^{\frac{\gamma^2}{4}} \right) e^{\pi\gamma Py} \\ & \quad \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} q^{\frac{\gamma^2}{12}} \eta(q)^{\frac{\gamma^2}{2}} \frac{\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}}}{|\Theta_\tau(x-y)|^{\frac{\gamma^2}{2}}} \left((1-t)\Theta_\tau(x)^{\frac{\gamma^2}{4}} + t\Theta_\tau(u+x)^{\frac{\gamma^2}{4}} \right) e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma} - \frac{1}{2}} \right] dy \\ &= \lim_{u \rightarrow 0} \pi^{-1} \sin(\pi u)^{-2l_0} \int_0^{u^{-\delta}} \Theta_\tau(uz)^{-\frac{\alpha\gamma}{2}} \left(\Theta_\tau(u+uz)^{\frac{\gamma^2}{4}} - \Theta_\tau(uz)^{\frac{\gamma^2}{4}} \right) e^{\pi\gamma Puz} \\ & \quad \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} q^{\frac{\gamma^2}{12}} \eta(q)^{\frac{\gamma^2}{2}} \frac{\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}}}{|\Theta_\tau(x-uz)|^{\frac{\gamma^2}{2}}} \left((1-t)\Theta_\tau(x)^{\frac{\gamma^2}{4}} + t\Theta_\tau(u+x)^{\frac{\gamma^2}{4}} \right) e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma} - \frac{1}{2}} \right] dz \\ &= \pi^{-1} e^{2i\pi l_0 - \frac{i\pi\gamma^2}{2}} q^{-\frac{\alpha\gamma}{2} - \frac{\gamma^2}{24}} \eta(q)^{-\frac{\alpha\gamma}{2} - \frac{\gamma^2}{4}} \int_0^\infty \lim_{u \rightarrow 0} \sin(\pi u)^{-2l_0} \Theta_\tau(uz)^{-\frac{\alpha\gamma}{2}} \left(\Theta_\tau(u+uz)^{\frac{\gamma^2}{4}} - \Theta_\tau(uz)^{\frac{\gamma^2}{4}} \right) e^{\pi\gamma Puz} dz \\ & \quad \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2} - \frac{\gamma^2}{4}} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma} - \frac{1}{2}} \right], \end{aligned}$$

where in the last step we have again applied dominated convergence. For $|z| \leq u^{-\delta}$, we have

$$\begin{aligned} & \lim_{u \rightarrow 0} \sin(\pi u)^{-2l_0} \Theta_\tau(uz)^{-\frac{\alpha\gamma}{2}} \left(\Theta_\tau(u+uz)^{\frac{\gamma^2}{4}} - \Theta_\tau(uz)^{\frac{\gamma^2}{4}} \right) e^{\pi\gamma Puz} \\ &= \pi^{-2l_0} \lim_{u \rightarrow 0} u^{-2l_0} (uz\Theta'_\tau(0))^{-\frac{\alpha\gamma}{2}} \left((u(1+z)\Theta'_\tau(0))^{\frac{\gamma^2}{4}} - (uz\Theta'_\tau(0))^{\frac{\gamma^2}{4}} \right) \\ &= \pi^{-2l_0} \Theta'_\tau(0)^{2l_0} z^{-\frac{\alpha\gamma}{2}} \left((1+z)^{\frac{\gamma^2}{4}} - z^{\frac{\gamma^2}{4}} \right). \end{aligned}$$

Substituting this equation into the line above yields a value of

$$\begin{aligned} & \pi^{-1-2l_0} e^{2i\pi l_0 - \frac{i\pi\gamma^2}{2}} \Theta'_\tau(0)^{2l_0} q^{-\frac{\alpha\gamma}{12} - \frac{\gamma^2}{24}} \eta(q)^{-\frac{\alpha\gamma}{2} - \frac{\gamma^2}{4}} \int_0^\infty z^{-\frac{\alpha\gamma}{2}} \left((1+z)^{\frac{\gamma^2}{4}} - z^{\frac{\gamma^2}{4}} \right) dz \\ & \quad \times \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2} - \frac{\gamma^2}{4}} e^{\pi\gamma P x} dx \right)^{-\frac{\alpha}{\gamma} - \frac{1}{2}} \right]. \end{aligned}$$

The integral over z appearing above is absolutely convergent because $\alpha \in (\frac{\gamma}{2}, \frac{2}{\gamma})$ and can be explicitly evaluated as

$$\int_0^\infty z^{-\frac{\alpha\gamma}{2}} \left((1+z)^{\frac{\gamma^2}{4}} - z^{\frac{\gamma^2}{4}} \right) dz = \frac{\Gamma(1 - \frac{\alpha\gamma}{2}) \Gamma(-1 + \frac{\alpha\gamma}{2} - \frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})}.$$

The same analysis for the integration interval $y \in [1 - u^{1-\delta}, 1]$ yields the same limit multiplied by the phase $-e^{\pi\gamma P - 2i\pi l_0}$. The conclusion of these computations is thus that

$$\begin{aligned} & \lim_{u \rightarrow 0} \sin(\pi u)^{-2l_0-1} \mathbb{E}[(g(u) - g(0))g(t, u)^{-\frac{\alpha}{\gamma} - \frac{1}{2}}] \\ & = \pi^{-2l_0-1} e^{2i\pi l_0 - \frac{i\pi\gamma^2}{2}} q^{-\frac{\alpha\gamma}{12} - \frac{\gamma^2}{24}} \eta(q)^{-\frac{\alpha\gamma}{2} - \frac{\gamma^2}{4}} \Theta'_\tau(0)^{2l_0} \frac{\Gamma(1 - \frac{\alpha\gamma}{2}) \Gamma(-1 + \frac{\alpha\gamma}{2} - \frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})} (1 - e^{\pi\gamma P - 2i\pi l_0}) \\ & \quad \times \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2} - \frac{\gamma^2}{4}} e^{\pi\gamma P x} dx \right)^{-\frac{\alpha}{\gamma} - \frac{1}{2}} \right]. \end{aligned}$$

Substituting everything into (D.1) then implies the final claim

$$\begin{aligned} & \lim_{u \rightarrow 0} \sin(\pi u)^{-2l_0-1} \left(\phi_{\frac{\alpha}{2}}^\alpha(u, q) - \phi_{\frac{\alpha}{2}}^\alpha(0, q) \right) \\ & = e^{2i\pi l_0 - \frac{i\pi\gamma^2}{2}} q^{\frac{P^2}{2} + \frac{l_0}{6} (1 - \frac{4l_0}{\gamma^2})} q^{-\frac{\alpha\gamma}{12} - \frac{\gamma^2}{24}} \eta(q)^{-\frac{\alpha\gamma}{2} - \frac{\gamma^2}{4}} \Theta'_\tau(0)^{\frac{8}{3} \frac{l_0(1-l_0)}{\gamma^2} + \frac{4}{3} l_0} \pi^{-l_0-1} \left(-\frac{\alpha}{\gamma} + \frac{1}{2} \right) (1 - e^{\pi\gamma P - 2i\pi l_0}) \\ & \quad \times \frac{\Gamma(1 - \frac{\alpha\gamma}{2}) \Gamma(-1 + \frac{\alpha\gamma}{2} - \frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})} \mathbb{E} \left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2} - \frac{\gamma^2}{4}} e^{\pi\gamma P x} dx \right)^{-\frac{\alpha}{\gamma} - \frac{1}{2}} \right] \\ & = e^{2i\pi l_0 - \frac{i\pi\gamma^2}{2}} q^{\frac{P^2}{2} + \frac{l_0}{6} - \frac{2}{3} \frac{l_0(1+l_0)}{\gamma^2}} \eta(q)^{\frac{8l_0+24l_0^2}{\gamma^2} - 6l_0} \Theta'_\tau(0)^{\frac{8}{3} \frac{l_0(1-l_0)}{\gamma^2} + \frac{4}{3} l_0} \pi^{-l_0-1} \frac{4l_0}{\gamma^2} (1 - e^{\pi\gamma P - 2i\pi l_0}) \\ & \quad \times \frac{\Gamma(1 - \frac{\alpha\gamma}{2}) \Gamma(-1 + \frac{\alpha\gamma}{2} - \frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})} \mathcal{A}_{\gamma, P}^q \left(\alpha + \frac{\gamma}{2} \right) \\ & = e^{2i\pi l_0 - \frac{i\pi\gamma^2}{2}} q^{\frac{P^2}{2} + \frac{l_0}{6} - \frac{2}{3} \frac{l_0(1+l_0)}{\gamma^2}} (2\pi e^{i\pi})^{-\frac{8l_0+24l_0^2}{3\gamma^2} + 2l_0} \Theta'_\tau(0)^{\frac{16}{3} \frac{l_0(1+l_0)}{\gamma^2} - \frac{2}{3} l_0} \pi^{-l_0-1} \frac{4l_0}{\gamma^2} (1 - e^{\pi\gamma P - 2i\pi l_0}) \\ & \quad \times \frac{\Gamma(1 - \frac{\alpha\gamma}{2}) \Gamma(-1 + \frac{\alpha\gamma}{2} - \frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})} \mathcal{A}_{\gamma, P}^q \left(\alpha + \frac{\gamma}{2} \right). \quad \square \end{aligned}$$

Proof of Lemma 5.3. Recall the notation $\tilde{l}_0 := l_{\frac{2}{\gamma}}$ from (5.4), and set $s = -\frac{\alpha}{\gamma} + \frac{2}{\gamma^2}$. This proof follows the strategy detailed in [RZ20] closely and thus we will be quite brief on each estimate required. We write $u = it$ and work with small $t > 0$. For a Borel set $I \subseteq [0, 1]$, we introduce the notation

$$(D.2) \quad K_I(it) := \int_I e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} \Theta_\tau(it+x)^{\frac{\gamma\chi}{2}} e^{\pi\gamma P x} dx.$$

We now study the asymptotics of the quantity

$$(D.3) \quad \mathbb{E}[K_{[0,1]}(it)^s] - \mathbb{E}[K_{[0,1]}(0)^s] =: T_1 + T_2,$$

where we define

$$(D.4) \quad T_1 := \mathbb{E}[K_{(t,1-t)}(it)^s] - \mathbb{E}[K_{[0,1]}(0)^s] \quad \text{and} \quad T_2 := \mathbb{E}[K_{[0,1]}(it)^s] - \mathbb{E}[K_{(t,1-t)}(it)^s].$$

By direct inequalities one can show that there exists $\alpha_0 > 0$ depending only on γ such that for $\alpha > Q - \alpha_0$ we have

$$(D.5) \quad T_1 = o(t^{\chi(Q-\alpha)}).$$

We now focus on T_2 . The goal is to restrict $[0, 1]$ to $[0, t^{1+h}] \cup (t, 1-t) \cup (1-t^{1+h}, 1]$ for a small $h > 0$ to be fixed later so that the GMCs on the three disjoint parts will be weakly correlated. Choosing h small enough, the arguments of [RZ20] show that

$$(D.6) \quad \mathbb{E}[K_{[0,1]}(\mathbf{it})^s] - \mathbb{E}[K_{[0,t^{1+h}] \cup (t,1-t) \cup (1-t^{1+h},1]}(\mathbf{it})^s] = o(t^{\chi(Q-\alpha)}).$$

It remains to evaluate $\mathbb{E}[K_{[0,t^{1+h}] \cup (t,1-t) \cup (1-t^{1+h},1]}(\mathbf{it})^s] - \mathbb{E}[K_{(t,1-t)}(\mathbf{it})^s]$. Using [JSW19, Theorem A], in a small neighborhood of 0, it is possible to write $Y_\infty(x) = Y_1(x) + Y_2(x)$ with $Y_1(x)$ an exactly log-correlated Gaussian field and $Y_2(x)$ a continuous Gaussian process such that $Y_2(0) = 0$ almost surely. (Note that Y_1 and Y_2 may not be independent, unlike $Y_\infty(x)$ and $F_\tau(x)$.) Furthermore, for $x \in \mathbb{R}$ with $|x|$ small enough we can decompose $Y_1(x)$ as

$$Y_1(x) = B_{-2 \ln |2\pi x|} + Z(x),$$

where $Z(x)$ is the Gaussian process with covariance given by (5.5). Therefore, for $x \in [-t, t]$ with t small enough, we have a decomposition

$$(D.7) \quad Y_\tau(x) = B_{-2 \ln |2\pi x|} + Z(x) + Y_2(x) + F_\tau(x).$$

Define now the processes

$$\begin{aligned} P(x) &:= (B_{-2 \ln |2\pi x|} + Z(x) + Y_2(x) + F_\tau(x)) \mathbf{1}_{|x| \leq t^{1+h}} + Y_\tau(x) \mathbf{1}_{|x| \geq t}, \\ \tilde{P}(x) &:= (B_{-2 \ln |2\pi x|} + \tilde{Z}(x) + F_\tau(0)) \mathbf{1}_{|x| \leq t^{1+h}} + Y_\tau(x) \mathbf{1}_{|x| \geq t}, \end{aligned}$$

where \tilde{Z} is an independent copy of Z . We may write

$$\begin{aligned} K_{[0,t^{1+h}] \cup (t,1-t) \cup (1-t^{1+h},1]}(\mathbf{it}) &= \int_{[0,t^{1+h}] \cup (t,1-t) \cup (1-t^{1+h},1]} e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} \Theta_\tau(\mathbf{it} + x)^{\frac{\chi\gamma}{2}} e^{\pi\gamma P x} dx \\ &= \int_{(-\frac{1}{2}, t) \cup (-t^{1+h}, t^{1+h}) \cup (t, \frac{1}{2})} e^{\frac{\gamma}{2} Y_\tau(x)} e^{-\frac{i\pi\alpha\gamma}{2}} |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} \Theta_\tau(\mathbf{it} + x)^{\frac{\chi\gamma}{2}} \left(e^{\pi\gamma P x} \mathbf{1}_{\{x \geq 0\}} + e^{-\frac{i\pi\chi\gamma}{2}} e^{\pi\gamma P(x+1)} \mathbf{1}_{\{x < 0\}} \right) dx. \end{aligned}$$

Consider now for $v \in [0, 1]$ the quantities

$$\begin{aligned} P_v(x) &= \sqrt{1-v} P(x) + \sqrt{v} \tilde{P}(x), \\ K(\mathbf{it}, v) &= \int_{(-\frac{1}{2}, t) \cup (-t^{1+h}, t^{1+h}) \cup (t, \frac{1}{2})} e^{\frac{\gamma}{2} P_v(x)} e^{-\frac{i\pi\alpha\gamma}{2}} |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} \Theta_\tau(\mathbf{it} + x)^{\frac{\chi\gamma}{2}} \\ &\quad \times \left(e^{\pi\gamma P x} \mathbf{1}_{\{x \geq 0\}} + e^{-\frac{i\pi\chi\gamma}{2}} e^{\pi\gamma P(x+1)} \mathbf{1}_{\{x < 0\}} \right) dx. \end{aligned}$$

Let $\tilde{K}_{[0,t^{1+h}] \cup (1-t^{1+h},1]}(\mathbf{it})$ be defined in exactly the same way as $K_{[0,t^{1+h}] \cup (1-t^{1+h},1]}(\mathbf{it})$ but using the independent copy \tilde{Z} of Z instead of Z in the decomposition (D.7) of $Y_\tau(x)$. By applying Kahane's inequality of Theorem B.3 as performed in [RZ20], we have for some constant $c > 0$ that

$$\begin{aligned} &\left| \mathbb{E}[K_{[0,t^{1+h}] \cup (t,1-t) \cup (1-t^{1+h},1]}(\mathbf{it})^s] - \mathbb{E}\left[\left(K_{(t,1-t)}(\mathbf{it}) + \tilde{K}_{[0,t^{1+h}] \cup (1-t^{1+h},1]}(\mathbf{it})\right)^s\right] \right| \\ &\leq 2|q(q-1)|t^h \sup_{u \in [0,1]} \mathbb{E}[|K(\mathbf{it}, v)|^s] \\ (D.8) \quad &\leq c t^h. \end{aligned}$$

When $h > \chi(Q - \alpha)$, we can bound the previous term by $o(t^{\chi(Q-\alpha)})$. Let us now look more closely at $\tilde{K}_{[0,t^{1+h}] \cup (1-t^{1+h},1]}(\mathbf{it})$. We can write this term as

$$\begin{aligned} &\tilde{K}_{[0,t^{1+h}] \cup (1-t^{1+h},1]}(\mathbf{it}) \\ &= \int_{(-t^{1+h}, t^{1+h})} e^{\frac{\gamma}{2} \tilde{P}(x)} e^{-\frac{i\pi\alpha\gamma}{2}} |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} \Theta_\tau(\mathbf{it} + x)^{\frac{\chi\gamma}{2}} \left(e^{\pi\gamma P x} \mathbf{1}_{\{x \geq 0\}} + e^{-\frac{i\pi\chi\gamma}{2}} e^{\pi\gamma P(x+1)} \mathbf{1}_{\{x < 0\}} \right) dx \end{aligned}$$

with

$$e^{\frac{\gamma}{2}\tilde{P}(x)} dx = |2\pi x|^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}B_{-2\ln|2\pi x|}} e^{\frac{\gamma}{2}\tilde{Z}(x) - \frac{\gamma^2}{8}\mathbb{E}[\tilde{Z}(x)^2]} e^{\frac{\gamma}{2}F_\tau(0) - \frac{\gamma^2}{8}\mathbb{E}[F_\tau(0)^2]} dx.$$

Up to an asymptotically negligible error, it is possible to replace $\tilde{K}_{[0, t^{1+h}] \cup (1-t^{1+h}, 1]}(\mathbf{it})$ by

$$\Theta'_\tau(0)^{\frac{\gamma_X}{2} - \frac{\alpha\gamma}{2}} \int_{(-t^{1+h}, t^{1+h})} e^{\frac{\gamma}{2}\tilde{P}(x)} |x|^{-\frac{\alpha\gamma}{2}} (\mathbf{it} + x)^{\frac{\gamma_X}{2}} \left(\mathbf{1}_{\{x \geq 0\}} + e^{-\frac{i\pi X \gamma}{2}} e^{\pi\gamma P} \mathbf{1}_{\{x < 0\}} \right) dx.$$

Apply now the change of variable $x = \pm t^{1+h} e^{-s/2}$ for the above quantity to obtain

$$\begin{aligned} & \Theta'_\tau(0)^{\frac{\gamma_X}{2} - \frac{\alpha\gamma}{2}} \int_{(0, t^{1+h})} |x|^{-\frac{\alpha\gamma}{2}} \left(e^{\frac{\gamma}{2}\tilde{P}(x)} (\mathbf{it} + x)^{\frac{\gamma_X}{2}} \mathbf{1}_{\{x \geq 0\}} + e^{\frac{\gamma}{2}\tilde{P}(-x)} (\mathbf{it} - x)^{\frac{\gamma_X}{2}} e^{-\frac{i\pi X \gamma}{2}} e^{\pi\gamma P} \mathbf{1}_{\{x < 0\}} \right) dx \\ &= \frac{1}{2} \Theta'_\tau(0)^{\frac{\gamma_X}{2} - \frac{\alpha\gamma}{2}} |2\pi|^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}F_\tau(0) - \frac{\gamma^2}{8}\mathbb{E}[F_\tau(0)^2]} t^{(1+h)(\frac{\gamma^2}{4} - \frac{\alpha\gamma}{2} + 1)} e^{\frac{\gamma}{2}B_{-2\ln|2\pi t^{1+h}|}} \\ & \times \int_0^\infty e^{\frac{\gamma}{2}\tilde{B}_s} e^{-\frac{s}{2}(\frac{\gamma^2}{4} - \frac{\alpha\gamma}{2} + 1)} \left(e^{\frac{\gamma}{2}\tilde{Z}(e^{-s/2})} (\mathbf{it} + t^{1+h} e^{-s/2})^{\frac{\gamma_X}{2}} + e^{\frac{\gamma}{2}\tilde{Z}(-e^{-s/2})} (\mathbf{it} - t^{1+h} e^{-s/2})^{\frac{\gamma_X}{2}} e^{-\frac{i\pi X \gamma}{2} + \pi\gamma P} \right) ds. \end{aligned}$$

Again up to an asymptotically negligible error, we can replace the expression above by

$$\frac{\mathbf{i}^{\frac{\gamma_X}{2}}}{2} \Theta'_\tau(0)^{\frac{\gamma_X}{2} - \frac{\alpha\gamma}{2}} |2\pi|^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}F_\tau(0) - \frac{\gamma^2}{8}\mathbb{E}[F_\tau(0)^2]} t^{\frac{\gamma_X}{2} + (1+h)(\frac{\gamma^2}{4} - \frac{\alpha\gamma}{2} + 1)} e^{\frac{\gamma}{2}B_{-2\ln|2\pi t^{1+h}|}} \int_0^\infty e^{\frac{\gamma}{2}(\tilde{B}_s - \frac{s}{2}(Q-\alpha))} d\mu_{\tilde{Z}}(s)$$

where we have introduced

$$d\mu_{\tilde{Z}}(s) := \left(e^{\frac{\gamma}{2}\tilde{Z}(e^{-s/2})} + e^{\frac{\gamma}{2}\tilde{Z}(-e^{-s/2})} e^{-\frac{i\pi X \gamma}{2} + \pi\gamma P} \right) ds$$

and used the Markov property of the Brownian motion and stationarity of $d\mu_{\tilde{Z}}(s)$. We define the quantities

$$\begin{aligned} \sigma_t &:= \Theta'_\tau(0)^{\frac{\gamma_X}{2} - \frac{\alpha\gamma}{2}} |2\pi|^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}F_\tau(0) - \frac{\gamma^2}{8}\mathbb{E}[F_\tau(0)^2]} t^{\frac{\gamma_X}{2} + (1+h)(\frac{\gamma^2}{4} - \frac{\alpha\gamma}{2} + 1)} e^{\frac{\gamma}{2}B_{-2\ln|2\pi t^{1+h}|}}, \\ V &:= \frac{1}{2} \int_0^\infty e^{\frac{\gamma}{2}(\tilde{B}_s - \frac{s}{2}(Q-\alpha))} d\mu_{\tilde{Z}}(s). \end{aligned}$$

By simple inequalities, we can prove that

$$\left| \mathbb{E}[(K_{(t, 1-t)}(\mathbf{it}) + \tilde{K}_{[0, t^{1+h}] \cup (1-t^{1+h}, 1]}(\mathbf{it}))^s] - \mathbb{E}[(K_{(t, 1-t)}(\mathbf{it}) + \mathbf{i}^{\frac{\gamma_X}{2}} \sigma_t V)^s] \right| = o(t^{\chi(Q-\alpha)}).$$

By the Williams path decomposition of Theorem B.4 we can write

$$(D.9) \quad V = e^{\frac{\gamma}{2}M} \frac{1}{2} \int_{-L_M}^\infty e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} \mu_{\tilde{Z}}(ds),$$

where $M = \sup_{s \geq 0} (\tilde{B}_s - \frac{Q-\alpha}{2}s)$ and L_M is the last time $\left(\mathcal{B}_{-s}^{\frac{Q-\alpha}{2}} \right)_{s \geq 0}$ hits $-M$. Recall that the law of M is known and satisfies

$$(D.10) \quad \mathbb{P}(e^{\frac{\gamma}{2}M} > v) = \frac{1}{v^{\frac{2}{\gamma}(Q-\alpha)}}$$

for $v \geq 1$, and also recall from definition (5.7) the quantity

$$(D.11) \quad \rho(\alpha, 1, e^{-i\pi \frac{\gamma_X}{2} + \pi\gamma P}) := \frac{1}{2} \int_{-\infty}^\infty e^{\frac{\gamma}{2}\mathcal{B}_s^{\frac{Q-\alpha}{2}}} \mu_{\tilde{Z}}(ds).$$

Next we can show that

$$\left| \mathbb{E}[(K_{(t, 1-t)}(\mathbf{it}) + \mathbf{i}^{\frac{\gamma_X}{2}} \sigma_t V)^s] - \mathbb{E}[(K_{(t, 1-t)}(\mathbf{it}) + \mathbf{i}^{\frac{\gamma_X}{2}} \sigma_t e^{\frac{\gamma}{2}M} \rho(\alpha, 1, e^{-i\pi \frac{\gamma_X}{2} + \pi\gamma P})^s] \right| = o(t^{\chi(Q-\alpha)}).$$

In summary,

$$(D.12) \quad T_2 = \mathbb{E}[(K_{(t, 1-t)}(\mathbf{it}) + \mathbf{i}^{\frac{\gamma_X}{2}} \sigma_t e^{\frac{\gamma}{2}M} \rho(\alpha, 1, e^{-i\pi \frac{\gamma_X}{2} + \pi\gamma P})^s] - \mathbb{E}[K_{(t, 1-t)}(\mathbf{it})^s] + o(t^{\chi(Q-\alpha)}).$$

Finally, we evaluate the above difference at first order explicitly using the fact that the density of $e^{\frac{\gamma}{2}M}$ is known

$$\begin{aligned} & \mathbb{E}[(K_{(t,1-t)}(\mathbf{it}) + \mathbf{i}^{\frac{\gamma\chi}{2}} \sigma_t e^{\frac{\gamma}{2}M} \rho(\alpha, 1, e^{-i\pi\frac{\gamma\chi}{2} + \pi\gamma P}))^s] - \mathbb{E}[K_{(t,1-t)}(\mathbf{it})^s] \\ &= \frac{2}{\gamma}(Q-\alpha)\mathbb{E}\left[\int_1^\infty \frac{dv}{v^{\frac{2}{\gamma}(Q-\alpha)+1}} \left(\left(K_{(t,1-t)}(\mathbf{it}) + \mathbf{i}^{\frac{\gamma\chi}{2}} \sigma_t \rho(\alpha, 1, e^{-i\pi\frac{\gamma\chi}{2} + \pi\gamma P}) v \right)^s - K_{(t,1-t)}(\mathbf{it})^s \right)\right] \\ &= \mathbf{i}^{\chi(Q-\alpha)} \frac{2}{\gamma}(Q-\alpha) \frac{\Gamma(\frac{2}{\gamma}(\alpha-Q))\Gamma(\frac{2}{\gamma}(Q-\alpha)-s)}{\Gamma(-s)} \bar{R}(\alpha, 1, e^{\pi\gamma P - i\pi\frac{\gamma\chi}{2}}) \mathbb{E}\left[\sigma_t^{\frac{2}{\gamma}(Q-\alpha)} K_{(t,1-t)}(\mathbf{it})^{s-\frac{2}{\gamma}(Q-\alpha)}\right] \\ &+ o(t^{\chi(Q-\alpha)}). \end{aligned}$$

To obtain the desired answer, we perform the manipulation

$$\begin{aligned} \sigma_t^{\frac{2}{\gamma}(Q-\alpha)} &= \left(\Theta'_\tau(0)^{\frac{\gamma\chi}{2} - \frac{\alpha\gamma}{2}} |2\pi|^{\frac{\gamma^2}{4}} e^{\frac{\gamma}{2}F_\tau(0) - \frac{\gamma^2}{8}\mathbb{E}[F_\tau(0)^2]} t^{\frac{\gamma\chi}{2} + (1+h)(\frac{\gamma^2}{4} - \frac{\alpha\gamma}{2} + 1)} e^{\frac{\gamma}{2}B_{-2\ln|2\pi t^{1+h}|}} \right)^{\frac{2}{\gamma}(Q-\alpha)} \\ &= \left(\Theta'_\tau(0)^{\frac{\gamma\chi}{2} - \frac{\alpha\gamma}{2}} (2\pi)^{\frac{\gamma^2}{4}} t^{\frac{\gamma\chi}{2} + \frac{\gamma}{2}(1+h)(Q-\alpha)} q^{-\frac{\gamma^2}{24}} \eta(q)^{\frac{\gamma^2}{2}} \right)^{\frac{2}{\gamma}(Q-\alpha)} e^{(Q-\alpha)(B_{-2\ln|2\pi t^{1+h}|} + F_\tau(0))} \end{aligned}$$

and then

$$\begin{aligned} & e^{(Q-\alpha)(B_{-2\log|2\pi t^{1+h}|} + F_\tau(0))} \\ &= (2\pi)^{-(Q-\alpha)^2} t^{-(1+h)(Q-\alpha)^2} (q^{-1/12} \eta(q))^{-2(Q-\alpha)^2} e^{(Q-\alpha)(B_{-2\ln|2\pi t^{1+h}|} + F_\tau(0)) - \frac{(Q-\alpha)^2}{2}(\mathbb{E}[B_{-2\ln|2\pi t^{1+h}|}^2] + \mathbb{E}[F_\tau(0)^2])}. \end{aligned}$$

Further, for all $x \in [t, 1-t]$ with t small enough

$$\mathbb{E}[Y_\tau(x)(B_{-2\ln|2\pi t^{1+h}|} + F_\tau(0))] = -2\log|\Theta_\tau(x)| + 2\log|q^{1/6}\eta(q)|,$$

which implies by Girsanov's theorem (Theorem B.2) that

$$\begin{aligned} & \lim_{t \rightarrow 0} \mathbb{E}\left[e^{(Q-\alpha)(B_{-2\ln|2\pi t^{1+h}|} + F_\tau(0)) - \frac{(Q-\alpha)^2}{2}(\mathbb{E}[B_{-2\ln|2\pi t^{1+h}|}^2] + \mathbb{E}[F_\tau(0)^2])} \right. \\ & \quad \left. \times \left(\int_t^{1-t} e^{\frac{\gamma}{2}Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} \Theta_\tau(\mathbf{it}+x)^{\frac{\chi\gamma}{2}} e^{\pi\gamma Px} dx \right)^{\frac{\alpha+\chi-2Q}{\gamma}} \right] \\ &= e^{i\pi(Q-\alpha)(\alpha+\chi-2Q)} (q^{\frac{1}{6}}\eta(q))^{(Q-\alpha)(\alpha+\chi-2Q)} \mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)} \Theta_\tau(x)^{-\frac{\gamma}{2}(2Q-\alpha-\chi)} e^{\pi\gamma Px} dx \right)^{\frac{\alpha+\chi-2Q}{\gamma}} \right]. \end{aligned}$$

From this we obtain the final claim

$$\begin{aligned} & \mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} \Theta_\tau(u+x)^{\frac{\chi\gamma}{2}} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] - \mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2} + \frac{\gamma\chi}{2}} e^{\pi\gamma Px} dx \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right] \\ &= -u^{1+2l_\chi} (2\pi)^{(\alpha-Q)(\frac{2}{\gamma}-\alpha)} \frac{\Gamma(\frac{2\alpha}{\gamma} - \frac{4}{\gamma^2})\Gamma(\frac{2Q-\alpha-\chi}{\gamma})}{\Gamma(\frac{\alpha}{\gamma} - \frac{\chi}{\gamma})} (q^{\frac{1}{6}}\eta(q))^{(Q-\alpha)(\alpha+\chi-2Q)} \Theta'_\tau(0)^{(Q-\alpha)(\chi-\alpha)} e^{i\pi(Q-\alpha)(\alpha+\chi-2Q)} \\ & \quad \times (q^{-\frac{1}{12}}\eta(q))^{(Q-\alpha)(2\alpha-\frac{4}{\gamma})} \bar{R}(\alpha, 1, e^{-i\pi\frac{\gamma\chi}{2} + \pi\gamma P}) \mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)} \Theta_\tau(x)^{-\frac{\gamma}{2}(2Q-\alpha-\chi)} e^{\pi\gamma Px} dx \right)^{\frac{\alpha+\chi-2Q}{\gamma}} \right] \\ &+ o(u^{1+2l_\chi}), \end{aligned}$$

where we can simplify the prefactors to

$$\begin{aligned} & (2\pi)^{(\alpha-Q)(\frac{2}{\gamma}-\alpha)} (q^{\frac{1}{6}}\eta(q))^{(Q-\alpha)(\alpha+\chi-2Q)} \Theta'_\tau(0)^{(Q-\alpha)(\chi-\alpha)} e^{i\pi(Q-\alpha)(\alpha+\chi-2Q)} (q^{-\frac{1}{12}}\eta(q))^{(Q-\alpha)(2\alpha-\frac{4}{\gamma})} \\ &= (2\pi)^{(\alpha-Q)(\frac{2}{\gamma}-\alpha)} q^{\frac{1}{6}(Q-\alpha)(\chi+\frac{2}{\gamma}-2Q)} \eta(q)^{(Q-\alpha)(3\alpha+\chi-2Q-\frac{4}{\gamma})} \Theta'_\tau(0)^{(Q-\alpha)(\chi-\alpha)} e^{i\pi(Q-\alpha)(\alpha+\chi-2Q)} \\ &= (2\pi)^{(Q-\alpha)(\frac{\gamma}{3}-\frac{\chi}{3}+\frac{2}{3\gamma})} q^{\frac{1}{6}(Q-\alpha)(\chi+\frac{2}{\gamma}-2Q)} \Theta'_\tau(0)^{(Q-\alpha)(\frac{2\chi}{3}-\frac{4}{3\gamma}-\frac{2}{3\chi})} e^{i\pi(Q-\alpha)(\frac{4}{3\gamma}-\frac{2\chi}{3}-\frac{4}{3\chi})}. \quad \square \end{aligned}$$

REFERENCES

- [AGT09] L. Alday, D. Gaiotto, and Y. Tachikawa. Liouville correlation functions from four-dimensional gauge theories. 2009. <https://arxiv.org/abs/0906.3219>.
- [Bav19] G. Bavezez. Modular bootstrap agrees with the path integral in the large moduli limit. *Electron. J. Probab.*, 24:22 pp., 2019.
- [Ber88] D. Bernard. On the Wess-Zumino-Witten models on the torus. *Nuclear Phys. B*, 303(1):77–93, 1988.
- [BPZ84] A. Belavin, A. Polyakov, and A. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Phys. B*, 241(2):333–380, 1984.
- [CCY19] M. Cho, S. Collier, and X. Yin. Recursive representations of arbitrary Virasoro conformal blocks. *Journal of High Energy Physics*, 2019(4):18, Apr 2019.
- [DF84] V. Dotsenko and V. Fateev. Conformal algebra and multipoint correlation functions in 2D statistical models. *Nuclear Phys. B*, 240(3):312–348, 1984.
- [DF85] V. Dotsenko and V. Fateev. Operator algebra of two-dimensional conformal theories with central charge $C \leq 1$. *Phys. Lett. B*, 154(4):291–295, 1985.
- [DKRV16] F. David, A. Kupiainen, R. Rhodes, and V. Vargas. Liouville quantum gravity on the Riemann sphere. *Comm. Math. Phys.*, 342(3):869–907, 2016.
- [DLMF] *NIST Digital Library of Mathematical Functions*. <http://dlmf.nist.gov/>, Release 1.0.24 of 2019-09-15. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, B. V. Saunders, H. S. Cohl, and M. A. McClain, eds.
- [DMS14] B. Duplantier, J. Miller, and S. Sheffield. Liouville quantum gravity as a mating of trees. *arXiv e-prints*, page arXiv:1409.7055, Sep 2014.
- [DO94] H. Dorn and H. Otto. Two- and three-point functions in Liouville theory. *Nuclear Phys. B*, 429(2):375–388, 1994.
- [DRV16] F. David, R. Rhodes, and V. Vargas. Liouville quantum gravity on complex tori. *J. Math. Phys.*, 57(2):022302, 25, 2016.
- [DV09] R. Dijkgraaf and C. Vafa. Toda theories, matrix models, topological strings, and $N = 2$ gauge systems, 2009.
- [FL10] V. Fateev and A. Litvinov. On AGT conjecture. *Journal of High Energy Physics*, 2010(2):14, 2010.
- [FLNO09] V. Fateev, A. Litvinov, A. Neveu, and E. Onofri. A differential equation for a four-point correlation function in Liouville field theory and elliptic four-point conformal blocks. *J. Phys. A*, 42(30):304011, 29, 2009.
- [FML18] G. Felder and M. Müller-Lennert. Analyticity of Nekrasov partition functions. *Comm. Math. Phys.*, 364(2):683–718, 2018.
- [GRV19] C. Guillarmou, R. Rhodes, and V. Vargas. Polyakov’s formulation of 2d bosonic string theory. *Publications mathématiques de l’IHÉS*, 130(1):111–185, 2019.
- [HJS10] L. Hadasz, Z. Jaskólski, and P. Suchanek. Recursive representation of the torus 1-point conformal block. *J. High Energy Phys.*, (1):063, 13, 2010.
- [HRV18] Y. Huang, R. Rhodes, and V. Vargas. Liouville quantum gravity on the unit disk. *Ann. Inst. H. Poincaré Probab. Statist.*, 54(3):1694–1730, 08 2018.
- [JSW19] J. Junnila, E. Saksman, and C. Webb. Decompositions of log-correlated fields with applications. *Ann. Appl. Probab.*, 29(6):3786–3820, 12 2019.
- [KRV19a] A. Kupiainen, R. Rhodes, and V. Vargas. Integrability of Liouville theory: proof of the DOZZ formula, 2019. To appear.
- [KRV19b] A. Kupiainen, R. Rhodes, and V. Vargas. Local conformal structure of Liouville quantum gravity. *Comm. Math. Phys.*, 371(3):1005–1069, 2019.
- [MMS10] A. Mironov, A. Morozov, and S. Shakirov. Conformal blocks as Dotsenko-Fateev integral discriminants. *Internat. J. Modern Phys. A*, 25(16):3173–3207, 2010.
- [MMS11] A. Mironov, A. Morosov, and S. Shakirov. On the ‘Dotsenko-Fateev’ representation of the toric conformal blocks. *J. Phys. A*, 44(8):085401, 11, 2011.
- [MY11] K. Maruyoshi and F. Yagi. Seiberg-Witten curve via generalized matrix model. *J. High Energy Phys.*, (1):042, 18, 2011.
- [Pog09] R. Poghossian. Recursion relations in CFT and $N = 2$ SYM theory. *J. High Energy Phys.*, (12):038, 14, 2009.
- [Rem20] G. Remy. The Fyodorov-Bouchaud formula and Liouville conformal field theory. *Duke Math. J.*, 169(1):177–211, 01 2020.
- [RZ18] G. Remy and T. Zhu. The distribution of Gaussian multiplicative chaos on the unit interval. *Ann. Probab.*, 2018. To appear.
- [RZ20] G. Remy and T. Zhu. Integrability of boundary Liouville conformal field theory. 2020. [arXiv:2002.05625](https://arxiv.org/abs/2002.05625).
- [Tes95] J. Teschner. On the Liouville three-point function. *Physics Letters B*, 363(1-2):6570, Nov 1995.
- [Wil74] D. Williams. Path decomposition and continuity of local time for one-dimensional diffusions, i. *Proceedings of the London Mathematical Society*, s3-28(4):738–768, 1974.
- [Zam84] A. Zamolodchikov. Conformal symmetry in two dimensions: an explicit recurrence formula for the conformal partial wave amplitude. *Comm. Math. Phys.*, 96(3):419–422, 1984.
- [Zam87] A. Zamolodchikov. Conformal symmetry in two-dimensional space: on a recurrent representation of the conformal block. *Teoret. Mat. Fiz.*, 73(1):103–110, 1987.
- [ZZ96] A. Zamolodchikov and Al. Zamolodchikov. Conformal bootstrap in Liouville field theory. *Nuclear Phys. B*, 477(2):577–605, 1996.

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