PROBABILISTIC CONFORMAL BLOCKS FOR LIOUVILLE CFT ON THE TORUS

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Abstract. Virasoro conformal blocks are a family of important functions defined as power series via the Virasoro algebra. They are a fundamental input to the conformal bootstrap program for 2D conformal field theory (CFT) and are closely related to four dimensional supersymmetric gauge theory through the Alday-Gaiotto-Tachikawa correspondence. The present work provides a probabilistic construction of the 1-point toric Virasoro conformal block for central change greater than 25. More precisely, we construct an analytic function using a probabilistic tool called Gaussian multiplicative chaos (GMC) and prove that its power series expansion coincides with the 1-point toric Virasoro conformal block. The range $(25, \infty)$ of central charges corresponds to Liouville CFT, an important CFT originating from 2D quantum gravity and bosonic string theory. Our work reveals a new integrable structure underlying GMC and opens the door to the study of non-perturbative properties of Virasoro conformal blocks such as their analytic continuation and modular symmetry. Our proof combines an analysis of GMC with tools from CFT such as Belavin-Polyakov-Zamolodchikov differential equations, operator product expansions, and Dotsenko-Fateev type integrals.

1. Introduction

A conformal field theory (CFT) is a way to construct random functions on Riemannian manifolds that transform covariantly under conformal (i.e. angle preserving) mappings. Since the seminal work of Belavin-Polyakov-Zamolodchikov in [BPZ84], two dimensional (2D) CFT has grown into one of the most prominent branches of theoretical physics, with applications to 2D statistical physics and string theory, as well as far reaching consequences in mathematics; see e.g. [DFMS97]. The paper [BPZ84] introduced a schematic program called the *conformal bootstrap* to exactly solve correlation functions of a given 2D CFT in terms of its 3-point sphere correlation functions and certain power series called **conformal blocks**. These conformal blocks are completely specified by the Virasoro algebra that encodes the infinitesimal local conformal symmetries, and they only depend on the specific CFT through a single parameter called the *central charge*. Outside of CFT, conformal blocks are related to Nekrasov partition functions in gauge theory via the Alday-Gaiotto-Tachikawa (AGT) correspondence [AGT09], solutions to Painlevé-type equations [GIL12], and quantum Teichmüller theory and representation of quantum groups [PT99, PT01, TV15], among other things.

In this paper, we initiate a probabilistic approach to study the conformal blocks appearing in the conformal bootstrap for an important 2D CFT called **Liouville conformal field theory** (LCFT). LCFT arose from Polyakov's work on 2D quantum gravity and bosonic string theory in [Pol81a]; it was rigorously constructed from the path integral formalism of quantum field theory on the sphere in [DKRV16] and on other surfaces in [DRV16, HRV18, GRV19]. The construction is via **Gaussian multiplicative chaos** (GMC), which are random measures defined by exponentiating the Gaussian free field (see e.g. [RV14, Ber17]). LCFT depends on a coupling constant $\gamma \in (0, 2)$ which is in bijection with the central charge c via

$$c = 1 + 6Q^2 \in (25, \infty), \qquad \text{where } Q = \frac{\gamma}{2} + \frac{2}{\gamma}. \tag{1.1}$$

The present work gives a GMC representation of the conformal blocks with central charge $c \in (25, \infty)$ for a torus with one marked point. Given τ in the upper half plane, let $\mathbb{T}_\tau$ be the flat torus with modular parameter τ. The 1-point toric correlation of LCFT, rigorously constructed in [DRV16], has the form $\langle e^{\alpha\phi(0)}\rangle_\tau$, where $\langle \cdots \rangle_\tau$ is the average over the random field ϕ for LCFT on $\mathbb{T}_\tau$ and α is called the vertex insertion weight.

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The conformal bootstrap gives rise to the conjectural *modular bootstrap equation* expressing $\langle e^{\alpha\phi(0)}\rangle_\tau$ in terms of the 1-*point toric conformal block* $\mathcal{F}^\alpha_{\gamma,P}(q)$

$$\langle e^{\alpha\phi(0)}\rangle_\tau = \frac{1}{|\eta(q)|^2}\int_{-\infty}^{\infty} C_\gamma(\alpha, Q-\mathbf{i}P, Q+\mathbf{i}P)|q|^{P^2}\mathcal{F}^\alpha_{\gamma,P}(q)\mathcal{F}^\alpha_{\gamma,P}(\bar{q})dP. \tag{1.2}$$

Here, $q = e^{\mathbf{i}\pi\tau}$, $\eta(q)$ is the Dedekind eta function, and $C_\gamma(\alpha_1,\alpha_2,\alpha_3)$ is the DOZZ formula for the Liouville 3-point sphere correlation function first proposed in [DO94, ZZ96] and proved by [KRV19a]. The conformal block $\mathcal{F}^\alpha_{\gamma,P}(q)$ is a q-power series defined via the Virasoro algebra in [BPZ84]; see Appendix A. In this paper, we use GMC to construct a function of q analytic around 0 whose series expansion is given by $\mathcal{F}^\alpha_{\gamma,P}(q)$.

1.1. **Summary of results.** To state our results, we first give two ways to characterize the 1-point toric conformal block $\mathcal{F}^\alpha_{\gamma,P}(q)$ as a formal q-series with parameters γ, P, α: Zamolodchikov's recursion and the AGT correspondence. The original definition based on the Virasoro algebra will be reviewed in Appendix A.

It was shown in [Zam84, Zam87, HJS10] that $\mathcal{F}^\alpha_{\gamma,P}(q)$, viewed as a formal q-series, is the unique solution to Zamolodchikov's recursion

$$\mathcal{F}^\alpha_{\gamma,P}(q) = \sum_{n,m=1}^{\infty} q^{2mn}\frac{R_{\gamma,m,n}(\alpha)}{P^2 - P^2_{m,n}}\mathcal{F}^\alpha_{\gamma,P_{-m,n}}(q) + q^{\frac{1}{12}}\eta(q)^{-1}, \tag{1.3}$$

where $R_{\gamma,m,n}(\alpha)$ and $P_{m,n}$ are explicit constants defined in (2.20) and (2.21). We give more details about (1.3) in Section 1.3.

The AGT correspondence stated in [AGT09] and proven in [FL10] asserts that the conformal block may be represented explicitly in terms of the instanton part of the Nekrasov partition function $\mathcal{Z}^\alpha_{\gamma,P}(q)$ as

$$\mathcal{F}^\alpha_{\gamma,P}(q) = \left(q^{-\frac{1}{12}}\eta(q)\right)^{1-\alpha(Q-\frac{\alpha}{2})}\mathcal{Z}^\alpha_{\gamma,P}(q). \tag{1.4}$$

Here, $\mathcal{Z}^\alpha_{\gamma,P}(q)$ is a formal series coming from four-dimensional $SU(2)$ supersymmetric gauge theory given by

$$\mathcal{Z}^\alpha_{\gamma,P}(q) := 1 + \sum_{k=1}^{\infty} q^{2k} \sum_{\substack{(Y_1,Y_2)\text{ Young diagrams}\\ |Y_1|+|Y_2|=k}} \prod_{i,j=1}^{2}\prod_{s\in Y_i}\frac{(E_{ij}(s,P)-\alpha)(Q-E_{ij}(s,P)-\alpha)}{E_{ij}(s,P)(Q-E_{ij}(s,P))}, \tag{1.5}$$

where $E_{ij}(s,P)$ is an explicit product given by (2.18). We also note that by (B.2), $q^{-\frac{1}{12}}\eta(q) = \prod_{n=1}^\infty (1-q^{2n})$ has an explicit q-series expansion. We give more details about the AGT correspondence in Section 1.3.

Having specified $\mathcal{F}^\alpha_{\gamma,P}(q)$, we are ready to state our main result. For $\gamma\in(0,2)$, consider the GMC measure $e^{\frac{\gamma}{2}Y_\tau(x)}dx$ on $[0,1]$. It is a random measure defined as the regularized exponential of the Gaussian field $Y_\tau(x)$ on $[0,1]$ with covariance

$$\mathbb{E}[Y_\tau(x)Y_\tau(y)] = -2\log|\Theta_\tau(x-y)| + 2\log|q^{\frac{1}{6}}\eta(q)|,$$

where $\Theta_\tau(x)$ is the Jacobi theta function (see Appendix B). For $\alpha\in(-\frac{4}{\gamma},Q)$, $q\in(0,1)$, and $P\in\mathbb{R}$, define the *probabilistic* 1-*point toric conformal block* by

$$\mathcal{G}^\alpha_{\gamma,P}(q) := \frac{1}{Z}\mathbb{E}\left[\left(\int_0^1 \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}}e^{\pi\gamma P x}e^{\frac{\gamma}{2}Y_\tau(x)}dx\right)^{-\frac{\alpha}{\gamma}}\right], \tag{1.6}$$

where Z is an explicit constant defined in Definition 2.6 (also see Remark 2.7) for which $\lim_{q\to0}\mathcal{G}^\alpha_{\gamma,P}(q) = 1$ and $\lim_{P\to+\infty}\mathcal{G}^\alpha_{\gamma,P}(q) = q^{\frac{1}{12}}\eta(q)^{-1}$. Our main result Theorem 1.1 shows that (1.6) gives a probabilistic construction of $\mathcal{F}^\alpha_{\gamma,P}(q)$ which is non-perturbative, in contrast to Zamolodchikov's recursion, the AGT correspondence, and the original definition of conformal blocks from the Virasoro algebra.

Theorem 1.1. *For $\gamma\in(0,2)$, $\alpha\in(-\frac{4}{\gamma},Q)$, and $P\in\mathbb{R}$, the probabilistic conformal block $\mathcal{G}^\alpha_{\gamma,P}(q)$ admits an analytic extension on a complex neighborhood of $q=0$, whose q-series expansion around $q=0$ agrees with $\mathcal{F}^\alpha_{\gamma,P}(q)$ defined in (1.4). In particular, the conformal block $\mathcal{F}^\alpha_{\gamma,P}(q)$ has a positive radius of convergence.*

Moreover, when $\alpha\in[0,Q)$, the analytic extension of $\mathcal{G}^\alpha_{\gamma,P}(q)$ exists on a complex neighborhood of $[0,1)$, and the radius of convergence of $\mathcal{F}^\alpha_{\gamma,P}(q)$ is at least $\frac{1}{2}$.

The range $(-\frac{4}{\gamma}, Q)$ for α is the range in which the 1-point correlation function $\langle e^{\alpha\phi(0)}\rangle_\tau$ in (1.2) has a GMC expression from the path integral formalism [DRV16]. Although our probabilistic construction of conformal blocks also relies on GMC, we are not aware of a natural path integral interpretation. From this perspective, our Theorem 1.1 reveals a new integrable structure underlying GMC.

Theorem 1.1 opens the door to the study of analytic properties of conformal blocks. In Section 1.5, we will describe our work in progress on the modular transformation rule for conformal blocks, which will allows us to analytically continue $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ and prove that the convergence radius of $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ is in fact 1 for $\alpha \in [0, Q)$. We believe that this holds for all $\alpha \in (-\frac{4}{\gamma}, Q)$.

The remainder of this introduction gives additional motivation and background for our results and outlines our methods. All notations and results will be reintroduced in full detail in later sections.

1.2. **Relation to probabilistic Liouville theory.** There are two important and fruitful lines of research in probability inspired by Polyakov's work on 2D quantum gravity [Pol81b]. One is random planar geometry, which includes Liouville quantum gravity and the scaling limits of random planar maps; see [LG13, Mie13, She16, DMS14b, HS19, GHS19] and references therein. The other is the rigorous path integral formalism of LCFT, which is more recent. For the sphere and disk, it was proved in [AHS17, Cer19] that LCFT indeed describes the surfaces that arise in random planar geometry, linking these two lines of research which share the same origin. We now review the second line of research, which is closely related to our work.

The path integral formalism was used to rigorously construct LCFT on various surfaces in [DKRV16, DRV16, HRV18, GRV19], which opened the door for probabilists to carry out the conformal bootstrap program for LCFT at a mathematical level of rigor. In [KRV19b], Kupiainen-Rhodes-Vargas proved that the BPZ equations translating the constraints of local conformal invariance of a CFT hold for correlation functions on the sphere with a degenerate insertion. Building upon this work, the same authors proved in [KRV19a] the DOZZ formula for the 3-point function of LCFT on the sphere, first proposed in physics in [DO94, ZZ96]. Similar methods were used in the recent works [Rem20, RZ18, RZ20] to study LCFT on a simply connected domain with boundary and solve several open problems about the distribution of one-dimensional GMC measures.

Completing our understanding of the integrable structure of LCFT requires giving a mathematical treatment of conformal blocks and bootstrap equations such as (1.2) in the case of the torus with one point or the sphere with four points, where we start to see nontrivial structure of moduli. Very recently, Guillarmou-Kupiainen-Rhodes-Vargas [GKRV20] proved the bootstrap equation for the sphere with four points for $\gamma \in (0, \sqrt{2})$. Their approach makes rigorous sense of the *operator product expansion* of [BPZ84] for $\gamma \in (0, \sqrt{2})$, which is the algebraic origin of the conformal bootstrap. A byproduct of their proof is the convergence of the power series for the corresponding four-point spherical conformal block for $\gamma \in (0, \sqrt{2})$ for almost every real value of the parameter P. From this perspective, our work gives an unexpected complementary approach to conformal blocks from GMC that is able to handle all real values of the parameter P as well as the full range of coupling constant $\gamma \in (0, 2)$. In future work, we hope to leverage this to prove (1.2) and similar bootstrap equations in this full range; see Section 1.5 for more details.

1.3. **Relation to existing approaches to conformal blocks in mathematical physics.** Conformal blocks have been studied from many different perspectives in mathematical physics, beginning with their definition in [BPZ84]. For the reader's convenience, we provide a brief overview of the physical origins of conformal blocks in Appendix A. If a 2D CFT has a larger symmetry algebra than the Virasoro algebra, such as the Wess-Zumino-Witten (WZW) model ([DFMS97, Chapter 15]), there is a corresponding notion of conformal blocks for the larger algebra; for WZW model, it is the affine Lie algebra. In light of this, the conformal blocks considered in our paper, which are most relevant to LCFT, are sometimes called the Virasoro conformal blocks. We now relate our results to a few directions in the mathematical physics literature on Virasoro conformal blocks.

- **Dotsenko-Fateev integrals:** When $N = -\frac{\alpha}{\gamma}$ is a positive integer, up to a normalizing constant, our GMC expression (1.6) for the conformal block $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ equals

$$\left(\int_0^1\right)^N \prod_{1\le i<j\le N} |\Theta_\tau(x_i - x_j)|^{-\frac{\gamma^2}{4}} \prod_{i=1}^N \Theta_\tau(x_i)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x_i} \prod_{i=1}^N dx_i. \tag{1.7}$$

This can be seen by a direct Gaussian computation on $\mathcal{G}^\alpha_{\gamma,P}(q)$; see Lemma 6.8. This integral representation of $\mathcal{F}^\alpha_{\gamma,P}(q)$ was first proposed in [FLNO09]. Such an integral is an example of a Dotsenko-Fateev integral [DF84, DF85]. Our formula (1.6) can therefore be viewed as an extension of the integral (1.7) to the case when N is not a positive integer. This is in the same spirit as that of the DOZZ formula that extends a certain Selberg type multiple integral; see [DKRV16, Section 5.1]. Dotsenko-Fateev integral representations are available under certain specializations of parameters for more general conformal blocks, including the 4-point spherical case; see [MMS10, DV09].

- **Zamolodchikov's recursion:** In [Zam84, Zam87], Zamolodchikov derived recursion relations for conformal blocks on the sphere which uniquely specify their formal series expansions and provide a rapidly converging method to compute their numerical value. In [Pog09], Poghossian conjectured the analogous recursion (1.3) for the toric case, which was proven for 1-point toric conformal blocks in [FL10, HJS10] and for multipoint toric conformal blocks in [CCY19]; we give a sketch of the proof given in [FL10] in Appendix A. One important step of our proof of Theorem 1.1 is to establish an analogue of (1.3) for the Dotsenko-Fateev integral expression (1.6) of the probabilistic conformal block when $N = -\frac{\alpha}{\gamma}$ is a positive integer; see Theorem 6.5.
- **AGT correspondence:** In [AGT09], Alday-Gaiotto-Tachikawa conjectured a general correspondence between LCFT and four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theory. In particular, conformal blocks correspond to the so-called Nekrasov partition function on the gauge theory side, which has been the topic of extensive study in both mathematics and physics; see e.g. [Nek03, NO06]. In our setting of 1-point toric conformal blocks, this correspondence is given by (1.4), which was proven in [FL10]; another proof was later given in [Neg16] using the work of [CO12]. Both proofs consider the conformal block as a formal power series and ignore convergence. From this perspective, our Theorem 1.1 proves that the Nekrasov partition function (1.5) is analytic in q, resolving a conjecture of [FML18][1].

1.4. **Summary of method.** We first use Girsanov's theorem to show that the GMC expression $\mathcal{G}^\alpha_{\gamma,P}(q)$ in (1.6) has the desired analytic properties in q prescribed by Theorem 1.1. To prove that its Taylor series is given by the conformal block $\mathcal{F}^\alpha_{\gamma,P}(q)$ in (1.4), we show that the q-series coefficients of both $\mathcal{G}^\alpha_{\gamma,P}(q)$ and $\mathcal{F}^\alpha_{\gamma,P}(q)$ are solutions to the coupled system of two difference equations (6.3) in the α variable. These *shift equations* are inhomogeneous first order difference equations with difference 2χ for $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$, which have unique solutions when γ^2 is irrational. A pair of similar homogeneous shift equations were proposed for the DOZZ formula in [Tes95] and used in its proof in [KRV19a], while other versions played a similar role in [Rem20, RZ18, RZ20].

To establish the shift equations for the series coefficients of $\mathcal{G}^\alpha_{\gamma,P}(q)$, for $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$ we define a deformed GMC expressions $\psi^\alpha_\chi(u, q)$ in (3.4) corresponding to adding a degenerate insertion with weight χ at the point u. We then prove in Theorem 3.5 that $\psi^\alpha_\chi(u, q)$ satisfies the *BPZ equation*, which for $l_\chi = \frac{\chi^2}{2} - \frac{\alpha\chi}{2}$ and $\wp$ denoting Weierstrass's elliptic function is the PDE

$$\Big(\partial_{uu} - l_\chi(l_\chi + 1)\wp(u) + 2\mathbf{i}\pi\chi^2\partial_\tau\Big)\psi^\alpha_\chi(u, q) = 0 \tag{1.8}$$

relating variation in the modular parameter τ and the additional parameter u. This equation was proposed for Dotsenko-Fateev type integral expressions for conformal blocks in [FLNO09] and coincides with the KZB heat equation described in [Ber88] for the WZW model on the torus.

We then apply separation of variables to the BPZ equation (1.8), obtaining that the q-series coefficients of $\psi^\alpha_\chi(u, q)$ satisfy a system of coupled inhomogeneous hypergeometric ODEs after a proper normalization. Each ODE in this system has a two dimensional solution space, and we obtain the shift equations in Theorem 6.1 by analyzing the solution space near $u = 0$ and $u = 1$ using the *operator product expansions* (OPEs) of Theorem 5.4, which characterize the behavior of the deformed blocks $\psi^\alpha_\chi(u, q)$ near $u = 0, 1$. This argument is a generalization of the one used in [KRV19a] to prove the DOZZ formula, although that case only involved a single homogeneous hypergeometric ODE. We mention also that the OPE for $\chi = \frac{2}{\gamma}$ requires an intricate reflection argument making use of the results and the techniques of [RZ20].

[1]More precisely, they state their conjecture for the 4-point spherical conformal block. In light of [FLNO09, Pog09, HJS10], the 1-point toric conformal block is a special case of the 4-point spherical conformal block under a parameter change.

Finally, to show that the series coefficients of $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ satisfies the shift equations, we leverage the integral expression (1.7) for $\mathcal{G}^{\alpha}_{\gamma,P}(q)$ when $N := -\frac{\alpha}{\gamma}$ is a positive integer. First, the integral expression (1.7) allows us to check that $\mathcal{G}^{\alpha}_{\gamma,P}(q)$ satisfies Zamolodchikov's recursion (1.3) and therefore equals $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ as a formal q-series when N is a positive integer. This implies that the series coefficients of $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ satisfies the shift equations with $\chi = \frac{\gamma}{2}$ on a sequence of γ's limiting to 0 by virtue of its equality with $\mathcal{G}^{\alpha}_{\gamma,P}(q)$. An analytic argument based on the meromorphicity of q-series coefficients of $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ in γ then shows that the shift equations for $\chi = \frac{\gamma}{2}$ hold for all values of γ. Finally, the shift equations for $\chi = \frac{2}{\gamma}$ follow from the fact that $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ is invariant under the exchange $\frac{\gamma}{2} \leftrightarrow \frac{2}{\gamma}$, yielding both shift equations for $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ and completing our proof. This procedure is carried out in detail in Section 6.

1.5. **Outlook.** We now outline a few directions that we are working on or will investigate in the future.

Modular transformations for conformal blocks. For τ in the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, the flat torus $\mathbb{T}_\tau$ with modular parameter τ is obtained by identifying the opposite edges of the parallelogram on $\mathbb{H}$ with vertices $0, 1, \tau$, and $\tau + 1$. Given $\tau, \tau' \in \mathbb{H}$, the tori $\mathbb{T}_\tau$ and $\mathbb{T}_{\tau'}$ are conformally equivalent if and only if τ and τ' are such that $\tau' = \frac{a\tau+b}{c\tau+d}$ for some integers a, b, c, d. Therefore, the *moduli space* of the torus, which is the space of Riemannian metrics on tori modulo conformal equivalence, is given by the quotient $\mathbb{H}/\mathrm{PSL}_2(\mathbb{Z})$ of $\mathbb{H}$ by the *modular group* $\mathrm{PSL}_2(\mathbb{Z})$, which is generated by $\tau \mapsto \tau + 1$ and $\tau \mapsto -\tau^{-1}$.

Conformal invariance implies that correlation functions of a 2D CFT on the torus transform simply under the action of the modular group. In particular, the 1-point function from (1.2) satisfies $\langle e^{\alpha\phi(0)}\rangle_\tau = \langle e^{\alpha\phi(0)}\rangle_{\tau+1}$ and moreover, $\langle e^{\alpha\phi(0)}\rangle_\tau$ and $\langle e^{\alpha\phi(0)}\rangle_{-\tau^{-1}}$ are related by a simple factor; see [DRV16]. Through the conformal bootstrap equation (1.2), the modular symmetry of $\langle e^{\alpha\phi(0)}\rangle_\tau$ translates into a quite nontrivial relation between $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ and $\mathcal{F}^{\alpha}_{\gamma,P}(\tilde{q})$ with $q = e^{\mathbf{i}\pi\tau}$ and $\tilde{q} = e^{-\mathbf{i}\pi\tau^{-1}}$. Following the insight of Verlinde [Ver88], Moore and Seiberg [MS89], it is believed that a deeper statement should hold: the modular group $\mathrm{PSL}_2(\mathbb{Z})$ induces a linear action on the linear span of $\{q^{\frac{P^2}{2}}\mathcal{F}^{\alpha}_{\gamma,P}(q) : P \in \mathbb{R}\}$, which is realized by the equation

$$\tilde{q}^{\frac{P^2}{2}}\mathcal{F}^{\alpha}_{\gamma,P}(\tilde{q}) = \int_{\mathbb{R}} \mathcal{M}_{\gamma,\alpha}(P,P')q^{\frac{P'^2}{2}}\mathcal{F}^{\alpha}_{\gamma,P'}(q)dP' \tag{1.9}$$

for a certain explicit *modular kernel* $\mathcal{M}_{\gamma,\alpha}(P,P')$. Given (1.9), the modular symmetry for the right hand side of the bootstrap equation (1.2) follows from the unitarity proven in [PT99, PT01] of the modular transformation under a certain inner product. The explicit formula of $\mathcal{M}_{\gamma,\alpha}(P,P')$ was derived by Ponsot and Teschner [PT99] under the assumption that there exists a kernel $\mathcal{M}_{\gamma,\alpha}(P,P')$ satisfying (1.9). However, the equation (1.9) itself is still open as a mathematical question.

In a work in progress, we plan to prove (1.9) for $\alpha \in [0, Q)$ based on our explicit probabilistic construction of $\mathcal{F}^{\alpha}_{\gamma,P}(q)$. More precisely, we will use the BPZ equation, GMC techniques, and the explicit form of the modular kernel to show that $\tilde{q}^{\frac{P^2}{2}}\mathcal{G}^{\alpha}_{\gamma,P}(\tilde{q}) = \int_{\mathbb{R}} \mathcal{M}_{\gamma,\alpha}(P,P')q^{\frac{P'^2}{2}}\mathcal{G}^{\alpha}_{\gamma,P'}(q)dP'$ for $q \in (0,1)$, where $\mathcal{G}^{\alpha}_{\gamma,P}(q)$ is the GMC in Theorem 1.1. Once this is done, we can use the $\mathrm{PSL}_2(\mathbb{Z})$ action to analytically continue $\mathcal{G}^{\alpha}_{\gamma,P}(q)$. Recall that one of the fundamental domains of $\mathrm{PSL}_2(\mathbb{Z})$ on $\mathbb{H}$ has interior $\{\tau \in \mathbb{H} : \operatorname{Re}\tau \in (-\frac{1}{2}, \frac{1}{2})$ and $|\tau| > 1\}$, which is contained in $\{\tau \in \mathbb{H} : |q| < \frac{1}{2}$ with $q = e^{\mathbf{i}\pi\tau}\}$. Thus when $\alpha \in [0, Q)$, since Theorem 1.1 shows $q \mapsto \mathcal{G}^{\alpha}_{\gamma,P}(q)$ is analytic for $|q| < \frac{1}{2}$, the function $\mathcal{G}^{\alpha}_{\gamma,P}(q)$ admits an analytic continuation to the whole unit disk. This means that $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ indeed has convergence radius 1 in this range of α, and (1.9) holds.

4-point spherical conformal blocks. As discussed in Section 1.3, the GMC expression of $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ specializes to a Dotsenko-Fateev type integral when $-\frac{\alpha}{\gamma} \in \mathbb{N}$. Such an integral representation is available under certain specializations of parameters for more general conformal blocks, including the 4-point spherical case; see [MMS10, DV09]. This allows us to propose a GMC expression for 4-point spherical conformal blocks and hence an analog of Theorem 1.1. We hope to prove this analog in a future work. Moreover, similar to (1.9), there is a linear transformation on the linear space spanned by the 4-point spherical conformal blocks called the *fusion transformation*, which is responsible for the so-called *crossing symmetry* of the conformal bootstrap for four-point sphere; see [GKRV20, Eq (1.16)]. We also hope to establish the fusion transformation and use it to study the analytic continuation of conformal blocks. As a long term goal, we hope to extend our GMC framework to conformal blocks on a genus-g surface with n points, and explore their symmetries predicted by Verlinde [Ver88], Moore and Seiberg [MS89], Ponsot and Teschner [PT99].

Conformal bootstrap for LCFT. The method in [GKRV20] is based on constructing the Hilbert space of LCFT and applying spectral theory to diagonalize the Liouville Hamiltonian, which has the potential to extend to prove the conformal bootstrap for sphere and torus with n points, assuming $\gamma \in (0, \sqrt{2})$. As announced in [GKRV20], the authors are currently working on proving (1.2) for $\gamma \in (0, \sqrt{2})$. However, the method presents an essential obstruction to extending their approach to $\gamma \in [\sqrt{2}, 2)$. We hope to prove (1.2) for all $\gamma \in (0, 2)$ in a future work using our probabilistic knowledge of $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ and a strategy similar to that of this paper. Namely, we plan to show that appropriate u-deformations of both sides of (1.2) obey the BPZ equation, satisfy certain OPEs, and have certain analytic properties in q allowing us to conclude equality by establishing a system of shift equations. More generally, once our framework is extended to other conformal blocks as discussed above, we hope to address the corresponding conformal bootstrap statement for LCFT.

1.6. **Organization of the paper.** The remainder of this paper is organized as follows. In Section 2, we prove the analytic continuation property of the probabilistic conformal block $\mathcal{G}^{\alpha}_{\gamma,P}(q)$ prescribed by Theorem 1.1, and then reduce Theorem 1.1 to a variant Theorem 2.13. In Section 3, we define deformed versions of $\mathcal{G}^{\alpha}_{\gamma,P}(q)$, characterize their analytic properties, and prove the BPZ equations stated in Theorem 3.5. In Section 4, we perform separation of variables for the deformed probabilistic conformal block and derive from the BPZ equations a system of coupled inhomogenous hypergeometric equations. In Section 5, we state the operator product expansions (OPEs) for these deformed conformal blocks in Theorem 5.4, and perform an analytic continuation in α leveraging crucially a reflection principle. In Section 6, we use the results derived in Sections 4 and 5 to obtain two shift equations on series coefficients of our probabilistic conformal blocks in Theorem 6.1. We then put everything together to prove Theorem 2.13 by deriving Theorem 6.5 giving Zamolodchikov's recursion for our probabilistic conformal block when $N = -\frac{\alpha}{\gamma}$ is an integer. Appendices A, B, C, D, and E respectively collect the definition of conformal blocks from the Virasoro algebra, facts and conventions on special functions, background on Gaussian multiplicative chaos, facts about the Gauss hypergeometric equation, and the proof of the OPE statements used in the main text.

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2. Probabilistic construction of the conformal block

In this section, we give the precise definition of the probabilistic conformal block $\mathcal{G}^{\alpha}_{\gamma,P}(q)$, prove its analytic continuation property prescribed by Theorem 1.1, and reduce Theorem 1.1 to a variant Theorem 2.13 whose proof occupies the rest of the paper.

We will use the following notations. Let $\mathbb{C}$ be the complex plane. If $K \subset U \subset \mathbb{C}$ and U is open, we say that U is a complex neighborhood of K. Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathbb{H}$ be the upper half plane and $\mathbb{D}$ be the unit disk. For $\tau \in \mathbb{H}$, let $q = q(\tau) = e^{\mathbf{i}\pi\tau} \in \mathbb{D}$. In particular, $\tau \in \mathbf{i}\mathbb{R}_{>0}$ if and only if $q \in (0, 1)$. We recall the Jacobi theta function Θ_τ and the Dedekind eta function η from Appendix B. Throughout Sections 2—6.1, we view $\gamma \in (0, 2)$ as a fixed parameter and set $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$ as in (1.1).

2.1. **Definition of Gaussian multiplicative chaos.** We begin by introducing Gaussian multiplicative chaos (GMC), the probabilistic object which enables our construction. Let $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n\}_{n\geq 1}$, $\{\alpha_{n,m}\}_{n,m\geq 1}$, $\{\beta_{n,m}\}_{n,m\geq 1}$ be sequences of i.i.d. standard real Gaussians. For $\tau \in \mathbb{H}$, the following series converge almost surely and define Gaussian fields Y_∞ and Y_τ on $[0, 1]$ by

$$Y_\infty(x) := \sum_{n\geq 1} \sqrt{\frac{2}{n}} \Big(\alpha_n \cos(2\pi n x) + \beta_n \sin(2\pi n x)\Big), \qquad x \in [0, 1]; \tag{2.1}$$

$$Y_\tau(x) := Y_\infty(x) + 2 \sum_{n,m\geq 1} \frac{q^{nm}}{\sqrt{n}} \Big(\alpha_{n,m} \cos(2\pi n x) + \beta_{n,m} \sin(2\pi n x)\Big), \qquad x \in [0, 1]. \tag{2.2}$$

We interpret the series in (2.1) in the sense of generalized functions while the one in (2.2) is a pointwise sum.

Both Y_∞ and Y_τ are examples of log-correlated fields, whose covariance kernels have a logarithmic singularity along the diagonal. Although Y_∞ is not pointwise defined, we use the intuitive notion $\mathbb{E}[Y_\infty(x)Y_\infty(y)]$ to represent its covariance kernel. See Appendix C for general background on log-correlated fields and more details on these conventions.

Lemma 2.1. *For $\gamma \in (0,2)$ and $\tau \in \mathbf{i}\mathbb{R}_{>0}$, the covariance kernel of Y_∞ and Y_τ are given by*

$$\mathbb{E}[Y_\infty(x)Y_\infty(y)] = -2\log|2\sin(\pi(x-y))|, \tag{2.3}$$

$$\mathbb{E}[Y_\tau(x)Y_\tau(y)] = -2\log|\Theta_\tau(x-y)| + 2\log|q^{1/6}\eta(q)|. \tag{2.4}$$

Remark 2.2. We emphasize that Lemma 2.1 does not hold if $\tau \notin \mathbf{i}\mathbb{R}_{>0}$. Note that $\mathbb{E}[Y_\tau(x)Y_\tau(y)]$ is analytic in $\tau \in \mathbb{H}$ while the right hand side of (2.4) is not.

Proof of Lemma 2.1. For the first covariance, notice that

$$\mathbb{E}[Y_\infty(x)Y_\infty(y)] = \mathbb{E}[Y_\tau(x)Y_\infty(y)] = \sum_{n\geq 1}\frac{2}{n}\cos(2\pi n(x-y)) = -2\log|2\sin(\pi(x-y))|,$$

where the last equality follows by computing Fourier series. For the second covariance, notice that

$$\begin{aligned}\mathbb{E}[Y_\tau(x)Y_\tau(y)] &= \mathbb{E}[Y_\infty(x)Y_\infty(y)] + \sum_{n,m\geq 1}\frac{4q^{2nm}}{n}\cos(2\pi n(x-y))\\ &= -2\log|2\sin(\pi(x-y))| - 2\sum_{m\geq 1}\log|(1-q^{2m}e^{2\mathbf{i}\pi(x-y)})(1-q^{2m}e^{-2\mathbf{i}\pi(x-y)})|\\ &= -2\log|\Theta_\tau(x-y)| + 2\log|q^{1/6}\eta(q)|.\end{aligned}$$

□

Remark 2.3. Let $X_{\mathbb{D}}$ be the Gaussian free field on $\mathbb{D}$ with free boundary conditions (see [DMS14a, Section 4.1.4]). Then Y_∞ can be viewed as the restriction to the unit circle of $X_{\mathbb{D}}$, under the identification $Y_\infty(x) = X_{\mathbb{D}}(e^{2\pi\mathbf{i}x})$. Similarly, suppose $\tau \in \mathbf{i}\mathbb{R}_{>0}$ and let $\mathbb{T}_\tau$ be the torus obtained by identifying the opposite sides of the rectangle with $0, \tau, 1, \tau+1$ as vertices. Let $\sqrt{2}X_\tau$ be distributed as the *Gaussian free field* on $\mathbb{T}_\tau$ (see definition in [Bav19, Equation (2.5)]). Then the restriction of X_τ to the loop parametrized by $[0,1]$ has the law of $Y_\tau + \mathcal{N}(0, -\frac{1}{3}\log q)$ where $\mathcal{N}(0,-\frac{1}{3}\log q)$ is a Gaussian random variable with variance $-\frac{1}{3}\log q$ independent of Y_τ. See Appendix C for more details.

We now introduce the Gaussian Multiplicative Chaos (GMC) measures $e^{\frac{\gamma}{2}Y_\infty(x)}dx$ and $e^{\frac{\gamma}{2}Y_\tau(x)}dx$ on $[0,1]$ for τ purely imaginary. Because the fields $Y_\infty(x)$ and $Y_\tau(x)$ live in the space of distributions, exponentiating them requires a regularization procedure, which we perform as follows. For $N \in \mathbb{N}$, define

$$\begin{aligned}Y_{\infty,N}(x) &= \sum_{n=1}^{N}\sqrt{\frac{2}{n}}\Big(\alpha_n\cos(2\pi nx) + \beta_n\sin(2\pi nx)\Big)\\ Y_{\tau,N}(x) &= Y_{\infty,N}(x) + 2\sum_{n,m=1}^{\infty}\frac{q^{nm}}{\sqrt{n}}\Big(\alpha_{n,m}\cos(2\pi nx) + \beta_{n,m}\sin(2\pi nx)\Big).\end{aligned}$$

Throughout this paper, when we consider the GMC measure $e^{\frac{\gamma}{2}Y_\tau(x)}dx$, we always assume $\tau \in \mathbf{i}\mathbb{R}_{>0}$. For more background on GMC we refer to [RV14, Ber17] and our Appendix C.

Definition 2.4 (Gaussian Multiplicative Chaos)**.** For $\gamma \in (0,2)$ and $\tau \in \mathbf{i}\mathbb{R}_{>0}$, we define the Gaussian multiplicative chaos measures $e^{\frac{\gamma}{2}Y_\infty(x)}dx$ and $e^{\frac{\gamma}{2}Y_\tau(x)}dx$ to be the weak limits of measures in probability

$$\begin{aligned}e^{\frac{\gamma}{2}Y_\infty(x)}dx &:= \lim_{N\to\infty} e^{\frac{\gamma}{2}Y_{\infty,N}(x)-\frac{\gamma^2}{8}\mathbb{E}[Y_{\infty,N}(x)^2]}dx\\ e^{\frac{\gamma}{2}Y_\tau(x)}dx &:= \lim_{N\to\infty} e^{\frac{\gamma}{2}Y_{\tau,N}(x)-\frac{\gamma^2}{8}\mathbb{E}[Y_{\tau,N}(x)^2]}dx.\end{aligned}$$

More precisely, for any continuous test function $f : [0,1] \to \mathbb{R}$, we have in probability that

$$\begin{aligned}\int_0^1 f(x)e^{\frac{\gamma}{2}Y_\infty(x)}dx &= \lim_{N\to\infty}\int_0^1 f(x)e^{\frac{\gamma}{2}Y_{\infty,N}(x)-\frac{\gamma^2}{8}\mathbb{E}[Y_{\infty,N}(x)^2]}dx\\ \int_0^1 f(x)e^{\frac{\gamma}{2}Y_\tau(x)}dx &= \lim_{N\to\infty}\int_0^1 f(x)e^{\frac{\gamma}{2}Y_{\tau,N}(x)-\frac{\gamma^2}{8}\mathbb{E}[Y_{\tau,N}(x)^2]}dx.\end{aligned}$$

For a general $\tau \in \mathbb{H}$, it will also be convenient to introduce the field

$$F_\tau(x) := Y_\tau(x) - Y_\infty(x) = 2 \sum_{n,m\geq 1} \frac{q^{nm}}{\sqrt{n}} \Big(\alpha_{n,m}\cos(2\pi n x) + \beta_{n,m}\sin(2\pi n x)\Big) \text{ for } x \in [0,1], \tag{2.5}$$

for which the following observation is straightforward.

Lemma 2.5. *Almost surely, for each $x \in [0,1]$, as a function of q, $F_\tau(x)$ is analytic in $q \in \mathbb{D}$. For a fixed $\tau \in \mathbb{H}$, $\{F_\tau(x)\}_{x\in[0,1]}$ is a continuous Gaussian field on $[0,1]$ independent of Y_∞. Moreover, if $\tau \in \mathbf{i}\mathbb{R}_{>0}$,*

$$\mathbb{E}[F_\tau(x)^2] = 4\sum_{n,m\geq 1} \frac{q^{2nm}}{n} = -4\log|q^{-1/12}\eta(q)| \qquad \text{for each } x \in [0,1].$$

Due to our normalization, the measures $e^{\frac{\gamma}{2}Y_\tau(x)}dx$ and $e^{\frac{\gamma}{2}F_\tau(x)}e^{\frac{\gamma}{2}Y_\infty(x)}dx$ do not coincide. Instead, by Definition 2.4 and Lemma 2.5, we have

$$e^{\frac{\gamma}{2}Y_\tau(x)}dx = e^{-\frac{\gamma^2}{8}\mathbb{E}[F_\tau(0)^2]} e^{\frac{\gamma}{2}F_\tau(x)}e^{\frac{\gamma}{2}Y_\infty(x)}dx \quad \text{for } \tau \in \mathbf{i}\mathbb{R}_{>0}. \tag{2.6}$$

2.2. **Definition and analyticity of $\mathcal{G}^\alpha_{\gamma,P}(q)$.** We are ready to give the precise definition of the probabilistic conformal block $\mathcal{G}^\alpha_{\gamma,P}(q)$. By Lemma C.4, for $\alpha \in (-\frac{4}{\gamma}, Q)$, $q \in (0,1)$, and $P \in \mathbb{R}$ we have

$$\mathbb{E}\left[\left(\int_0^1 |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\tau(x)}dx\right)^{-\frac{\alpha}{\gamma}}\right] < \infty. \tag{2.7}$$

Recalling Definition B.8 and (B.21), for $x \in (0,1)$ we have $\Theta_\tau(x)^{-\alpha\gamma/2} = e^{-\mathbf{i}\pi\alpha\gamma/2}|\Theta_\tau(x)|^{-\alpha\gamma/2}$. Therefore, for $\beta \in \mathbb{R}$, we should interpret $\left(\int_0^1 \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\tau(x)}dx\right)^\beta$ via

$$\left(\int_0^1 \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\tau(x)}dx\right)^\beta = e^{-\mathbf{i}\pi\alpha\gamma\beta/2}\left(\int_0^1 |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\tau(x)}dx\right)^\beta. \tag{2.8}$$

Definition 2.6 (Probabilistic conformal block)**.** For $\alpha \in (-\frac{4}{\gamma}, Q)$, $q \in (0,1)$, and $P \in \mathbb{R}$, let

$$\mathcal{G}^\alpha_{\gamma,P}(q) := \frac{1}{Z}\mathbb{E}\left[\left(\int_0^1 \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\tau(x)}dx\right)^{-\frac{\alpha}{\gamma}}\right], \tag{2.9}$$

where the normalization Z is

$$Z := q^{\frac{1}{12}(\frac{\alpha\gamma}{2}+\frac{\alpha^2}{2}-1)}\eta(q)^{\alpha^2+1-\frac{\alpha\gamma}{2}}\mathbb{E}\left[\left(\int_0^1 [-2\sin(\pi x)]^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\infty(x)}dx\right)^{-\frac{\alpha}{\gamma}}\right]. \tag{2.10}$$

We call $\mathcal{G}^\alpha_{\gamma,P}(q)$ the *probabilistic* 1*-point toric conformal block*.

Remark 2.7. In Equation (2.15) we relate the normalization Z in Definition 2.6 to a quantity $\mathcal{A}_{\gamma,P,0}(\alpha)$ which has an explicit formula given by Proposition 6.4. This will in turn give an explicit expression for Z.

In this section we prove the following proposition which says that $\mathcal{G}^\alpha_{\gamma,P}(q)$ has the desired analytic continuation property prescribed by Theorem 1.1.

Proposition 2.8. *For $\gamma \in (0,2)$, $\alpha \in (-\frac{4}{\gamma}, Q)$, and $P \in \mathbb{R}$, the probabilistic conformal block $\mathcal{G}^\alpha_{\gamma,P}(q)$ admits an analytic extension on a complex neighborhood of $q = 0$. Moreover, when $\alpha \in [0,Q)$, the analytic extension of $\mathcal{G}^\alpha_{\gamma,P}(q)$ exists on both $\{|q| < \frac{1}{2}\}$ and a complex neighborhood of $[0,1)$.*

Before proving Proposition 2.8, we now introduce a variant of $\mathcal{G}^\alpha_{\gamma,P}(q)$ which is more convenient to work with. For $\alpha \in (-\frac{4}{\gamma}, Q)$, $q \in (0,1)$, and $P \in \mathbb{R}$, define

$$\mathcal{A}^q_{\gamma,P}(\alpha) := q^{\frac{1}{12}(-\alpha\gamma-\frac{2\alpha}{\gamma}+2)}\eta(q)^{\alpha\gamma+\frac{2\alpha}{\gamma}-\frac{3}{2}\alpha^2-2}\mathbb{E}\left[\left(\int_0^1 \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\tau(x)}dx\right)^{-\frac{\alpha}{\gamma}}\right]. \tag{2.11}$$

Here we use the notation $\mathcal{A}^q_{\gamma,P}(\alpha)$ instead of $\mathcal{A}^\alpha_{\gamma,P}(q)$ because we mostly view $\mathcal{A}^q_{\gamma,P}(\alpha)$ as a function of α with a parameter q. Proposition 2.8 is an immediate consequence of the following lemma.

Lemma 2.9. *Fix $\gamma \in (0,2)$ and $P \in \mathbb{R}$. The quantity $\mathcal{A}^q_{\gamma,P}(\alpha)$ satisfies the following analytic properties.*

(a) *For $\alpha \in (-\frac{4}{\gamma}, Q)$, the function $q \mapsto \mathcal{A}^q_{\gamma,P}(\alpha)$ admits an analytic extension on a complex neighborhood of $q = 0$. Moreover, when $\alpha \in [0, Q)$, the analytic extension exists on both a complex neighborhood of $[0, 1)$ and on $\{|q| < \frac{1}{2}\}$.*

(b) *There exists an open set in $\mathbb{C}^2$ containing $\{(\alpha, q) : \alpha \in (-\frac{4}{\gamma}, Q)$ and $q = 0\}$ on which $(\alpha, q) \mapsto \mathcal{A}^q_{\gamma,P}(\alpha)$ admits an analytic extension.*

(c) *For $n \in \mathbb{N}_0$, the function $\alpha \mapsto \mathcal{A}_{\gamma,P,n}(\alpha)$ can be analytically extended to a complex neighborhood of $(-\frac{4}{\gamma}, Q)$, where $\{\mathcal{A}_{\gamma,P,n}(\alpha)\}_{n\geq 1}$ is defined by*

$$\mathcal{A}^q_{\gamma,P}(\alpha) = \sum_{n=0}^{\infty} \mathcal{A}_{\gamma,P,n}(\alpha) q^n \qquad \text{for } |q| \text{ sufficiently small.} \tag{2.12}$$

We postpone the proof of Lemma 2.9 to Section 2.5 and proceed to show how it implies Proposition 2.8. Define normalized versions of $\mathcal{A}^q_{\gamma,P}$ and $\mathcal{A}_{\gamma,P,n}$ from Lemma 2.9 by

$$\widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha) := \frac{\mathcal{A}^q_{\gamma,P}(\alpha)}{\mathcal{A}_{\gamma,P,0}(\alpha)} \qquad \text{and} \qquad \widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha) := \frac{\mathcal{A}_{\gamma,P,n}(\alpha)}{\mathcal{A}_{\gamma,P,0}(\alpha)}. \tag{2.13}$$

Proof of Proposition 2.8 given Lemma 2.9. Note that (2.11), (2.12), and (2.13) yield that

$$\mathcal{G}^\alpha_{\gamma,P}(q) = \left(q^{-\frac{1}{12}}\eta(q)\right)^{1-\alpha(Q-\frac{\alpha}{2})} \widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha) \tag{2.14}$$

and

$$Z = q^{\frac{1}{12}(\frac{\alpha\gamma}{2}+\frac{\alpha^2}{2}-1)}\eta(q)^{\alpha^2+1-\frac{\alpha\gamma}{2}}\mathcal{A}_{\gamma,P,0}(\alpha). \tag{2.15}$$

Recall from Lemma B.1 that $q^{-\frac{1}{12}(1-\alpha(Q-\frac{\alpha}{2}))}\eta(q)^{1-\alpha(Q-\frac{\alpha}{2})}$ is a convergent power series for $|q| < 1$. Using (2.14) and Lemma 2.9 (a), we get Proposition 2.8. □

2.3. **1-point toric conformal block and Nekrasov partition function.** In this section, we give a precise definition of the 1-point toric conformal block using the AGT correspondence and then review Zamolodchikov's recursion for it. We survey the original definition based on the Virasoro algebra in Appendix A, as it is not needed for the rest of the paper. We first define the 1-point Nekrasov partition function on the torus as the formal q-series

$$\mathcal{Z}^\alpha_{\gamma,P}(q) := 1 + \sum_{k=1}^{\infty} \mathcal{Z}_{\gamma,P,k}(\alpha) q^{2k}, \tag{2.16}$$

where

$$\mathcal{Z}_{\gamma,P,k}(\alpha) := \sum_{\substack{(Y_1,Y_2) \text{ Young diagrams} \\ |Y_1|+|Y_2|=k}} \prod_{i,j=1}^{2} \prod_{s\in Y_i} \frac{(E_{ij}(s,P)-\alpha)(Q-E_{ij}(s,P)-\alpha)}{E_{ij}(s,P)(Q-E_{ij}(s,P))} \tag{2.17}$$

for

$$E_{ij}(s,P) := \begin{cases} \mathbf{i}P - \frac{\gamma}{2}H_{Y_j}(s) + \frac{2}{\gamma}(V_{Y_i}(s)+1) & i=1, j=2 \\ -\frac{\gamma}{2}H_{Y_j}(s) + \frac{2}{\gamma}(V_{Y_i}(s)+1) & i=j \\ -\mathbf{i}P - \frac{\gamma}{2}H_{Y_j}(s) + \frac{2}{\gamma}(V_{Y_i}(s)+1) & i=2, j=1. \end{cases} \tag{2.18}$$

Here, we draw a Young diagram Y corresponding to a partition λ in the first quadrant with unit squares so that the top right corner of each square has positive coordinates. In (2.18), for a unit square s with top right corner (i, j), we define $H_Y(s) = \lambda'_j - i$ and $V_Y(s) = \lambda_i - j$, where λ' is the transposed partition to λ.

For the following definition, recall from Lemma B.1 that for each $\beta \in \mathbb{R}$, $[q^{-\frac{1}{12}}\eta(q))]^\beta$ is a power series in q convergent for $|q| < 1$.

Definition 2.10. The 1-*point toric conformal block* is the formal q-series given by

$$\mathcal{F}^\alpha_{\gamma,P}(q) := \left(q^{-\frac{1}{12}}\eta(q)\right)^{1-\alpha(Q-\frac{\alpha}{2})} \mathcal{Z}^\alpha_{\gamma,P}(q). \tag{2.19}$$

The original definition of $\mathcal{F}^{\alpha}_{\gamma,P}(q)$, reviewed in Appendix A, is via the Virasoro algebra. In this paper, we define $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ in terms of $\mathcal{Z}^{\alpha}_{\gamma,P}(q)$ instead for concreteness. The fact that these two definitions agree is precisely the AGT correspondence for 1-point torus proven in [FL10, Neg16]. We will not use the precise expression (2.17) beyond the following information it provides: $\mathcal{Z}_{\gamma,P,k}(\alpha)$ is a rational function in P, Q, α. In particular, $\mathcal{Z}_{\gamma,P,k}(\alpha)$ depends on γ through $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.

We now review another characterization of $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ which is a toric variant of Zamolodchikov's recursion [Zam84]. Define the quantity

$$R_{\gamma,m,n}(\alpha) := \frac{2\prod_{j=-m}^{m-1}\prod_{l=-n}^{n-1}(Q - \frac{\alpha}{2} + \frac{j\gamma}{2} + \frac{2l}{\gamma})}{\prod_{(j,l)\in S_{m,n}}(\frac{j\gamma}{2} + \frac{2l}{\gamma})} \tag{2.20}$$

for $S_{m,n} := \{(j,l) \in \mathbb{Z}^2 \mid 1-m \le j \le m, 1-n \le l \le n, (j,l) \notin \{(0,0),(m,n)\}\}$ and

$$P_{m,n} := \frac{2\mathbf{i}n}{\gamma} + \frac{\mathbf{i}\gamma m}{2}. \tag{2.21}$$

The q-series expansion of $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ can be characterized by the following recursive relation.

Proposition 2.11 (Zamolodchikov's recursion)**.** *The formal q-series $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ defined in (2.10) satisfies*

$$\mathcal{F}^{\alpha}_{\gamma,P}(q) = \sum_{n,m=1}^{\infty} q^{2mn}\frac{R_{\gamma,m,n}(\alpha)}{P^2 - P_{m,n}^2}\mathcal{F}^{\alpha}_{\gamma,P_{-m,n}}(q) + q^{\frac{1}{12}}\eta(q)^{-1}. \tag{2.22}$$

Proposition 2.11 is a concrete identity in terms of the rational functions $\mathcal{Z}_{\gamma,P,k}(\alpha)$ defined in (2.17), which asserts that the q-series $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ defined through $q^{-\frac{1}{12}(1-\alpha(Q-\frac{\alpha}{2}))}\eta(q)^{1-\alpha(Q-\frac{\alpha}{2})}\mathcal{Z}^{\alpha}_{\gamma,P}(q)$ satisfies (2.22). This is proven rigorously in an elementary way in [FL10, Section 2], although overall [FL10] is a theoretical physics paper. On the other hand, it is not hard to prove the recursion (2.22) from the Virasoro algebra definition of $\mathcal{F}^{\alpha}_{\gamma,P}(q)$. We include a proof sketch in Appendix A. This combined with the proof of the AGT correspondence in [Neg16] yields an alternative proof of Proposition 2.11.

Remark 2.12. We parametrize the conformal block as a function of P and α because these are convenient coordinates for our GMC expressions. In mathematical physics, it is more common to represent it as a function of *conformal dimension* $\Delta_\alpha = \frac{\alpha}{2}(Q - \frac{\alpha}{2})$ corresponding to *momentum* α and the *intermediate dimension* $\Delta = \frac{1}{4}(Q^2 + P^2)$ corresponding to momentum $Q + \mathbf{i}P$.

2.4. **A one-step reduction.** Using Proposition 2.8 we can reduce Theorem 1.1 to Theorem 2.13, whose proof will occupy the remainder of this paper. Recall (2.12) and (2.13), which give that $\widetilde{\mathcal{A}}^{q}_{\gamma,P}(\alpha) = \sum_{n=0}^{\infty}\widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha)q^n$ for q small enough.

Theorem 2.13. *For $\gamma \in (0,2)$, $\alpha \in (-\frac{4}{\gamma}, Q)$, and $P \in \mathbb{R}$, as formal q-series we have*

$$\mathcal{Z}^{\alpha}_{\gamma,P}(q) = \widetilde{\mathcal{A}}^{q}_{\gamma,P}(\alpha). \tag{2.23}$$

Namely, $\mathcal{Z}_{\gamma,P,k}(\alpha) = \widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha)$ for all $n \ge 1$.

Proof of Theorem 1.1. For $\gamma \in (0,2)$, $\alpha \in (-\frac{4}{\gamma}, Q)$, and $P \in \mathbb{R}$, by Theorem 2.13, (2.14) and (2.19), we have $\mathcal{F}^{\alpha}_{\gamma,P}(q) = \mathcal{G}^{\alpha}_{\gamma,P}(q)$ as formal q-series. Combined with Proposition 2.8, we obtain Theorem 1.1. □

Remark 2.14. Notice that $P \in \mathbb{R}$ in Definition 2.6 is the most relevant range of P as it corresponds exactly to the domain of integration for the bootstrap integral (1.2). One may wonder if Definition 2.6 extends to other values of P. For $P \in \mathbb{C}$ such that $\mathrm{Im}(P) \in (-\frac{1}{2\gamma}, \frac{1}{2\gamma})$ and $x \in (0,1)$, we find that $\mathrm{Im}(\gamma\pi P x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, which implies that $|\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}}e^{\pi\gamma P x}$ and thus $\int_0^1 |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}}e^{\pi\gamma P x}e^{\frac{\gamma}{2}Y_\tau(x)}dx$ a.s. have positive real part. Therefore we can take an $-\frac{\alpha}{\gamma}$ power using a branch cut along $(-\infty, 0]$ so that the expression in (2.7) is well defined. This allows us to extend the definition (2.9) of $P \mapsto \mathcal{G}^{q}_{\gamma,P}(\alpha)$ to an analytic function on the set $\{P \in \mathbb{C} | \mathrm{Im}(P) \in (-\frac{1}{2\gamma}, \frac{1}{2\gamma})\}$. Furthermore this implies that both Theorem 2.13 and Theorem 1.1 hold in this extended region of P. On another note, from the conjectured modular transformation for conformal blocks (1.9), we expect that for $\alpha \in (0,Q)$ and $q \in \mathbb{D}$, the function $P \mapsto \mathcal{F}^{\alpha}_{\gamma,P}(q)$ is meromorphic on $\mathbb{C}$, with poles at $\pm P_{m,n}$ as predicted by Zamolodchikov's recursion. See Section 1.5 for more details on our work in progress proving (1.9).

2.5. **Analyticity: proof of Lemma 2.9.** We first record a basic fact on analyticity of expectations.

Lemma 2.15. *Let $f(\cdot)$ be a random analytic function on a planar domain D. Suppose for each compact $K \subset D$ we have $\max_{z\in K} \mathbb{E}[|f(z)|] < \infty$. Then $\mathbb{E}[f(\cdot)]$ is analytic on D. Moreover, $\mathbb{E}[|\frac{d^n}{dz} f(z)|] < \infty$ and $\frac{d^n}{d^n z}\mathbb{E}[f(z)] = \mathbb{E}[\frac{d^n}{d^n z} f(z)]$ for each $z \in D$ and $n \in \mathbb{N}$.*

Proof. Consider $K_0 = \{z : |z - z_0| \le r\} \subset D$ for some $z_0 \in D$ and $r > 0$. Let $M_0 = \max_{z\in K_0} \mathbb{E}[|f(z)|] < \infty$. Since $\frac{d^n}{dz} f(z_0) = \frac{n!}{2\pi \mathbf{i}} \oint_{\partial K_0} f(w)(w - z_0)^{-n-1}dw$, we have $\mathbb{E}[|\frac{d^n}{dz} f(z_0)|] \le n!M_0r^{-n}$. Therefore, by Fubini's Theorem, if $|z - z_0| < r$ then $\mathbb{E}[f(z)] = \sum_0^\infty \frac{1}{n!}\mathbb{E}[\frac{d^n}{d^n z} f(z_0)](z - z_0)^n$. Varying z_0 and r, we conclude. □

Proof of Lemma 2.9 (a). Notice the definition (2.11) is originally only valid for $q \in (0, 1)$. To find the analytic continuation in q, we will apply Girsanov's theorem (Theorem C.5) to rewrite (2.11) so that taking q complex produces a holomorphic function. For this, notice that

$$\mathbb{E}[\alpha_n Y_\infty(x)] = \sqrt{\frac{2}{n}}\cos(2\pi nx) \qquad \text{and} \qquad \mathbb{E}[\beta_n Y_\infty(x)] = \sqrt{\frac{2}{n}}\sin(2\pi nx). \tag{2.24}$$

In the following computation, we will use the decomposition $Y_\tau(x) = Y_\infty(x) + F_\tau(x)$. Notice that Y_∞ and F_τ are independent. By Girsanov's theorem (Theorem C.5), Lemmas 2.1 and 2.5, and (2.6), we can write

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^1 |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma Px} e^{\frac{\gamma}{2}Y_\tau(x)}dx\right)^{-\frac{\alpha}{\gamma}}\right]\\
&= \left(q^{1/6}\eta(q)\right)^{\frac{\alpha^2}{2}} \mathbb{E}\left[\left(\int_0^1 (2\sin(\pi x))^{-\alpha\gamma/2} e^{\pi\gamma Px} e^{\frac{\gamma}{2}Y_\tau(x)+\frac{\gamma}{2}\mathbb{E}[Y_\tau(x)\cdot\frac{\alpha}{2}F_\tau(0)]}dx\right)^{-\frac{\alpha}{\gamma}}\right]\\
&= \left(q^{1/6}\eta(q)\right)^{\frac{\alpha^2}{2}} e^{-\frac{\alpha^2}{8}\mathbb{E}[F_\tau(0)^2]}\mathbb{E}\left[e^{\frac{\alpha}{2}F_\tau(0)}\left(\int_0^1 (2\sin(\pi x))^{-\alpha\gamma/2} e^{\pi\gamma Px} e^{\frac{\gamma}{2}Y_\tau(x)}dx\right)^{-\frac{\alpha}{\gamma}}\right]\\
&= \left(q^{1/6}\eta(q)\right)^{\frac{\alpha^2}{2}} e^{(\frac{\alpha\gamma}{8}-\frac{\alpha^2}{8})\mathbb{E}[F_\tau(0)^2]}\hat{\mathcal{A}}^q_{\gamma,P}(\alpha) = \left(q^{1/6}\eta(q)\right)^{\frac{\alpha^2}{2}} (q^{-1/12}\eta(q))^{\frac{\alpha^2-\alpha\gamma}{2}}\hat{\mathcal{A}}^q_{\gamma,P}(\alpha),
\end{aligned} \tag{2.25}$$

where $\hat{\mathcal{A}}^q_{\gamma,P}(\alpha) := \mathbb{E}\left[e^{\frac{\alpha}{2}F_\tau(0)}\left(\int_0^1 e^{\frac{\gamma}{2}F_\tau(x)}(2\sin(\pi x))^{-\alpha\gamma/2}e^{\pi\gamma Px}e^{\frac{\gamma}{2}Y_\infty(x)}dx\right)^{-\frac{\alpha}{\gamma}}\right]$.

We claim the following lemma with its proof postponed, and conclude the proof of (a) right after.

Lemma 2.16. *Assertion (a) in Lemma 2.9 holds with $\hat{\mathcal{A}}^q_{\gamma,P}(\alpha)$ in place of $\mathcal{A}^q_{\gamma,P}(\alpha)$.*

Recall from (B.2) that $q^{-\frac{1}{12}}\eta(q)$ is analytic and nonzero on the unit disk $\mathbb{D}$. Therefore, the function

$$q^{\frac{1}{12}(-\alpha\gamma-\frac{2\alpha}{\gamma}+2)}\eta(q)^{\alpha\gamma+\frac{2\alpha}{\gamma}-\frac{3}{2}\alpha^2-2}\left(q^{1/6}\eta(q)\right)^{\frac{\alpha^2}{2}}(q^{-1/12}\eta(q))^{\frac{\alpha^2-\alpha\gamma}{2}} = (q^{-\frac{1}{12}}\eta(q))^{\alpha(Q-\frac{\alpha}{2})-2} \tag{2.26}$$

is analytic on $\mathbb{D}$. By the definition of $\mathcal{A}^q_{\gamma,P}(\alpha)$, (2.8), and Lemma 2.16, we conclude the proof. □

Remark 2.17. From (2.25), (2.26) and the definition of $\mathcal{A}^q_{\gamma,P}(\alpha)$, $\widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha)$, and $\hat{\mathcal{A}}^q_{\gamma,P}(\alpha)$, we have

$$\widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha) = (q^{-\frac{1}{12}}\eta(q))^{\alpha(Q-\frac{\alpha}{2})-2}\frac{\hat{\mathcal{A}}^q_{\gamma,P}(\alpha)}{\hat{\mathcal{A}}^0_{\gamma,P}(\alpha)}. \tag{2.27}$$

Proof of Lemma 2.16. We start by assuming $q \in (0, 1)$. Using (2.24) again, we have

$$F_\tau = \sqrt{2}\sum_{m,n=1}^\infty q^{nm}(\alpha_{n,m}\mathbb{E}[\alpha_n Y_\infty(x)] + \beta_{n,m}\mathbb{E}[\beta_n Y_\infty(x)]). \tag{2.28}$$

Applying Girsanov's theorem (Theorem C.5) to Y_∞ while conditioning on $\{\alpha_{m,n}, \beta_{m,n}\}$, we obtain

$$\hat{A}^q_{\gamma,P}(\alpha) = \mathbb{E}\left[e^{\frac{\alpha}{2}F_\tau(0)}\left(\int_0^1 (\sin(\pi x))^{-\alpha\gamma/2}e^{\pi\gamma P x}e^{\frac{\gamma}{2}Y_\infty(x)+\frac{\gamma}{\sqrt{2}}\sum_{m,n=1}^\infty q^{nm}(\alpha_{n,m}\mathbb{E}[\alpha_n Y_\infty(x)]+\beta_{n,m}\mathbb{E}[\beta_n Y_\infty(x)])}dx\right)^{-\frac{\alpha}{\gamma}}\right]$$

(2.29)

$$= \mathbb{E}\left[e^{\frac{\alpha}{2}F_\tau(0)}\mathcal{Q}(q)\left(\int_0^1 (\sin(\pi x))^{-\alpha\gamma/2}e^{\pi\gamma P x}e^{\frac{\gamma}{2}Y_\infty(x)}dx\right)^{-\frac{\alpha}{\gamma}}\right],$$

where

$$\mathcal{Q}(q) := \exp\left(\sqrt{2}\sum_{m,n=1}^\infty q^{nm}(\alpha_{n,m}\alpha_n + \beta_{n,m}\beta_n) - \sum_{n=1}^\infty\Big(\sum_{m=1}^\infty q^{nm}\alpha_{m,n}\Big)^2 - \sum_{n=1}^\infty\Big(\sum_{m=1}^\infty q^{nm}\beta_{m,n}\Big)^2\right). \tag{2.30}$$

Although $\hat{A}^q_{\gamma,P}(\alpha)$ is originally only defined for $q \in (0,1)$, the function $e^{\frac{\alpha}{2}F_\tau(0)}\mathcal{Q}(q)$ contains all the q dependence and is clearly a random analytic function defined for $|q| < 1$. In light of Lemma 2.15, for each $\alpha \in (-\frac{4}{\gamma}, Q)$ and open set $U \subset \mathbb{D}$, the function $q \mapsto \hat{A}^q_{\gamma,P}(\alpha)$ admits an analytic extension on U if we have

$$\mathbb{E}\left[\left|e^{\frac{\alpha}{2}F_\tau(0)}\mathcal{Q}(q)\right|\left(\int_0^1 (\sin(\pi x))^{-\alpha\gamma/2}e^{\pi\gamma P x}e^{\frac{\gamma}{2}Y_\infty(x)}dx\right)^{-\frac{\alpha}{\gamma}}\right] < \infty \tag{2.31}$$

uniformly on compact subsets of U. The remainder of the proof is devoted to verifying (2.31) for α in different domains and different sets U. In each case, we will choose $p_1, p_2, p_3 \in (1,\infty)$ with $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} = 1$ so that the upper bound

$$\begin{aligned}\mathbb{E}&\left[\left|e^{\frac{\alpha}{2}F_\tau(0)}\mathcal{Q}(q)\right|\left(\int_0^1 (\sin(\pi x))^{-\alpha\gamma/2}e^{\pi\gamma P x}e^{\frac{\gamma}{2}Y_\infty(x)}dx\right)^{-\frac{\alpha}{\gamma}}\right] \\ &\le \mathbb{E}\left[e^{\frac{p_1|\alpha|}{2}|F_\tau(0)|}\right]^{\frac{1}{p_1}} \cdot \mathbb{E}\left[|\mathcal{Q}(q)|^{p_2}\right]^{\frac{1}{p_2}} \cdot \mathbb{E}\left[\left(\int_0^1 (\sin(\pi x))^{-\alpha\gamma/2}e^{\pi\gamma P x}e^{\frac{\gamma}{2}Y_\infty(x)}dx\right)^{-\frac{\alpha p_3}{\gamma}}\right]^{\frac{1}{p_3}}\end{aligned} \tag{2.32}$$

provided by Holder's inequality is finite.

Case 1: $\alpha \in [0,Q)$. In this case, we wish to prove (2.31) for both $U = \{q \in \mathbb{C} : |q| < \frac{1}{2}\}$ and U being a complex neighborhood of $[0,1)$. By Lemma C.4 and the fact that the expectation of an exponentiated Gaussian random variable is finite, for all $|q| < 1$, $p_1 \in (1,\infty)$, and $p_3 \in (1,\infty)$ we have

$$\mathbb{E}\left[e^{\frac{p_1|\alpha|}{2}|F_\tau(0)|}\right]^{\frac{1}{p_1}} < \infty, \quad \text{and} \quad \mathbb{E}\left[\left(\int_0^1 (\sin(\pi x))^{-\alpha\gamma/2}e^{\pi\gamma P x}e^{\frac{\gamma}{2}Y_\infty(x)}dx\right)^{-\frac{\alpha p_3}{\gamma}}\right]^{\frac{1}{p_3}} < \infty. \tag{2.33}$$

Since p_1, p_3 can be chosen arbitrarily large in (2.33), we can choose them to make p_2 arbitrarily close to 1 in the constraint $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} = 1$. It therefore remains only to prove that $\lim_{p_2\to 1^+}\mathbb{E}\left[|\mathcal{Q}(q)|^{p_2}\right] < \infty$.

By (2.30) and the independence of $(\alpha_n)_{n\ge1}$, $(\alpha_{n,m})_{n,m\ge1}$, $(\beta_n)_{n\ge1}$, and $(\beta_{n,m})_{n,m\ge1}$, we can write $\mathbb{E}[|\mathcal{Q}(q)|^{p_2}]$ as

$$\begin{aligned}\mathbb{E}\left[|\mathcal{Q}(q)|^{p_2}\right] &= \mathbb{E}\left[\left|e^{\sqrt{2}\sum_{m,n=1}^\infty q^{nm}(\alpha_{n,m}\alpha_n+\beta_{n,m}\beta_n)}e^{-\sum_{n=1}^\infty\left(\sum_{m=1}^\infty q^{nm}\alpha_{m,n}\right)^2-\sum_{n=1}^\infty\left(\sum_{m=1}^\infty q^{nm}\beta_{m,n}\right)^2}\right|^{p_2}\right] \\ &= \mathbb{E}\left[\left|e^{\sqrt{2}\sum_{m,n=1}^\infty q^{nm}(\alpha_{n,m}\alpha_n+\beta_{n,m}\beta_n)}e^{-\sum_{n=1}^\infty\sum_{m_1,m_2=1}^\infty q^{nm_1+nm_2}(\alpha_{m_1,n}\alpha_{m_2,n}+\beta_{m_1,n}\beta_{m_2,n})}\right|^{p_2}\right] \\ &= \prod_{n=1}^\infty A_n(q,p_2)^2,\end{aligned}$$

where we define $A_n(q,p_2)$ by

$$A_n(q,p_2) := \mathbb{E}\left[\left|e^{\sqrt{2}\sum_{m=1}^\infty q^{nm}\alpha_{n,m}\alpha_n}e^{-\sum_{m_1,m_2=1}^\infty q^{nm_1+nm_2}\alpha_{m_1,n}\alpha_{m_2,n}}\right|^{p_2}\right]$$

and note that replacing α_n with β_n and $\alpha_{n,m}$ by $\beta_{n,m}$ in $A_n(q,p_2)$ does not change its value. We now compute that

$$\begin{aligned} A_n(q,p_2) &= \mathbb{E}\left[e^{\sqrt{2}\sum_{m=1}^\infty p_2 \mathrm{Re}(q^{nm})\alpha_{n,m}\alpha_n} e^{-\sum_{m_1,m_2=1}^\infty p_2\mathrm{Re}(q^{nm_1+nm_2})\alpha_{m_1,n}\alpha_{m_2,n}}\right] \\ &= \mathbb{E}\left[e^{\sum_{m_1,m_2=1}^\infty \left(p_2^2\mathrm{Re}(q^{nm_1})\mathrm{Re}(q^{nm_2}) - p_2\mathrm{Re}(q^{nm_1+nm_2})\right)\alpha_{n,m_2}\alpha_{n,m_2}}\right] \\ &= \mathbb{E}\left[e^{\sum_{m_1,m_2=1}^\infty \left((p_2^2-p_2)\mathrm{Re}(q^{nm_1})\mathrm{Re}(q^{nm_2}) + p_2\,\mathrm{Im}(q^{nm_1})\,\mathrm{Im}(q^{nm_2})\right)\alpha_{n,m_2}\alpha_{n,m_2}}\right], \end{aligned}$$

where in the second line we compute the expectation over α_n. Define now the Gaussian random variables

$$X_n := \sum_{m=1}^\infty \mathrm{Re}(q^{nm})\alpha_{n,m} \qquad \text{and} \qquad Y_n := \sum_{m=1}^\infty \mathrm{Im}(q^{nm})\alpha_{n,m}$$

so that (X_n, Y_n) is a bivariate Gaussian with covariance matrix $\begin{bmatrix} R_n & S_n \\ S_n & T_n \end{bmatrix}$ for

$$R_n = \sum_{m=1}^\infty \mathrm{Re}(q^{nm})^2 \qquad S_n = \sum_{m=1}^\infty \mathrm{Re}(q^{nm})\,\mathrm{Im}(q^{nm}) \qquad T_n = \sum_{m=1}^\infty \mathrm{Im}(q^{nm})^2.$$

We find that $(X_n, Y_n) \overset{d}{=} (\sqrt{R_n}Z, \frac{S_n}{\sqrt{R_n}}Z + \frac{\sqrt{R_nT_n - S_n^2}}{\sqrt{R_n}}W)$, where Z, W are independent standard Gaussians. In these terms, we have that

$$\begin{aligned} A_n(q,p_2) &= \mathbb{E}\left[e^{(p_2^2-p_2)X_n^2 + p_2Y_n^2}\right] \\ &= \mathbb{E}\left[\exp\left(((p_2^2-p_2)R_n + p_2\frac{S_n^2}{R_n})Z^2 + \frac{2p_2S_n\sqrt{R_nT_n - S_n^2}}{R_n}ZW + p_2\frac{R_nT_n - S_n^2}{R_n}W^2\right)\right] \\ &= \mathbb{E}\left[\exp\left(\begin{bmatrix} Z \\ W\end{bmatrix}^\intercal M_n \begin{bmatrix} Z \\ W\end{bmatrix}\right)\right] \end{aligned}$$

for the matrix

$$M_n = \begin{bmatrix} (p_2^2-p_2)R_n + p_2\frac{S_n^2}{R_n} & \frac{p_2S_n\sqrt{R_nT_n-S_n^2}}{R_n} \\ \frac{p_2S_n\sqrt{R_nT_n-S_n^2}}{R_n} & p_2\frac{R_nT_n-S_n^2}{R_n}\end{bmatrix}. \tag{2.34}$$

Notice that $\mathrm{Tr}(M_n) > 0$ and $\det(M_n) > 0$ and further that

$$\mathrm{Tr}(M_n) = (p_2^2 - p_2)R_n + p_2T_n, \quad \det(M_n) = p_2^2(p_2-1)(R_nT_n - S_n^2).$$

For $|q| < \frac{1}{2}$, we have uniformly in q and n that

$$T_n \leq \sum_{m=1}^\infty |q|^{2nm} = \frac{|q|^{2n}}{1-|q|^{2n}} < \frac{1}{2}, \tag{2.35}$$

which implies that $\lim_{p_2\to1^+}\mathrm{Tr}(M_n) < \frac{1}{2}$ uniformly in q and n for $|q| < \frac{1}{2}$. Similarly, for $q \in [0,1)$, we have that $T_n = 0$, which implies by continuity in q that on a complex neighborhood of $[0,1)$, we have $\lim_{p_2\to1^+}\mathrm{Tr}(M_n) < \frac{1}{2}$ uniformly in n as well.

We conclude that M_n is symmetric and may be orthogonally diagonalized with eigenvalues λ_1, λ_2 with value less than $\frac{1}{2}$. Hence for independent standard Gaussians W_1, W_2 we have

$$A_n(q,p_2) = \mathbb{E}[e^{\lambda_1W_1^2 + \lambda_2W_2^2}] = \frac{1}{\sqrt{(1-2\lambda_1)(1-2\lambda_2)}} = \frac{1}{\sqrt{1 - 2\,\mathrm{Tr}(M_n) + 4\det(M_n)}}.$$

Because $\lim_{p_2\to1^+}\mathrm{Tr}(M_n) < \frac{1}{2}$ uniformly in n, we have that that $\lim_{p_2\to1^+}\frac{1}{1-2\,\mathrm{Tr}(M_n)} \leq e^{2C\,\mathrm{Tr}(M_n)}$ for some $C > 0$ uniformly in n and uniformly on compact subsets in q. We conclude that

$$\lim_{p_2\to1^+} A_n(q,p_2) \leq e^{2CT_n}.$$

Using (2.35), we find that $\sum_{n=1}^\infty T_n < \infty$ uniformly in n and uniformly on compact subsets of either $|q| < \frac{1}{2}$ or q in a complex neighborhood of $[0, 1)$. This implies that

$$\lim_{p_2 \to 1^+} \mathbb{E}[|\mathcal{Q}(q)|^{p_2}] = \lim_{p_2 \to 1^+} \prod_{n=1}^\infty A_n(q, p_2)^2 \leq e^{2C \sum_{n=1}^\infty T_n} < \infty \tag{2.36}$$

for either $|q| < \frac{1}{2}$ or q in a complex neighborhood of $[0, 1)$. Continuity in p_2 then implies (2.31) for $p_2 \in (1, \infty)$ close to 1 and hence Lemma 2.16 for $\alpha \in [0, Q)$.

Case 2: $\alpha \in (-\frac{4}{\gamma}, 0)$. In this case, we wish to prove (2.31) for q in a neighborhood U of 0. Define $\bar{p}_2 = \frac{1}{1+\frac{\gamma\alpha}{4}}$. By Lemma C.4, there exists $p_1 > 1$ large and $p_3 > 1$ near $-\frac{4}{\alpha\gamma}$ so that (2.33) holds for all $|q| < 1$ and p_2 determined from $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ lies in $(\bar{p}_2, \infty)$ and is arbitrarily close to $\bar{p}_2$. It remains to check that $\lim_{p_2 \to \bar{p}_2^+} \mathbb{E}[|\mathcal{Q}(q)|^{p_2}] < \infty$.

We argue similarly to Case 1. The maximum eigenvalue of M_n is bounded above by

$$\mathrm{Tr}(M_n) = (p_2^2 - p_2) R_n + p_2 T_n \leq p_2^2 \sum_{m=1}^\infty |q|^{2nm} < \frac{1}{2} \tag{2.37}$$

uniformly in n and q for p_2 near $\bar{p}_2$ and $|q|$ sufficiently small. This implies as before that, uniformly in n and q, we have some $C > 0$ such that

$$\begin{aligned}
\lim_{p_2 \to \bar{p}_2^+} A_n(q, p_2) &= \frac{1}{\sqrt{1 - 2\,\mathrm{Tr}(M_n) + 4\det(M_n)}} \\
&= \frac{1}{\sqrt{1 - 2(\bar{p}_2^2 - \bar{p}_2) R_n - 2\bar{p}_2 T_n + 4\bar{p}_2(\bar{p}_2^2 - \bar{p}_2)(R_n T_n - S_n^2)}} \\
&\leq \exp\Big(C[(\bar{p}_2^2 - \bar{p}_2) R_n + \bar{p}_2 T_n]\Big).
\end{aligned}$$

We thus find that

$$\lim_{p_2 \to \bar{p}_2^+} \mathbb{E}[|\mathcal{Q}(q)|^{p_2}] = \lim_{p_2 \to \bar{p}_2^+} \prod_{n=1}^\infty A_n(q, p_2)^2 \leq e^{2C \sum_{n=1}^\infty [(\bar{p}_2^2 - \bar{p}_2) R_n + \bar{p}_2 T_n]} < \infty, \tag{2.38}$$

which is finite for $|q|$ sufficiently small. This implies (2.31) for our choice of p_1, p_2, p_3 and hence implies Lemma 2.16 for $\alpha \in (-\frac{4}{\gamma}, 0)$. ☐

Based on the proof of Lemma 2.16, in Definition 2.18 we define $r_\alpha > 0$ so that Lemma 2.19 below holds for $|q| < r_\alpha$. The proof of Lemma 2.16 shows that $r_\alpha > \frac{1}{2}$ for $\alpha \in [0, Q)$ and $r_\alpha > 0$ for $\alpha \in (-\frac{4}{\gamma}, 0)$, though we do not attempt to find more optimal bounds. In what follows, the precise value of r_α will not be important.

Definition 2.18. For $\alpha \in (-\frac{4}{\gamma}, Q)$, define

$$r_\alpha := \begin{cases} \sup\{r > 0 : \text{(2.35) and (2.36) hold uniformly in } q \text{ and } n \text{ for } |q| < r\} & \alpha \in [0, Q) \\ \sup\{r > 0 : \text{(2.37) and (2.38) hold uniformly in } q \text{ and } n \text{ for } |q| < r\} & \alpha \in (-\frac{4}{\gamma}, 0), \end{cases}$$

where we recall that (2.37) holds for p_2 near $\bar{p}_2 = \frac{1}{1+\frac{\gamma\alpha}{4}}$.

Lemma 2.19. *Define $\bar{p}_2(\alpha) = 1$ if $\alpha \in [0, Q)$ and $\bar{p}_2(\alpha) = \frac{1}{1+\frac{\gamma\alpha}{4}}$ if $\alpha \in (-\frac{4}{\gamma}, 0)$. We have*

$$\lim_{p_2 \to \bar{p}_2(\alpha)^+} \mathbb{E}[|\mathcal{Q}(q)|^{p_2}] < \infty \qquad \textit{for } |q| < r_\alpha.$$

Proof. This follows from the proof of Lemma 2.16. ☐

Proof of Lemma 2.9 (b). Following the notation of the proof of Lemma 2.16, by (2.28), it suffices to establish the analytic extension for

$$\hat{A}^q_{\gamma,P}(\alpha) = \mathbb{E}\left[e^{\frac{\alpha}{2}F_\tau(0)} \mathcal{Q}(q) \left(\int_0^1 (\sin(\pi x))^{-\alpha\gamma/2} e^{\pi\gamma P x} e^{\frac{\gamma}{2} Y_\infty(x)} dx\right)^{-\frac{\alpha}{\gamma}}\right].$$

Thanks to Lemma 2.16, we have the desired analyticity with respect to q. The analyticity in α of moments of Gaussian multiplicative chaos has already been shown to hold in several works such as [KRV19a, RZ20]. To reduce our GMC to the one studied in [RZ20], one can map the unit disk $\mathbb{D}$ to the upper-half plane $\mathbb{H}$

by the map $z \mapsto -\mathbf{i}\frac{z-1}{z+1}$. The circle parametrized by $x \in [0,1]$ becomes the real line $\mathbb{R}$ and the point x goes to $y = \phi(x) := -\mathbf{i}\frac{e^{2\pi \mathbf{i} x}-1}{e^{2\pi \mathbf{i} x}+1}$. The field $Y_\infty(x)$ is mapped to the restriction to the real line of the Gaussian field $X_{\mathbb{H}}$ with covariance given by

$$\mathbb{E}[X_{\mathbb{H}}(y)X_{\mathbb{H}}(y')] = \log \frac{1}{|y-y'||y-\overline{y}'|} - \log|y+\mathbf{i}|^2 - \log|y'+\mathbf{i}|^2 + 2\log 2$$

for $y, y' \in \mathbb{H}$. At the level of GMC measures, Lemma C.2 implies the measure $|(\phi^{-1}(y)')|e^{\frac{\gamma}{2}X_{\mathbb{H}}(y)}dy$ on $\mathbb{R}$ is the pushforward of the measure $e^{\frac{\gamma}{2}Y_\infty(x)}dx$ under ϕ.

By performing this change of variable one gets

(2.39)
$$\mathbb{E}\left[e^{\frac{\alpha}{2}F_\tau(0)}\mathcal{Q}(q)\left(\int_0^1 (\sin(\pi x))^{-\alpha\gamma/2}e^{\pi\gamma P x}e^{\frac{\gamma}{2}Y_\infty(x)}dx\right)^{-\frac{\alpha}{\gamma}}\right] = \mathbb{E}\left[e^{\frac{\alpha}{2}F_\tau(0)}\mathcal{Q}(q)\left(\int_{\mathbb{R}} |y|^{-\frac{\alpha\gamma}{2}}f_1(y)e^{\frac{\gamma}{2}X_{\mathbb{H}}(y)}dy\right)^{-\frac{\alpha}{\gamma}}\right]$$

where $f_1 : \mathbb{R} \mapsto (0,\infty)$ is such that the measure $|y|^{-\frac{\alpha\gamma}{2}}f_1(y)dy$ is the pushforward of $\sin(\pi x)^{-\frac{\alpha\gamma}{2}}e^{\pi\gamma P x}dx$ under ϕ. We can check that f_1 is bounded and continuous.

According to [RZ20, Lemma 5.6], $\mathbb{E}\left[\left(\int_{\mathbb{R}}|y|^{-\frac{\alpha\gamma}{2}}f(y)e^{\frac{\gamma}{2}X_{\mathbb{H}}(y)}dy\right)^{-\frac{\alpha}{\gamma}+p}\right]$ admits an analytic continuation as a function of α on a complex open neighborhood of $(-\frac{4}{\gamma}, Q)$, with $p = \frac{1}{\gamma}(2Q-\beta_2-\beta_3)$, $f(y) = |y-1|^{-\frac{\gamma\beta_2}{2}}|y|_+^{\frac{\gamma}{2}(\alpha+\gamma(p-1))}$, $|y|_+ := \max(|y|,1)$, and $\beta_2, \beta_3 \in U \subset \mathbb{R}^2$ for some open set U. If the random quantity $e^{\frac{\alpha}{2}F_\tau(0)}\mathcal{Q}(q)$ were deterministic, analyticity for the right hand side of (2.39) would follow in the same way as [RZ20, Lemma 5.6].

We now sketch how to adapt the proof to account for the fact that $e^{\frac{\alpha}{2}F_\tau(0)}\mathcal{Q}(q)$ is random. Applying Girsanov's Theorem C.5 to the right hand side of (2.39) yields

$$\mathbb{E}\left[e^{\frac{\alpha}{2}(F_\tau(0)+H_\tau)}\mathcal{Q}(q)e^{\frac{\alpha}{2}X_{\mathbb{H}}(0)-\frac{\alpha^2}{8}\mathbb{E}[X_{\mathbb{H}}(0)^2]}\left(\int_{\mathbb{R}} f_2(y)e^{\frac{\gamma}{2}X_{\mathbb{H}}(y)}dy\right)^{-\frac{\alpha}{\gamma}}\right] \tag{2.40}$$

where $f_2 : \mathbb{R} \mapsto (0,\infty)$ is again bounded and continuous and

$$H_\tau := -\sqrt{2}\sum_{m,n\geq 1} q^{nm}\left(\alpha_{n,m}\mathbb{E}[\alpha_n X_{\mathbb{H}}(0)] + \beta_{n,m}\mathbb{E}[\beta_n X_{\mathbb{H}}(0)]\right).$$

Fix $r > 0$. To show (2.40) is analytic in α in a complex neighborhood of $(-\frac{4}{\gamma}, Q)$, we realize it as the $r \to \infty$ limit of

$$g_r(\alpha) := \mathbb{E}\left[e^{\frac{\alpha}{2}(F_\tau(0)+H_\tau)}\mathcal{Q}(q)e^{\frac{\alpha}{2}\overline{X}(e^{-r/2})-\frac{\alpha^2}{8}\mathbb{E}[\overline{X}(e^{-r/2})^2]}\left(\int_{\mathbb{R}_r} f_2(y)e^{\frac{\gamma}{2}X_{\mathbb{H}}(y)}dy\right)^{-\frac{\alpha}{\gamma}}\right],$$

where $\mathbb{R}_r := \mathbb{R}\setminus(-e^{-r/2}, e^{-r/2})$ and $\overline{X}(e^{-r/2})$ is the mean of $X_{\mathbb{H}}(0)$ on the half circle centered at 0 of radius $e^{-r/2}$. For any $r > 0$, $g_r(\alpha)$ is analytic in α on a complex neighborhood of $(-\frac{4}{\gamma}, Q)$, so it suffices to check that the $r \to \infty$ convergence of $g_r(\alpha)$ is locally uniform in α. For this, we check that $\sum_{r=1}^\infty |g_{r+1}(\alpha) - g_r(\alpha)| < \infty$ by first applying Hölder's inequality to decouple $e^{\frac{\alpha}{2}(F_\tau(0)+H_\tau)}$ and $\mathcal{Q}(q)$ from the remaining part of $g_r(\alpha)$ as in the proof of Lemma 2.16 and then bounding this remaining part following the proof of [RZ20, Lemma 5.6] precisely. □

Proof of Lemma 2.9 (c). By Lemma 2.9 (a), $\mathcal{A}^q_{\gamma,P}(\alpha)$ is analytic in q. For a small enough contour $\mathcal{C}$ around the origin, we have by Cauchy integral that $\mathcal{A}_{\gamma,P,n}(\alpha) = \frac{n!}{2\pi\mathbf{i}}\oint_{\mathcal{C}}\mathcal{A}^q_{\gamma,P}(\alpha)q^{-n-1}dq$. Combined with Lemma 2.9 (b), we get the desired analyticity in α for $\mathcal{A}_{\gamma,P,n}(\alpha)$. □

3. BPZ equation for deformed conformal blocks

In this section we introduce a certain deformation of the conformal block and show that it satisfies the so called *BPZ equation* on the torus; see Theorem 3.5. Recall $\gamma \in (0,2)$ and $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$. Throughout Sections 3—6.1, we view $P \in \mathbb{R}$ as a fixed parameter as well. For $\alpha \in (-\frac{4}{\gamma}, Q)$, and $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$, define

$$l_\chi = \frac{\chi^2}{2} - \frac{\alpha\chi}{2}. \tag{3.1}$$

Recall the domain $\mathfrak{B} := \{z : 0 < \mathrm{Im}(z) < \frac{3}{4}\mathrm{Im}(\tau)\}$ from Appendix B.5 and q_0 in Lemma B.4. Let

$$\nu(dx) := |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\tau(x)} dx. \tag{3.2}$$

Fix $q \in (0, q_0)$. Let f_ν be defined as in (B.22) in Appendix B.5 with ν from (3.2) and $c = \frac{\gamma\chi}{2}$; namely, we have

$$f_\nu(u) := \int_0^1 \Theta_\tau(u+x)^{\frac{\gamma\chi}{2}} |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\tau(x)} dx \qquad \text{for } u \in \mathfrak{B}.$$

By Lemma B.7, f_ν is almost surely analytic and nonzero on $\mathfrak{B}$, meaning we can define its fractional power according to Definition B.8. Our deformed conformal block will be a moment of f_ν up to an explicit prefactor.

Lemma 3.1. *For $\alpha \in (-\frac{4}{\gamma}+\chi, Q)$ and $q \in (0, q_0)$, we have $\mathbb{E}\left[|f_\nu(u)|^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right] < \infty$ for each $u \in \mathfrak{B}$. Moreover, $\mathbb{E}\left[f_\nu(u)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right]$ is analytic in u on $\mathfrak{B}$.*

Proof. By Lemma C.4, $\max_{u\in K} \mathbb{E}\left[|f_\nu(u)|^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right] < \infty$ for each compact $K \subset \mathfrak{B}$. By Lemma 2.15, $\mathbb{E}\left[f_\nu(u)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right]$ is analytic in u on $\mathfrak{B}$. □

Our next proposition provides a useful analytic property of the deformed conformal block. Recall that in Definition 2.18, we defined the number $r_\alpha > 0$ to be maximal so that for all $|q| < r_\alpha$ the application of Hölder's inequality in the proof of Lemma 2.16 gives a finite upper bound. In the next proposition we will need the domain

$$D_\chi^\alpha := \{(q, u) : |q| < r_{\alpha-\chi} \text{ and } u \in \mathfrak{B}\}, \tag{3.3}$$

where $r_{\alpha-\chi}$ appears because the exponent of $f_\nu(u)$ is $-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}$ as opposed to $-\frac{\alpha}{\gamma}$ for the non-deformed conformal block (see the proof of Proposition 3.2 in Section 3.1). The only feature of r_α relevant to the rest of the paper is that $r_\alpha > 0$ for $\alpha \in (-\frac{4}{\gamma}, Q)$.

Proposition 3.2. *For $\alpha \in (-\frac{4}{\gamma} + \chi, Q)$ and $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$, let*

$$\hat{\psi}_\chi^\alpha(u, q) := C(q) e^{\chi P u \pi} \Theta_\tau(u)^{-l_\chi} \mathbb{E}\left[f_\nu(u)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right], \qquad \text{for } q \in (0, q_0 \wedge r_{\alpha-\chi}) \text{ and } u \in \mathfrak{B}, \tag{3.4}$$

where $C(q) := q^{\frac{\gamma l_\chi}{12\chi} - \frac{1}{6}\frac{l_\chi^2}{\chi^2} - \frac{1}{6\chi^2}l_\chi(l_\chi+1)} \Theta_\tau'(0)^{-\frac{2l_\chi^2}{3\chi^2} + \frac{l_\chi}{3} + \frac{4l_\chi}{3\gamma\chi}} e^{-\frac{1}{2}\mathbf{i}\pi\alpha\gamma(-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma})}$. Then $\hat{\psi}_\chi^\alpha$ admits a bi-holomorphic extension to D_χ^α.

We defer the proof of Proposition 3.2 to Section 3.1 and define the deformed conformal block now.

Definition 3.3. For $\alpha \in (-\frac{4}{\gamma} + \chi, Q)$ and $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$, define

$$\psi_\chi^\alpha(u, \tau) := e^{\left(\frac{P^2}{2} + \frac{1}{6\chi^2}l_\chi(l_\chi+1)\right)\mathbf{i}\pi\tau} \hat{\psi}_\chi^\alpha(u, e^{\mathbf{i}\pi\tau}) \quad \text{for } (u, e^{\mathbf{i}\pi\tau}) \in D_\chi^\alpha, \tag{3.5}$$

where $\hat{\psi}_\chi^\alpha$ is extended to D_χ^α as in Proposition 3.2. We call $\psi_\chi^\alpha(u,\tau)$ the *u-deformed conformal block*.

Remark 3.4. Recall from Section 2.2 that $\Theta_\tau(x)^{-\alpha\gamma/2} = e^{-\mathbf{i}\pi\alpha\gamma/2}|\Theta_\tau(x)|^{-\alpha\gamma/2}$ for $x \in (0,1)$ and $q \in (0,1)$. By Definition 3.3, when $q \in (0, q_0 \wedge r_{\alpha-\chi})$, the deformed block $\psi_\chi^\alpha(u,\tau)$ can be expressed as

$$\psi_\chi^\alpha(u,\tau) = \mathcal{W}(q) e^{\chi P u \pi} \mathbb{E}\left[\left(\int_0^1 \mathcal{T}(u,x) e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\tau(x)} dx\right)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right], \tag{3.6}$$

where $\mathcal{W}(q) := q^{\frac{P^2}{2} + \frac{\gamma l_\chi}{12\chi} - \frac{1}{6}\frac{l_\chi^2}{\chi^2}} \Theta_\tau'(0)^{-\frac{2l_\chi^2}{3\chi^2} + \frac{l_\chi}{3} + \frac{4l_\chi}{3\gamma\chi}}$ and $\mathcal{T}(u,x) := \Theta_\tau(u)^{-\frac{\gamma\chi}{2}} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} \Theta_\tau(u+x)^{\frac{\gamma}{2}\chi}$.

We now state the BPZ equation for $\psi_\chi^\alpha(u,\tau)$. The proof will be given in Section 3.3.

Theorem 3.5. *For $\alpha \in (-\frac{4}{\gamma} + \chi, Q)$ and $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$, we have*

$$\left(\partial_{uu} - l_\chi(l_\chi+1)\wp(u) + 2\mathbf{i}\pi\chi^2\partial_\tau\right)\psi_\chi^\alpha(u,\tau) = 0 \qquad \text{for } (u, e^{\mathbf{i}\pi\tau}) \in D_\chi^\alpha, \tag{3.7}$$

where $\wp$ is Weierstrass elliptic function from Appendix B.

Both the proof of Theorem 3.5 and the rest of our paper require the following analytic extension of the deformed block, which we prove in Section 3.2.

Lemma 3.6. *Given $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$, there exists an open set in $\mathbb{C}^3$ containing $\{(\alpha, u, q) : \alpha \in (-\frac{4}{\gamma} + \chi, Q), u \in \mathfrak{B}, q = 0\}$ on which $(\alpha, u, q) \mapsto \psi^\alpha_\chi(u, q)$ has an analytic continuation.*

3.1. **Proof of Proposition 3.2.** We start by assuming $q \in (0, q_0)$. Similarly to the proof of Lemma 2.9(a), to obtain analyticity in q, we manipulate the expression to remove the q-dependence from the GMC moment. By (2.24) and Girsanov's theorem (Theorem C.5), we have

$$\begin{aligned}\mathbb{E}\left[(f_\nu(u))^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right] &= \mathbb{E}\left[\left(\int_0^1 |\Theta_\tau(x)|^{-\frac{\alpha\gamma}{2}} \Theta_\tau(u+x)^{\frac{\gamma}{2}\chi} e^{\pi\gamma P x} e^{\frac{\gamma}{2} Y_\tau(x)} dx\right)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right] \\ &= C_1(q)\mathbb{E}\left[e^{\frac{\alpha}{2}F_\tau(0)}\left(\int_0^1 e^{\frac{\gamma}{2}F_\tau(x)}(2\sin(\pi x))^{-\frac{\alpha\gamma}{2}}\Theta_\tau(x+u)^{\frac{\chi\gamma}{2}} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\infty(x)} dx\right)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right]\end{aligned}$$

where $C_1(q) := \left(q^{1/6}\eta(q)\right)^{\frac{\alpha(\alpha-\chi)}{2}} e^{(\frac{\alpha\gamma}{8} - \frac{\gamma\chi}{8} - \frac{\alpha^2}{8})\mathbb{E}[F_\tau(0)^2]}$. Recall $\mathcal{Q}(q)$ in (2.30). By (2.28) and Girsanov's theorem (Theorem C.5), we get the following analog of (2.29)

$$\mathbb{E}\left[(f_\nu(u))^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right] = C_1(q)\mathbb{E}\left[e^{\frac{\alpha}{2}F_\tau(0)}\mathcal{Q}(q)\left(\int_0^1 (2\sin(\pi x))^{-\frac{\alpha\gamma}{2}}\Theta_\tau(x+u)^{\frac{\chi\gamma}{2}} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\infty(x)} dx\right)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right]. \tag{3.8}$$

Next, we would like to also apply Girsanov's theorem to the term $\Theta_\tau(u+x)^{\frac{\gamma}{2}\chi}$ in the GMC integral, but this is not completely straightforward because a priori $\Theta_\tau(u+x)^{\frac{\gamma}{2}\chi}$ is a complex number that differs by a phase from $|\Theta_\tau(u+x)|^{\frac{\gamma}{2}\chi}$. For this purpose, for $q \in \mathbb{D}$ and $u \in \mathfrak{B}$, define

$$\mathcal{X}(u, q) := -\chi \sum_{n,m=1}^\infty \frac{1}{\sqrt{2n}}\left((\alpha_n + \mathbf{i}\beta_n)q^{(2m-2)n}e^{2\pi\mathbf{i}un} + (\alpha_n - \mathbf{i}\beta_n)q^{2nm}e^{-2\pi\mathbf{i}un}\right). \tag{3.9}$$

Since $|q|^{3/2} < |e^{2\pi\mathbf{i}u}| < 1 < |e^{-2\pi\mathbf{i}u}| < |q|^{-3/2}$ when $u \in \mathfrak{B}$, the series converges almost surely in $q \in \mathbb{D}$. Moreover, $e^{\mathcal{X}(u,q)}$ has finite moments of all orders.

We claim that

$$\Theta_\tau(u+x) = -\mathbf{i}e^{-\mathbf{i}\pi u}q^{\frac{1}{6}}\eta(q)e^{\frac{1}{\chi}\mathbb{E}[Y_\infty(x)\mathcal{X}(u,q)]}. \tag{3.10}$$

To see (3.10), set $u' = u - \frac{\tau}{2}$. By (B.15), we have

$$\Theta_\tau(u+x) = -\mathbf{i}e^{-\mathbf{i}\pi u}q^{\frac{1}{6}}\eta(q)\prod_{m=1}^\infty(1 - q^{2m-1}e^{2\pi\mathbf{i}(u'+x)})(1 - q^{2m-1}e^{-2\pi\mathbf{i}(u'+x)}). \tag{3.11}$$

Using $1 - z = \exp\{\sum_{n=1}^\infty \frac{z^n}{n}\}$ for $|z| < 1$ and recalling (2.24), we have

$$\begin{aligned}\prod_{m=1}^\infty(1 - q^{2m-1}e^{2\pi\mathbf{i}(u'+x)})(1 - q^{2m-1}e^{-2\pi\mathbf{i}(u'+x)}) &= \exp\left\{-2\sum_{n,m=1}^\infty \frac{q^{(2m-1)n}}{n}\cos(2\pi(x+u')n)\right\} \\ = \exp\left\{-\sqrt{2}\sum_{n,m=1}^\infty \frac{q^{(2m-1)n}}{\sqrt{n}}\left(\cos(2\pi u'n)\mathbb{E}[\alpha_n Y_\infty(x)] - \sin(2\pi u'n)\mathbb{E}[\beta_n Y_\infty(x)]\right)\right\}&.\end{aligned}$$

Now, (3.10) follows from the observation that

$$\mathcal{X}(u, q) = -\chi\sqrt{2}\sum_{n,m=1}^\infty \frac{q^{(2m-1)n}}{\sqrt{n}}\left(\cos(2\pi u'n)\alpha_n - \sin(2\pi u'n)\beta_n\right). \tag{3.12}$$

Moreover, (3.12) also implies that

$$\mathcal{X}(u, q) \in \mathbb{R} \text{ and } \mathbb{E}[\mathcal{X}(u, q)^2] = 2\chi^2\sum_{n,m=1}^\infty \frac{q^{2(2m-1)n}}{n} \qquad \text{if } \operatorname{Im} u = \frac{1}{2}\operatorname{Im}\tau. \tag{3.13}$$

Thus let us first assume $\operatorname{Im} u = \frac{1}{2}\operatorname{Im}\tau$ so that $\mathcal{X}(u,q) \in \mathbb{R}$. By (3.10), we have

$$\left(\int_0^1 (2\sin(\pi x))^{-\frac{\alpha\gamma}{2}} \Theta_\tau(x+u)^{\frac{\chi\gamma}{2}} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\infty(x)} dx\right)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}} = \left(-\mathbf{i}e^{-\mathbf{i}\pi u} q^{1/6}\eta(q)\right)^{\frac{\chi}{2}(\chi-\alpha)} \left(\int_0^1 (2\sin(\pi x))^{-\frac{\alpha\gamma}{2}} e^{\frac{\gamma}{2}\mathbb{E}[Y_\infty(x)\mathcal{X}(u,q)]} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\infty(x)} dx\right)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}. \tag{3.14}$$

Let

$$\tilde{\psi}_\chi^\alpha(u,q) := \mathbb{E}\left[e^{\frac{\alpha}{2}F_\tau(0)} \mathcal{Q}(q) \left(\int_0^1 (2\sin(\pi x))^{-\frac{\alpha\gamma}{2}} e^{\frac{\gamma}{2}\mathbb{E}[Y_\infty(x)\mathcal{X}(u,q)]} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\infty(x)} dx\right)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right]. \tag{3.15}$$

Since $\mathcal{X}(u,q) \in \mathbb{R}$, applying Girsanov's theorem (Theorem C.5) with respect to the randomness of $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 1}$ while freezing the variables $(\alpha_{n,m})_{n,m\geq 1}$, $(\beta_{n,m})_{n,m\geq 1}$ gives that

$$\tilde{\psi}_\chi^\alpha(u,q) = \mathbb{E}\left[e^{\frac{\alpha}{2}F_\tau(0)} \mathcal{Q}(q) e^{\mathcal{Y}(u,q)} e^{\mathcal{X}(u,q) - \frac{1}{2}\mathbb{E}[\mathcal{X}(u,q)^2]} \left(\int_0^1 (2\sin(\pi x))^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\infty(x)} dx\right)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right],$$

where we have introduced

$$\mathcal{Y}(u,q) := 2\chi \sum_{m,n,k\geq 1} \alpha_{n,m} \frac{q^{(2k-1+m)n}}{\sqrt{n}} \cos(2\pi u' n) - 2\chi \sum_{m,n,k\geq 1} \beta_{n,m} \frac{q^{(2k-1+m)n}}{\sqrt{n}} \sin(2\pi u' n).$$

This term $\mathcal{Y}(u,q)$ comes from the fact that $\mathcal{Q}(q)$ depends on the $(\alpha_n)_{n\geq 1}$, $(\beta_n)_{n\geq 1}$ which produces the $e^{\mathcal{Y}(u,q)}$ when applying Girsanov's Theorem. For the reason explained below (3.9), just like for $\mathcal{X}(u,q)$, the series of $\mathcal{Y}(u,q)$ converges almost surely in $q \in \mathbb{D}$, and $e^{\mathcal{Y}(u,q)}$ has finite moments of all orders.

Now $\hat{\psi}_\chi^\alpha(u,q)$ and $\tilde{\psi}_\chi^\alpha(u,q)$ will be related by a simple factor, see (3.18) below, meaning we can focus on the analytic extension of $\tilde{\psi}_\chi^\alpha(u,q)$. We repeat the argument using Holder's inequality and Lemma 2.15 used in the proof of Lemma 2.9(a) to show that $\tilde{\psi}_\chi^\alpha(u,q)$ admits a bi-holomorphic extension to the domain D_χ^α defined in (3.3). As before, we will choose $p_1, p_2, p_3 \in (1,\infty)$ with $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3} = 1$ so that the upper bound

$$\mathbb{E}\left[\left|e^{\frac{\alpha}{2}F_\tau(0)} e^{\mathcal{Y}(u,q)} e^{\mathcal{X}(u,q)-\frac{1}{2}\mathbb{E}[\mathcal{X}(u,q)^2]} \mathcal{Q}(q)\right| \left(\int_0^1 (2\sin(\pi x))^{-\alpha\gamma/2} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\infty(x)} dx\right)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right] \tag{3.16}$$

$$\leq \mathbb{E}\left[\left|e^{\frac{\alpha}{2}F_\tau(0)} e^{\mathcal{Y}(u,q)} e^{\mathcal{X}(u,q)-\frac{1}{2}\mathbb{E}[\mathcal{X}(u,q)^2]}\right|^{p_1}\right]^{\frac{1}{p_1}} \cdot \mathbb{E}\left[|\mathcal{Q}(q)|^{p_2}\right]^{\frac{1}{p_2}} \cdot \mathbb{E}\left[\left(\int_0^1 (2\sin(\pi x))^{-\alpha\gamma/2} e^{\pi\gamma P x} e^{\frac{\gamma}{2}Y_\infty(x)} dx\right)^{\frac{(\chi-\alpha)p_3}{\gamma}}\right]^{\frac{1}{p_3}}$$

provided by Holder's inequality is finite. We now choose ranges for p_1, p_2, p_3 for which each of the three terms on the right side of (3.16) is finite. For the first term, because a Gaussian random variable has finite exponential moments, for any $p_1 > 1$ we have

$$\mathbb{E}\left[\left|e^{\frac{\alpha}{2}F_\tau(0)} e^{\mathcal{Y}(u,q)} e^{\mathcal{X}(u,q)-\frac{1}{2}\mathbb{E}[\mathcal{X}(u,q)^2]}\right|^{p_1}\right]^{\frac{1}{p_1}} < \infty.$$

As in the proof of Lemma 2.9(a), to analyze the other two terms we divide into cases based on the sign of $\alpha - \chi$. For $\alpha - \chi$ positive, the exponent $\frac{(\chi-\alpha)p_3}{\gamma}$ in the third term is negative, meaning that the third term is finite for arbitrarily large p_3. Also choosing p_1 arbitrarily large, it remains to check that the second term is finite for p_2 close to 1, which follows by Lemma 2.19 applied with $\alpha - \chi$ in place of α.

For $\alpha - \chi$ negative, the third term is finite if $1 < p_3 < -\frac{4}{(\alpha-\chi)\gamma}$. Choosing p_1 arbitrarily large and p_3 close to $-\frac{4}{(\alpha-\chi)\gamma}$, it suffices to check that the second term is finite for p_2 close to $\frac{1}{1+\frac{\gamma(\alpha-\chi)}{4}}$, which again follows by Lemma 2.19 applied with $\alpha - \chi$ in place of α. We conclude that the right side of (3.16) is finite for $\alpha \in (-\frac{4}{\gamma}+\chi, Q)$ and $|q| < r_{\alpha-\chi}$ and thus that $\tilde{\psi}_\chi^\alpha(u,q)$ admits a bi-holomorphic extension to D_χ^α.

Collecting (3.8) and (3.14), when $q \in (0, q_0 \wedge r_{\alpha-\chi})$ and $\operatorname{Im} u = \frac{1}{2}\operatorname{Im}\tau$, we have

$$\mathbb{E}\left[(f_\nu(u))^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right] = C_1(q)\left(-\mathbf{i}e^{-\mathbf{i}\pi u} q^{1/6}\eta(q)\right)^{\frac{\chi}{2}(\chi-\alpha)} \tilde{\psi}_\chi^\alpha(u,q). \tag{3.17}$$

By Lemma 3.1, $\mathbb{E}\left[(f_\nu(u))^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right]$ is analytic in $u \in \mathfrak{B}$. Thus for all $q \in (0, q_0 \wedge r_{\alpha-\chi})$, by analyticity in u (3.17) holds not only for $\mathrm{Im}(u) = \frac{1}{2}\mathrm{Im}(\tau)$ but for all $u \in \mathfrak{B}$. Since we know $\tilde{\psi}^\alpha_\chi(u,q)$ admits a bi-holomorphic extension to D^α_χ, the right hand side of (3.17) provides the desired analytic continuation of $\mathbb{E}\left[(f_\nu(u))^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right]$.

Lastly we just need to check the global prefactor relating $\tilde{\psi}^\alpha_\chi(u,q)$ to $\hat{\psi}^\alpha_\chi(u,q)$ is also bi-holomorphic on D^α_χ; for this, recall (3.4). One can check by Lemma 2.5 and (B.2) that for $q \in (0,1)$, the quantity

$$C(q)e^{\chi P u\pi}\Theta_\tau(u)^{-l_\chi}C_1(q)\left(-\mathbf{i}e^{-\mathbf{i}\pi u}q^{1/6}\eta(q)\right)^{\frac{\chi}{2}(\chi-\alpha)} \tag{3.18}$$

equals the product of $q^{\frac{\gamma}{12\chi}-\frac{1}{6\chi^2}+\frac{1}{3\chi\gamma}-\frac{1}{6}}$ with a power series in q which converges in $\mathbb{D}$. Moreover, we have $\frac{\gamma}{12\chi}-\frac{1}{6\chi^2}+\frac{1}{3\chi\gamma}-\frac{1}{6} = 0$ when $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$. This together with (3.17) concludes the proof of Proposition 3.2.

3.2. **Proof of Lemma 3.6.** Following the notation of the proof of Proposition 3.2, analytically extending $\psi^\alpha_\chi(u,q)$ reduces to analytically extending $\mathbb{E}\left[(f_\nu(u))^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right]$. Thanks to Proposition 3.2, we have the desired analyticity with respect to u and q. For the analyticity in α, we repeat the argument given in Lemma 2.9. Recall the map ϕ from Appendix C, which is also used in the proof of Lemma 2.9. By Girsanov's theorem (Theorem C.5) the analyticity in α of $\mathbb{E}\left[(f_\nu(u))^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right]$ reduces to the analyticity of

$$\begin{aligned}\mathbb{E}&\left[e^{\frac{\alpha}{2}F_\tau(0)}\mathcal{Q}(q)e^{\mathcal{Y}(u,q)}e^{\mathcal{X}(u,q)-\frac{1}{2}\mathbb{E}[\mathcal{X}(u,q)^2]}\left(\int_0^1 (2\sin(\pi x))^{-\frac{\alpha\gamma}{2}}e^{\pi\gamma P x}e^{\frac{\gamma}{2}Y_\infty(x)}dx\right)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right]\\ &= \mathbb{E}\left[e^{\frac{\alpha}{2}F_\tau(0)}\mathcal{Q}(q)e^{\mathcal{Y}(u,q)}e^{\mathcal{X}(u,q)-\frac{1}{2}\mathbb{E}[\mathcal{X}(u,q)^2]}\left(\int_{\mathbb{R}} |y|^{-\frac{\alpha\gamma}{2}}g_1(y)e^{\frac{\gamma}{2}X_{\mathbb{H}}(y)}dy\right)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right],\end{aligned}$$

where g_1 is the bounded continuous function such that $|y|^{-\frac{\alpha\gamma}{2}}g_1(y)dy$ is the pushforward of $(2\sin(\pi x))^{-\frac{\alpha\gamma}{2}}e^{\pi\gamma P x}dx$ under ϕ. The analyticity in α is now again a straightforward adaptation of the proof of [RZ20, Lemma 5.6], as performed in the proof Lemma 2.9 (b).

3.3. **Proof of Theorem 3.5.** Since $\hat{\psi}^\alpha_\chi(u,q)$ is bi-holomorphic in (u,q) by Proposition 3.2, it suffices to verify (3.7) for $q \in (0, q_0 \wedge r_{\alpha-\chi})$ and $u \in \mathfrak{B}$, where (3.6) applies. Ignoring the expectation symbol, (3.6) is a smooth function in (q,u) in this range. Moreover, Lemma 2.15 allows us to interchange the expectation and derivatives. Therefore, checking Theorem 3.5 is conceptually straightforward. However, as we will see, the proof requires an application of integration by parts and delicate manipulation of the theta function.

Recall $\mathcal{T}(u,x)$ from (3.6). Define $s := -\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}$ and introduce the notations

$$\mathcal{V}_1(u,y)dy := \mathbb{E}\left[\left(\int_0^1 \mathcal{T}(u,x)e^{\pi\gamma P x}e^{\frac{\gamma}{2}Y_\tau(x)}dx\right)^{s-1}e^{\frac{\gamma}{2}Y_\tau(y)}dy\right]; \tag{3.19}$$

$$\mathcal{V}_2(u,y,z)dydz := \mathbb{E}\left[\left(\int_0^1 \mathcal{T}(u,x)e^{\pi\gamma P x}e^{\frac{\gamma}{2}Y_\tau(x)}dx\right)^{s-2}e^{\frac{\gamma}{2}Y_\tau(y)}dye^{\frac{\gamma}{2}Y_\tau(z)}dz\right]. \tag{3.20}$$

Here we adopt the convention that for a random measure μ, $\mathbb{E}[\mu]$ is the measure satisfying $\int f\mathbb{E}[\mu] = \int \mathbb{E}[f\mu]$ for integrable test functions f. Moreover, $e^{\frac{\gamma}{2}Y_\tau(y)}dye^{\frac{\gamma}{2}Y_\tau(z)}dz$ means the product of $e^{\frac{\gamma}{2}Y_\tau(y)}dy$ and $e^{\frac{\gamma}{2}Y_\tau(z)}dz$.

By the Girsanov Theorem C.5,

$$\mathcal{V}_1(u,y) = \mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma^2}{4}\mathbb{E}[Y_\tau(x)Y_\tau(y)]}\mathcal{T}(u,x)e^{\pi\gamma P x}e^{\frac{\gamma}{2}Y_\tau(x)}dx\right)^{s-1}\right]. \tag{3.21}$$

Since $s<1$ so that $s-1<0$, by the third claim of Lemma C.4, for each fixed $u \in \mathfrak{B}$, the function $\mathcal{V}_1(u,\cdot)$ are bounded continuous on $[0,1]$. For the same reason, $\mathcal{V}_2(u,\cdot,\cdot)$ is bounded continuous on $[0,1]^2$.

We start by computing derivatives with respect to u; by direct differentiation in (3.6), we have

$$\partial_u \psi^\alpha_\chi(u,q) = \chi P\pi \psi^\alpha_\chi(u,q) + s\mathcal{W}(q)e^{\pi\chi P u}\int_0^1 \partial_u \mathcal{T}(u,y)e^{\pi\gamma P y}\mathcal{V}_1(u,y)dy.$$

$$\begin{aligned}\partial_{uu}\psi^\alpha_\chi(u,q) &= (\chi P\pi)^2\psi^\alpha_\chi(u,q) + 2\chi P\pi s\mathcal{W}(q)e^{\pi\chi P u}\int_0^1 \partial_u\mathcal{T}(u,y)e^{\pi\gamma P y}\mathcal{V}_1(u,y)dy \\ &\quad + s\mathcal{W}(q)e^{\pi\chi P u}\int_0^1 \partial_{uu}\mathcal{T}(u,y)e^{\pi\gamma P y}\mathcal{V}_1(u,y)dy \\ &\quad + s(s-1)\mathcal{W}(q)e^{\pi\chi P u}\int_0^1\int_0^1 \partial_u\mathcal{T}(u,y)\partial_u\mathcal{T}(u,z)e^{\pi\gamma P(y+z)}\mathcal{V}_2(u,y,z)dydz.\end{aligned} \tag{3.22}$$

Given $u \in \mathfrak{B}$, we can check that for some constant $c > 0$, as $y \to 0$ or $y \to 1$,

$$\partial_u\mathcal{T}(u,y) = \frac{\gamma\chi}{2}\left(\frac{\Theta'_\tau(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta'_\tau(u)}{\Theta_\tau(u)}\right)\mathcal{T}(u,y) \sim c\sin^{1-\frac{\alpha\gamma}{2}}(\pi y). \tag{3.23}$$

Similarly, $\partial_{uu}\mathcal{T}(u,y) \sim c\sin^{1-\frac{\alpha\gamma}{2}}(\pi y)$. Since we assume $\alpha < Q$, both $\partial_u\mathcal{T}$ and $\partial_{uu}\mathcal{T}$ have integrable singularities in the integrals appearing in (3.22).

We now compute the derivative in τ.

Lemma 3.7. *For $q \in (0, q_0 \wedge r_{\alpha-\chi})$ and $u \in \mathfrak{B}$, we have*

$$\begin{aligned}&\partial_\tau\psi^\alpha_\chi(u,q) = \mathbf{i}\pi\left(\frac{P^2}{2} + \frac{\gamma l_\chi}{12\chi} - \frac{1}{6}\frac{l_\chi^2}{\chi^2}\right)\psi^\alpha_\chi(u,q) \\ &+ \left(-\frac{2l_\chi^2}{3\chi^2} + \frac{l_\chi}{3} + \frac{2}{3}s\right)\frac{\partial_\tau\Theta'_\tau(0)}{\Theta'_\tau(0)}\psi^\alpha_\chi(u,q) + s\mathcal{W}(q)e^{\pi\chi P u}\int_0^1 \partial_\tau\mathcal{T}(u,y)e^{\pi\gamma P y}\mathcal{V}_1(u,y)dy \\ &+ \frac{\gamma^2 s(s-1)}{4}\mathcal{W}(q)e^{\pi\chi P u}\int_0^1\int_0^1\left(\frac{\mathbf{i}\pi}{6} - \frac{\partial_\tau\Theta_\tau(y-z)}{\Theta_\tau(y-z)} + \frac{1}{3}\frac{\partial_\tau\Theta'_\tau(0)}{\Theta'_\tau(0)}\right)\mathcal{T}(u,y)\mathcal{T}(u,z)e^{\pi\gamma P y + \pi\gamma P z}\mathcal{V}_2(u,y,z)dydz.\end{aligned}$$

Proof. Let $V(\tau) := \int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\mathcal{T}(u,x)e^{\pi\gamma P x}dx$. Taking the τ-derivative of (3.6), we obtain

$$\partial_\tau\psi^\alpha_\chi(u,q) = \partial_\tau(\log\mathcal{W}(q))\psi^\alpha_\chi(u,q) + s\mathcal{W}(q)e^{\pi\chi P u}\mathbb{E}\left[V(\tau)^{s-1}\partial_\tau V(\tau)\right]. \tag{3.24}$$

Note that

$$\partial_\tau(\log\mathcal{W}(q)) = \mathbf{i}\pi\left(\frac{P^2}{2} + \frac{\gamma l_\chi}{12\chi} - \frac{1}{6}\frac{l_\chi^2}{\chi^2}\right) + \left(-\frac{2l_\chi^2}{3\chi^2} + \frac{l_\chi}{3} + \frac{2}{3}s\right)\frac{\partial_\tau\Theta'_\tau(0)}{\Theta'_\tau(0)}, \tag{3.25}$$

$$\mathbb{E}\left[V(\tau)^{s-1}\partial_\tau V(\tau)\right] = \int_0^1 \partial_\tau\mathcal{T}(u,y)e^{\pi\gamma P y}\mathcal{V}_1(u,y)dy + \int_0^1 \mathcal{T}(u,y)e^{\pi\gamma P y}\mathbb{E}\left[V(\tau)^{s-1}\partial_\tau[e^{\frac{\gamma}{2}Y_\tau(y)}]dy\right], \tag{3.26}$$

where $\partial_\tau[e^{\frac{\gamma}{2}Y_\tau(y)}]dy$ means the measure (recall (2.6))

$$\partial_\tau[e^{\frac{\gamma}{2}Y_\tau(y)}]dy = \partial_\tau\left[e^{-\frac{\gamma^2}{8}\mathbb{E}[F_\tau(0)^2]}e^{\frac{\gamma}{2}F_\tau(y)}\right]e^{\frac{\gamma}{2}Y_\infty(y)}dy = \partial_\tau\left(\frac{\gamma}{2}F_\tau(y) - \frac{\gamma^2}{8}\mathbb{E}[F_\tau(0)^2]\right)\times e^{\frac{\gamma}{2}Y_\tau(y)}dy. \tag{3.27}$$

We claim that

$$\begin{aligned}&\mathbb{E}\left[V(\tau)^{s-1}\partial_\tau[e^{\frac{\gamma}{2}Y_\tau(y)}]dy\right] \\ &= \mathbb{E}\left[\frac{\gamma^2}{4}(s-1)\left(\int_0^1 \mathbb{E}[F_\tau(z)\partial_\tau F_\tau(y)]\mathcal{T}(u,z)e^{\pi\gamma P z}V(\tau)^{s-2}e^{\frac{\gamma}{2}Y_\tau(z)}dz\right)e^{\frac{\gamma}{2}Y_\tau(y)}dy\right].\end{aligned} \tag{3.28}$$

Computing using (2.2) and (B.3), we find that

$$\begin{aligned}\mathbb{E}[\partial_\tau F_\tau(y)F_\tau(z)] &= 4\pi\mathbf{i}\sum_{m,n=1}^\infty mq^{2nm}\cos(2\pi n(y-z)) = \frac{1}{2}\partial_\tau\mathbb{E}[F_\tau(y)F_\tau(z)] \\ &= -\partial_\tau\log\left|q^{-\frac{1}{6}}\frac{\Theta_\tau(y-z)}{\eta(q)}\right| = \frac{\mathbf{i}\pi}{6} - \frac{\partial_\tau\Theta_\tau(y-z)}{\Theta_\tau(y-z)} + \frac{\partial_\tau\eta(q)}{\eta(q)} = \frac{\mathbf{i}\pi}{6} - \frac{\partial_\tau\Theta_\tau(y-z)}{\Theta_\tau(y-z)} + \frac{1}{3}\frac{\partial_\tau\Theta'_\tau(0)}{\Theta'_\tau(0)}.\end{aligned}$$

Combining with (3.24)—(3.28), we obtain Lemma 3.7.

It remains to prove (3.28). For this we use the Girsanov Theorem C.5 to write:

$$\begin{aligned}
&\mathbb{E}\left[V(\tau)^{s-1}\partial_\tau F_\tau(y)e^{\frac{\gamma}{2}Y_\tau(y)}dy\right] = \frac{d}{d\epsilon}_{|\epsilon=0}\mathbb{E}\left[V(\tau)^{s-1}e^{\epsilon\partial_\tau F_\tau(y)-\frac{\epsilon^2}{2}\mathbb{E}[\partial_\tau F_\tau(y)^2]}e^{\frac{\gamma}{2}Y_\tau(y)}dy\right]\\
&= \frac{d}{d\epsilon}_{|\epsilon=0}\mathbb{E}\left[\left(\int_0^1 \mathcal{T}(u,z)e^{\pi\gamma Pz}e^{\frac{\gamma}{2}Y_\tau(z)+\frac{\gamma\epsilon}{2}\mathbb{E}[Y_\tau(z)\partial_\tau F_\tau(y)]}dz\right)^{s-1}e^{\frac{\gamma}{2}Y_\tau(y)+\frac{\gamma\epsilon}{2}\mathbb{E}[Y_\tau(y)\partial_\tau F_\tau(y)]}dy\right]\\
&= \frac{\gamma}{2}(s-1)\mathbb{E}\left[\left(\int_0^1 \mathbb{E}[F_\tau(z)\partial_\tau F_\tau(y)]\mathcal{T}(u,z)e^{\pi\gamma Pz}V(\tau)^{s-2}e^{\frac{\gamma}{2}Y_\tau(z)}dz\right)e^{\frac{\gamma}{2}Y_\tau(y)}dy\right]\\
&+\frac{\gamma}{2}\mathbb{E}[F_\tau(0)\partial_\tau F_\tau(0)]\mathbb{E}\left[V(\tau)^{s-1}e^{\frac{\gamma}{2}Y_\tau(y)}dy\right].
\end{aligned}$$

This computation combined with (3.27) and the fact that $\partial_\tau \mathbb{E}[F_\tau(0)^2] = 2\mathbb{E}[F_\tau(0)\partial_\tau F_\tau(0)]$ implies (3.28). □

By Lemma 3.6, it suffice to prove Theorem 3.5 assuming $\alpha \in (-\frac{4}{\gamma}+\chi, \frac{2}{\gamma})$. Then analyticity in α gives the full range of $\alpha \in (-\frac{4}{\gamma}+\chi, Q)$. We need to perform the following integration by parts on one of the terms in $\partial_{uu}\psi_\chi^\alpha(u,q)$. The assumption $\alpha \in (-\frac{4}{\gamma}+\chi, \frac{2}{\gamma})$ will allow us to ignoring the boundary terms involved.

Lemma 3.8. *Fix $\alpha \in (-\frac{4}{\gamma}+\chi, \frac{2}{\gamma})$. Then the three integrals below absolutely converge and satisfy*

$$\begin{aligned}
&\gamma P\pi\int_0^1 \partial_u\mathcal{T}(u,y)e^{\pi\gamma Py}\mathcal{V}_1(u,y)dy + \int_0^1 \partial_{uy}\mathcal{T}(u,y)e^{\pi\gamma Py}\mathcal{V}_1(u,y)dy\\
&= \frac{\chi\gamma^3(s-1)}{8}\int_0^1\int_0^1 \frac{\Theta_\tau'(y-z)}{\Theta_\tau(y-z)}\Big(\frac{\Theta_\tau'(u+y)}{\Theta_\tau(u+y)}-\frac{\Theta_\tau'(u+z)}{\Theta_\tau(u+z)}\Big)\mathcal{T}(u,y)\mathcal{T}(u,z)e^{\pi\gamma P(y+z)}\mathcal{V}_2(u,y,z)dydz.
\end{aligned} \tag{3.29}$$

Proof. Fix $\varepsilon > 0$ small. We first introduce a regularization of $Y_\tau, \mathcal{V}_1, \mathcal{V}_2$ for technical convenience. Recall $X_{\mathbb{H}}$ from Appendix C. For $x \in \mathbb{R}_{\geq 0}\times[0,1]$, let $\phi(x) = -\mathbf{i}\frac{e^{2\pi\mathbf{i}x}-1}{e^{2\pi\mathbf{i}x}+1} \in \mathbb{H}\cup\mathbb{R}$. Then ϕ conformally map the half cylinder $\mathcal{C}_+$ obtained by gluing the two vertical boundaries of $[0,1]\times\mathbb{R}_{\geq 0}$ to $\mathbb{H}\cup\mathbb{R}$. Let $Y_\infty(x) = X_{\mathbb{H}}(\phi(x))$ for $x \in \mathcal{C}_+$. Then $Y_\infty(x)$ is a free boundary GFF on $\mathcal{C}_+$. For $x \in (0,1)$, let $Y_\infty^\varepsilon(x)$ be the average of Y_∞ over the semi-circle $\{y \in \mathcal{C}_+ : |y-x| = \varepsilon\}$. Here $|\cdot|$ means the distance in the flat metric on $\mathcal{C}_+$. Let $Y_\tau^\varepsilon := Y_\infty^\varepsilon + F_\tau$. Since Green function is harmonic away from the diagonal, we have

$$\mathbb{E}[Y_\tau(x)Y_\tau(y)] - \mathbb{E}[Y_\tau(x)Y_\tau^\varepsilon(y)] = \left(-2\log\frac{|\sin(\pi(x-y))|}{\varepsilon}\right)1_{|x-y|<\varepsilon} \qquad \text{for } x,y\in[0,1]. \tag{3.30}$$

Let $K_\varepsilon(\cdot)$ be such that $\mathbb{E}[Y_\tau^\varepsilon(x)Y_\tau^\varepsilon(y)] = K_\varepsilon(x-y)$. Then we can check using (3.30) that $yK_\varepsilon'(y)$ is a bounded continuous function on $[0,1]$ and uniformly converge to $y\partial_y\mathbb{E}[Y_\tau(0)Y_\tau(y)]$. (See [Ber17, Lemma 3.5] for a similar calculation with $\log|\sin(\pi(x-y))|$ in (3.30) replaced by $\log|x-y|$.)

We define $\mathcal{V}_{1,\varepsilon}$ and $\mathcal{V}_{2,\varepsilon}$ as in (3.19) and (3.20) with each $e^{Y_\tau(\cdot)}$ replaced by $e^{Y_\tau^\varepsilon(\cdot)-\frac{1}{2}\mathbb{E}[Y_\tau^\varepsilon(\cdot)^2]}$. Since $s-1<0$, as $\varepsilon\to 0$, for a fixed $u\in\mathfrak{B}$, $\mathcal{V}_{1,\varepsilon}(u,\cdot)$ and $\mathcal{V}_{2,\varepsilon}(\cdot,\cdot)$ converge uniformly to $\mathcal{V}_1(u,\cdot)$ and $\mathcal{V}_2(u,\cdot,\cdot)$, respectively.

Recall (3.23). Since $\alpha < \frac{2}{\gamma}$, by integration by parts we have

$$\begin{aligned}
&\pi\gamma P\int_0^1 \partial_u\mathcal{T}(u,y)e^{\pi\gamma Py}\mathcal{V}_{1,\varepsilon}(u,y)dy = \int_0^1 \partial_u\mathcal{T}(u,y)[\partial_y e^{\pi\gamma Py}]\mathcal{V}_{1,\varepsilon}(u,y)dy\\
&= -\int_0^1 \partial_{uy}\mathcal{T}(u,y)e^{\pi\gamma Py}\mathcal{V}_{1,\varepsilon}(u,y)dy - \int_0^1 \partial_u\mathcal{T}(u,y)e^{\pi\gamma Py}\partial_y\mathcal{V}_{1,\varepsilon}(u,y)dy.
\end{aligned} \tag{3.31}$$

Note that (3.21) holds with $\mathcal{V}_{1,\epsilon}$ and $\mathbb{E}[Y_\tau(x)Y_\tau^\varepsilon(y)]$ in place of $\mathcal{V}_1$ and $\mathbb{E}[Y_\tau(x)Y_\tau(y)]$. Applying ∂_y to this modified (3.21) and using the Girsanov Theorem C.5 again, we find that

$$\partial_y\mathcal{V}_{1,\varepsilon}(u,y) = \frac{\gamma^2}{4}(s-1)\int_0^1 \mathcal{T}(u,z)\partial_y\mathbb{E}[Y_\tau^\varepsilon(y)Y_\tau(z)]e^{\pi\gamma Pz}\mathcal{V}_{2,\varepsilon}(u,y,z)dz.$$

Therefore $\int_0^1 \partial_u \mathcal{T}(u,y) e^{\pi\gamma P y} \partial_y \mathcal{V}_{1,\varepsilon}(u,y)dy$ equals

$$\frac{\gamma^2}{4}(s-1)\int_0^1\int_0^1 \frac{\partial_u \mathcal{T}(u,y)}{\mathcal{T}(u,y)} \partial_y \mathbb{E}[Y_\tau^\varepsilon(y)Y_\tau^\varepsilon(z)]\mathcal{T}(u,y)\mathcal{T}(u,z)e^{\pi\gamma P(y+z)}\mathcal{V}_{2,\varepsilon}(u,y,z)dydz.$$
$$=\frac{\gamma^2}{8}(s-1)\int_0^1\int_0^1 \left(\frac{\partial_u \mathcal{T}(u,y)}{\mathcal{T}(u,y)} - \frac{\partial_u \mathcal{T}(u,z)}{\mathcal{T}(u,z)}\right) K_\varepsilon'(y-z)\mathcal{T}(u,y)\mathcal{T}(u,z)e^{\pi\gamma P(y+z)}\mathcal{V}_{2,\varepsilon}(u,y,z)dydz$$

where we have used that $\mathcal{T}(u,y)\mathcal{T}(u,z)e^{\pi\gamma Py+\pi\gamma Pz}\mathcal{V}_{2,\varepsilon}(u,y,z)dydz$ is symmetric under interchange of y and z, and that the derivative K_ε' is an odd function.

Recall (3.23) for the expression for $\frac{\partial_u \mathcal{T}}{\mathcal{T}}$. By the discussion below (3.30), $\left(\frac{\partial_u \mathcal{T}(u,y)}{\mathcal{T}(u,y)} - \frac{\partial_u \mathcal{T}(u,z)}{\mathcal{T}(u,z)}\right) K_\varepsilon'(y-z)$ uniformly converge to

$$\frac{\gamma\chi}{2}\partial_y \mathbb{E}[Y_\tau(y)Y_\tau(z)]\Big(\frac{\Theta_\tau'(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta_\tau'(u+z)}{\Theta_\tau(u+z)}\Big) = -\chi\gamma\frac{\Theta_\tau'(y-z)}{\Theta_\tau(y-z)}\Big(\frac{\Theta_\tau'(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta_\tau'(u+z)}{\Theta_\tau(u+z)}\Big).$$

Therefore we may apply the dominated convergence theorem to obtain that $\int_0^1 \partial_u \mathcal{T}(u,y)e^{\pi\gamma Py}\partial_y \mathcal{V}_1(u,y)dy$ equals the right hand side of (3.29). Combing with (3.31), this concludes our proof. □

The expression for $\partial_{uu}\psi_\chi^\alpha(u,q)$ from (3.22) and $\partial_\tau \psi_\chi^\alpha(u,q)$ from Lemma 3.7 can be written as a summation of three types of terms: the product of an explicit function and $\psi_\chi^\alpha(u,q)$, 1-fold integrals over $[0,1]$, 2-fold integral over $[0,1]$. For the 1-fold integral term $2\chi P\pi s\mathcal{W}(q)e^{\pi\chi Pu}\int_0^1 \partial_u \mathcal{T}(u,y)e^{\pi\gamma Py}\mathcal{V}_1(u,y)dy$ from (3.22), we can further apply Lemma 3.8 to write it as a difference of a 2-fold integral and a 1-fold integral over $[0,1]$:

$$\frac{\chi^2\gamma^2 s(s-1)}{4}\mathcal{W}(q)e^{\pi\chi Pu}\int_0^1\int_0^1 \frac{\Theta_\tau'(y-z)}{\Theta_\tau(y-z)}\Big(\frac{\Theta_\tau'(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta_\tau'(u+z)}{\Theta_\tau(u+z)}\Big)\mathcal{T}(u,y)\mathcal{T}(u,z)e^{\pi\gamma P(y+z)}\mathcal{V}_2(u,y,z)dydz$$
$$-\frac{2\chi s}{\gamma}\mathcal{W}(q)e^{\pi\chi Pu}\int_0^1 \partial_{uy}\mathcal{T}(u,y)e^{\pi\gamma Py}\mathcal{V}_1(u,y)dy.$$

Under this substitution, we may now write

$$\Big(\partial_{uu} - l_\chi(l_\chi+1)\wp(u) + 2\mathbf{i}\pi\chi^2\partial_\tau\Big)\psi_\chi^\alpha(u,q) = \Xi_0+\Xi_1+\Xi_2,$$

where Ξ_k for $k=1,2$ contains all terms with a k-fold integral over $[0,1]$, and Ξ_0 contains all terms of the form of the product of an explicit function and $\psi_\chi^\alpha(u,q)$. To prove Theorem 3.5, we express $\frac{\Xi_0+\Xi_1+\Xi_2}{\psi_\chi^\alpha(u,q)}$ explicitly and check that it equals zero. We start by giving the following expression of Ξ_2.

Lemma 3.9. *We have* $\Xi_2 = \frac{\chi^2\gamma^2}{2}s(s-1)\mathcal{W}(q)e^{\pi\chi Pu}\int_0^1 \widetilde{\Delta}(u,y)\mathcal{T}(u,y)e^{\pi\gamma Py}\mathcal{V}_1(u,y)dy$, *where*

$$\widetilde{\Delta}(u,x) = \frac{1}{2}\frac{\Theta_\tau''(u+x)}{\Theta_\tau(u+x)} - \frac{\Theta_\tau'(u+x)}{\Theta_\tau(u+x)}\frac{\Theta_\tau'(u)}{\Theta_\tau(u)} + \frac{1}{2}\frac{\Theta_\tau'(u)^2}{\Theta_\tau(u)^2} - \frac{\pi^2}{6} - \frac{1}{6}\frac{\Theta_\tau'''(0)}{\Theta_\tau'(0)}.$$

Proof. Combining (3.22), Lemma 3.7, and Lemma 3.8 as explained above, we have

$$\Xi_2 = \frac{\chi^2\gamma^2}{2}s(s-1)\mathcal{W}(q)e^{\pi\chi Pu}\int_0^1\int_0^1 \Delta_2(y,z)\mathcal{T}(u,y)\mathcal{T}(u,z)e^{\pi\gamma Py+\pi\gamma Pz}\mathcal{V}_2(u,y,z)dydz \tag{3.32}$$

for

$$\Delta_2(y,z) := \Bigg[\frac{1}{2}\frac{\Theta_\tau'(y-z)}{\Theta_\tau(y-z)}\Big(\frac{\Theta_\tau'(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta_\tau'(u+z)}{\Theta_\tau(u+z)}\Big) + \frac{1}{2}\Big(\frac{\Theta_\tau'(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta_\tau'(u)}{\Theta_\tau(u)}\Big)\Big(\frac{\Theta_\tau'(u+z)}{\Theta_\tau(u+z)} - \frac{\Theta_\tau'(u)}{\Theta_\tau(u)}\Big)$$
$$-\frac{1}{4}\frac{\Theta_\tau''(y-z)}{\Theta_\tau(y-z)} - \frac{\pi^2}{6} + \frac{1}{12}\frac{\Theta_\tau'''(0)}{\Theta_\tau'(0)}\Bigg],$$

where we use (B.13). Applying the identity (B.14) with $(a,b)=(u+y,u+z)$, we have

$$\Delta_2(y,z) = \frac{1}{2}\left(\widetilde{\Delta}(u,y)+\widetilde{\Delta}(u,z)\right). \tag{3.33}$$

By (3.32) and (3.33), Lemma 3.9 is a consequence of the following observation:

$$\int_0^1\int_0^1 \Big(\widetilde{\Delta}(u,y)+\widetilde{\Delta}(u,z)\Big)\mathcal{T}(u,y)\mathcal{T}(u,z)e^{\pi\gamma Py+\pi\gamma Pz}\mathcal{V}_2(u,y,z)dydz = 2\int_0^1 \widetilde{\Delta}(u,y)\mathcal{T}(u,y)e^{\pi\gamma Py}\mathcal{V}_1(u,y)dy.$$

□

By Lemma 3.9, we have

$$\Xi_1 + \Xi_2 = s\mathcal{W}(q)e^{\pi\chi Pu}\int_0^1 \Delta_1(u,y)\mathcal{T}(u,y)e^{\pi\gamma Py}\mathcal{V}_1(u,y)dy$$

for

$$\Delta_1(u,y) = -\frac{2\chi}{\gamma}\frac{\partial_{uy}\mathcal{T}(u,y)}{\mathcal{T}(u,y)} + \frac{\partial_{uu}\mathcal{T}(u,y)}{\mathcal{T}(u,y)} + 2\mathbf{i}\pi\chi^2\frac{\partial_\tau\mathcal{T}(u,y)}{\mathcal{T}(u,y)} + (s-1)\frac{\chi^2\gamma^2}{2}\widetilde{\Delta}(u,y).$$

We compute

$$\begin{aligned}
\frac{\partial_{uu}\mathcal{T}(u,y)}{\mathcal{T}(u,y)} &= \frac{\gamma\chi}{2}\left(\frac{\Theta_\tau''(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta_\tau''(u)}{\Theta_\tau(u)} - \left(\frac{\Theta_\tau'(u+y)}{\Theta_\tau(u+y)}\right)^2 + \left(\frac{\Theta_\tau'(u)}{\Theta_\tau(u)}\right)^2\right) + \frac{\gamma^2\chi^2}{4}\left(\frac{\Theta_\tau'(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta_\tau'(u)}{\Theta_\tau(u)}\right)^2\\
\frac{\partial_{uy}\mathcal{T}(u,y)}{\mathcal{T}(u,y)} &= \frac{\gamma\chi}{2}\left(\frac{\Theta_\tau''(u+y)}{\Theta_\tau(u+y)} - \left(\frac{\Theta_\tau'(u+y)}{\Theta_\tau(u+y)}\right)^2\right) + \frac{\gamma\chi}{2}\left(\frac{\Theta_\tau'(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta_\tau'(u)}{\Theta_\tau(u)}\right)\left(\frac{\gamma\chi}{2}\frac{\Theta_\tau'(u+y)}{\Theta_\tau(u+y)} - \frac{\alpha\gamma}{2}\frac{\Theta_\tau'(y)}{\Theta_\tau(y)}\right)\\
\frac{\partial_\tau\mathcal{T}(u,y)}{\mathcal{T}(u,y)} &= \frac{1}{4\pi\mathbf{i}}\left(-\frac{\alpha\gamma}{2}\frac{\Theta_\tau''(y)}{\Theta_\tau(y)} + \frac{\gamma\chi}{2}\frac{\Theta_\tau''(u+y)}{\Theta_\tau(u+y)} - \frac{\gamma\chi}{2}\frac{\Theta_\tau''(u)}{\Theta_\tau(u)}\right).
\end{aligned}$$

The total prefactor of $\frac{\Theta_\tau'(u+y)^2}{\Theta_\tau(u+y)^2}$ in $\Delta_1(u,y)$ is therefore

$$-\frac{\gamma}{2}\chi + (1+\frac{\gamma^2}{4})\chi^2 - \frac{\gamma}{2}\chi^3 = -\frac{\gamma}{2}\chi(\chi-\frac{\gamma}{2})(\chi-\frac{2}{\gamma}) = 0.$$

Similarly, the total prefactor of $\frac{\Theta_\tau'(u)^2}{\Theta_\tau(u)^2}$ in $\Delta_1(u,y)$ is $\frac{\gamma}{2}\chi - \frac{\alpha\gamma}{4}\chi^2 + \frac{\gamma}{4}\chi^3$. We may therefore write

$$\Delta_1(u,y) = \frac{\gamma}{2}\Big(\chi - \frac{\alpha}{2}\chi^2 + \frac{1}{2}\chi^3\Big)\frac{\Theta_\tau'(u)^2}{\Theta_\tau(u)^2} + \chi\Delta_1^1(u,y) + \chi^2\Delta_1^2(u,y) + \chi^3\Delta_1^3(u,y)$$

for

$$\begin{aligned}
\Delta_1^1(u,y) &= \frac{\gamma}{2}\left(\frac{\Theta_\tau''(u+y)}{\Theta_\tau(u+y)} - \frac{\Theta_\tau''(u)}{\Theta_\tau(u)}\right)\\
\Delta_1^2(u,y) &= -(1+\frac{\gamma^2}{4}+\frac{\alpha\gamma}{4})\frac{\Theta_\tau''(u+y)}{\Theta_\tau(u+y)} - \frac{\alpha\gamma}{2}\frac{\Theta_\tau'(u)}{\Theta_\tau(u)}\frac{\Theta_\tau'(y)}{\Theta_\tau(y)} + \frac{\alpha\gamma}{2}\frac{\Theta_\tau'(y)\Theta_\tau'(u+y)}{\Theta_\tau(y)\Theta_\tau(u+y)} + \frac{\alpha\gamma}{2}\frac{\Theta_\tau'(u+y)\Theta_\tau'(u)}{\Theta_\tau(u+y)\Theta_\tau(u)}\\
&\quad - \frac{\alpha\gamma}{4}\frac{\Theta_\tau''(y)}{\Theta_\tau(y)} + \frac{\alpha\gamma\pi^2}{12} + \frac{\alpha\gamma}{12}\frac{\Theta_\tau'''(0)}{\Theta_\tau'(0)} + \frac{\pi^2\gamma^2}{12} + \frac{\gamma^2}{12}\frac{\Theta_\tau'''(0)}{\Theta_\tau'(0)}\\
&= -(1+\frac{\gamma^2}{4})\frac{\Theta_\tau''(u+y)}{\Theta_\tau(u+y)} + \frac{\alpha\gamma}{4}\frac{\Theta_\tau''(u)}{\Theta_\tau(u)} + \frac{\alpha\gamma\pi^2}{12} - \frac{\alpha\gamma}{6}\frac{\Theta_\tau'''(0)}{\Theta_\tau'(0)} + \frac{\pi^2\gamma^2}{12} + \frac{\gamma^2}{12}\frac{\Theta_\tau'''(0)}{\Theta_\tau'(0)}\\
\Delta_1^3(u,y) &= \frac{\gamma}{2}\frac{\Theta_\tau''(u+y)}{\Theta_\tau(u+y)} - \frac{\gamma}{4}\frac{\Theta_\tau''(u)}{\Theta_\tau(u)} - \frac{\pi^2\gamma}{12} - \frac{\gamma}{12}\frac{\Theta_\tau'''(0)}{\Theta_\tau'(0)},
\end{aligned}$$

where we apply (B.14) for $(a,b) = (u+y,y)$. Adding $0 = (-\frac{\gamma}{2}\chi + (1+\frac{\gamma^2}{4})\chi^2 - \frac{\gamma}{2}\chi^3)\frac{\Theta_\tau''(u+y)}{\Theta_\tau(u+y)}$, we obtain

$$\begin{aligned}
\Delta_1(u,y) &= \Big(\frac{\chi\gamma}{2} - \frac{\alpha\gamma}{4}\chi^2 + \frac{\gamma}{4}\chi^3\Big)\frac{\Theta_\tau'(u)^2}{\Theta_\tau(u)^2} - \Big(\frac{\chi\gamma}{2} - \frac{\alpha\gamma}{4}\chi^2 + \frac{\chi^3\gamma}{4}\Big)\frac{\Theta_\tau''(u)}{\Theta_\tau(u)}\\
&\quad + \Big(-\frac{\chi^2\alpha\gamma}{6} - \frac{\chi^3\gamma}{12} + \frac{\chi^2\gamma^2}{12}\Big)\frac{\Theta_\tau'''(0)}{\Theta_\tau'(0)} + \Big(\frac{\pi^2\alpha\gamma\chi^2}{12} - \frac{\pi^2\gamma\chi^3}{12} + \frac{\pi^2\chi^2\gamma^2}{12}\Big).
\end{aligned}$$

Hence $\Delta_1(u,y)$ does not depend on y so that by the definition of $\mathcal{V}_1(u,y)dy$ we have

$$\Xi_1 + \Xi_2 = s\mathcal{W}(q)e^{\pi\chi Pu}\Delta_1(u,y)\int_0^1 \mathcal{T}(u,y)e^{\pi\gamma Py}\mathcal{V}_1(u,y)dy = s\Delta_1(u,y)\psi_\chi^\alpha(u,q). \tag{3.34}$$

To conclude the proof of Theorem 3.5, we compute that

$$
\begin{aligned}
\frac{\Xi_0+\Xi_1+\Xi_2}{\psi^\alpha_\chi(u,q)} &= \chi^2P^2\pi^2 - \Big(\pi^2\chi^2P^2 + \frac{\pi^2\chi\gamma l_\chi}{6} - \frac{\pi^2 l_\chi^2}{3}\Big) + \Big(-\frac{l_\chi^2}{3} + \frac{1}{6}l_\chi\chi^2 + \frac{\chi^2}{3}s\Big)\frac{\Theta_\tau'''(0)}{\Theta_\tau'(0)} - l_\chi(l_\chi+1)\wp(u)\\
&\quad + s\frac{\gamma}{2}\Big(\chi - \frac{\alpha}{2}\chi^2 + \frac{1}{2}\chi^3\Big)\frac{\Theta_\tau'(u)^2}{\Theta_\tau(u)^2} - s\Big(\frac{\chi\gamma}{2} - \frac{\alpha\gamma}{4}\chi^2 + \frac{\chi^3\gamma}{4}\Big)\frac{\Theta_\tau''(u)}{\Theta_\tau(u)}\\
&\quad + s\Big(-\frac{\chi^2\alpha\gamma}{6} - \frac{\chi^3\gamma}{12} + \frac{\chi^2\gamma^2}{12}\Big)\frac{\Theta_\tau'''(0)}{\Theta_\tau'(0)} + s\Big(\frac{\pi^2\alpha\gamma\chi^2}{12} - \frac{\pi^2\gamma\chi^3}{12} + \frac{\pi^2\chi^2\gamma^2}{12}\Big)\\
&= -\frac{l_\chi}{3}(\chi-\frac{\gamma}{2})(\chi-\frac{2}{\gamma})\frac{\Theta_\tau'''(0)}{\Theta_\tau'(0)} + l_\chi(l_\chi+1)\Big(\frac{\Theta_\tau'(u)^2}{\Theta_\tau(u)^2} - \frac{\Theta_\tau''(u)}{\Theta_\tau(u)} + \frac{1}{3}\frac{\Theta_\tau'''(0)}{\Theta_\tau'(0)}\Big) - l_\chi(l_\chi+1)\wp(u)\\
&= 0,
\end{aligned}
$$

where we use (B.4) in the last step.

4. From the BPZ equation to hypergeometric differential equations

In this section, we apply separation of variables to the BPZ equation in Theorem 3.5 to show that, up to a renormalization and change of variable $w=\sin^2(\pi u)$, the coefficients of the q-series expansion of the u-deformed block satisfy the system of hypergeometric differential equations (4.4). These differential equations allow us to access certain analytic properties of the u-deformed block which are beyond the scope of GMC. We conclude the section with a construction of a particular solution to (4.4). Combining these analytic properties, the particular solution, and the OPE expansion in Section 5 will be used to show the existence of shift equations for the probabilistic conformal block in Section 6.

4.1. **Separation of variables for the BPZ equation.**

Recall $\hat\psi^\alpha_\chi(u,q)$ and $\mathfrak{B}$ from Proposition 3.2. Let $\hat\psi^\alpha_{\chi,n}(u)$ be the coefficients of the series expansion

$$
\hat\psi^\alpha_\chi(u,q) = \sum_{n=0}^\infty \hat\psi^\alpha_{\chi,n}(u)q^n \qquad \text{for } u\in\mathfrak{B}. \tag{4.1}
$$

For $\alpha\in(-\frac{4}{\gamma},Q)$, and $\chi\in\{\frac{\gamma}{2},\frac{2}{\gamma}\}$, recall $l_\chi=\frac{\chi^2}{2}-\frac{\alpha\chi}{2}$ from (3.1). To remove the singularities at $u\in\{0,1\}$ in $\hat\psi^\alpha_{\chi,n}(u)$ coming from the $\Theta(u)^{-l_\chi}$ factor in $\hat\psi^\alpha_\chi(u,q)$, we introduce the normalization

$$
\psi^\alpha_{\chi,n}(u) = \sin(\pi u)^{l_\chi}\hat\psi^\alpha_{\chi,n}(u) \qquad \text{for } n\ge 0. \tag{4.2}
$$

We will apply separation of variables to the BPZ equation from Theorem 3.5 to show that $\{\psi^\alpha_{\chi,n}(u)\}_{n\ge0}$ satisfy a hypergeometric system of differential equations after the change of variable $w=\sin^2(\pi u)$. We first clarify the nature of this change of variable by noting the following basic fact.

Lemma 4.1. *The map $u\mapsto\sin^2(\pi u)$ is a conformal map, (i.e. a holomorphic bijection) from $(0,1)\times(0,\infty)$ to $\mathbb{C}\setminus(-\infty,1]$ which maps $\{u:\operatorname{Re}u=1/2,\operatorname{Im}u>0\}$ to $(1,\infty)$.*

Define the domain

$$
D^w := \{w=\sin^2 u : u\in\mathfrak{B}\cap(0,1)\times(0,\infty)\}.
$$

For $w\in D^w$, let

$$
\phi^\alpha_{\chi,n}(w) := \psi^\alpha_{\chi,n}(u) \text{ for } w=\sin^2(\pi u), \quad \text{where } u\in\mathfrak{B}\cap((0,1)\times(0,\infty)).
$$

Define the differential operator

$$
\mathcal{H}_\chi := w(1-w)\partial_{ww} + (1/2-l_\chi-(1-l_\chi)w)\partial_w. \tag{4.3}
$$

Recall for $n\ge1$ the coefficients $\wp_n(u)$ in the q-series expansion of Weierstrass's elliptic function $\wp(u)$ from (B.6) and the polynomials $\widetilde\wp_n(w)$ such that $\widetilde\wp_n(w)=\wp_n(u)$ for $w=\sin^2(\pi u)$. Consider the system of equations on sequences of functions $\{\phi_n(w)\}_{n\ge0}$ given by

$$
\Big(\mathcal{H}_\chi - \Big(\frac{1}{4}l_\chi^2 + \frac{1}{4}\chi^2(P^2+2n)\Big)\Big)\phi_n(w) = \frac{l_\chi(l_\chi+1)}{4\pi^2}\sum_{l=1}^n\widetilde\wp_l(w)\phi_{n-l}(w) \qquad \text{for } n\ge0, \tag{4.4}
$$

where we adopt the convention that the empty summation $\sum_{l=1}^{0}$ is 0 so that (4.4) is homogeneous for $n=0$. The first result Proposition 4.2 of this section shows that separation of variables changes the BPZ equations to the system of equations (4.4).

Proposition 4.2. *The equations* (4.4) *hold for* $\{\phi^\alpha_{\chi,n}(w)\}_{n\geq 0}$ *on* D^w.

Proof. The BPZ equation (3.7) implies that

$$\sum_{n\geq 0}\left[\partial_{uu}\hat{\psi}^\alpha_{\chi,n}(u)-l_\chi(l_\chi+1)\sum_{l=0}^{n}\wp_l(u)\hat{\psi}^\alpha_{\chi,n-l}(u)-2\pi^2\chi^2 n\hat{\psi}^\alpha_{\chi,n}(u)-2\pi^2\chi^2\Big(\frac{P^2}{2}+\frac{1}{6\chi^2}l_\chi(l_\chi+1)\Big)\hat{\psi}^\alpha_{\chi,n}(u)\right]q^n=0.$$

Therefore for each $n=0,1,2..$, we have

$$\left(\partial_{uu}-l_\chi(l_\chi+1)\frac{\pi^2}{\sin^2(\pi u)}-\pi^2\chi^2(P^2+2n)\right)\hat{\psi}^\alpha_{\chi,n}(u)=l_\chi(l_\chi+1)\sum_{l=1}^{n}\wp_l(u)\hat{\psi}^\alpha_{\chi,n-l}(u),$$

where we make the convention that the summation $\sum_{l=1}^{0}$ gives 0. In terms of $\phi^\alpha_{\chi,n}(u)$, this yields

$$\begin{aligned}\Big(l_\chi(l_\chi+1)\pi^2\cos^2(\pi u)\sin(\pi u)^{-l_\chi-2}+l_\chi\pi^2\sin(\pi u)^{-l_\chi}-2l_\chi\pi\cos(\pi u)\sin(\pi u)^{-l_\chi-1}\partial_u\\
+\sin(\pi u)^{-l_\chi}\Big(\partial_{uu}-l_\chi(l_\chi+1)\frac{\pi^2}{\sin^2(\pi u)}-\pi^2\chi^2(P^2+2n)\Big)\Big)\phi^\alpha_{\chi,n}(u)\\
=l_\chi(l_\chi+1)\sum_{l=1}^{n}\wp_l(u)\sin(\pi u)^{-l_\chi}\phi^\alpha_{\chi,n-l}(u).\end{aligned}$$

Multiplying by $\sin(\pi u)^{l_\chi}$ yields

$$\left(\partial_{uu}-2\pi l_\chi\cot(\pi u)\partial_u-\pi^2 l_\chi^2-\pi^2\chi^2(P^2+2n)\right)\phi^\alpha_{\chi,n}(u)=l_\chi(l_\chi+1)\sum_{l=1}^{n}\wp_l(u)\phi^\alpha_{\chi,n-l}(u).\tag{4.5}$$

Notice that $2\pi\sqrt{w(1-w)}\partial_w=\partial_u$, hence for $n\geq 0$, we have

$$\begin{aligned}&\partial_{uu}-2\pi l_\chi\cot(\pi u)\partial_u-\pi^2 l_\chi^2-\pi^2\chi^2(P^2+2n)\\
&=4\pi^2 w(1-w)\partial_{ww}+2\pi^2(1-2w)\partial_w-4\pi^2 l_\chi(1-w)\partial_w-\pi^2 l_\chi^2-\pi^2\chi^2(P^2+2n)\\
&=4\pi^2\Big(\mathcal{H}_\chi-\Big(\frac{l_\chi^2}{4}+\frac{\chi^2}{4}(P^2+2n)\Big)\Big).\end{aligned}$$

This implies that the equations (4.4) hold for $\{\phi^\alpha_{\chi,n}(w)\}_{n\geq 0}$ on D^w. □

For $n\in\mathbb{N}_0$, equation (4.4) is an inhomogeneous *hypergeometric differential equation* with parameters $(A_{\chi,n},B_{\chi,n},C_\chi)$ defined by

$$A_{\chi,n}=-\frac{l_\chi}{2}+\mathbf{i}\frac{\chi}{2}\sqrt{P^2+2n},\qquad B_{\chi,n}=-\frac{l_\chi}{2}-\mathbf{i}\frac{\chi}{2}\sqrt{P^2+2n},\qquad C_\chi=\frac{1}{2}-l_\chi.\tag{4.6}$$

We summarize some basic facts about these equations in Appendix D for the reader's convenience. In particular, since $C_\chi-A_{\chi,n}-B_{\chi,n}=\frac{1}{2}\in(0,1)$, assumption (D.7) holds, hence results in Appendix D.2 apply here.

4.2. **Analytic properties of the deformed block.** We now use Proposition 4.2 to understand certain analytic properties of the deformed block which are difficult to access from the GMC perspective. Importantly, we can extend the domain of definition for $\phi^\alpha_{\chi,n}(w)$ as follows. For $i=1,2$, define the domain

$$D_i^w:=\left\{w=\sin^2(\pi u):u\in\mathfrak{B}\cap\Big((\frac{i-1}{2},\frac{i}{2})\times(0,\infty)\Big)\right\}.$$

Recall that $\mathbb{D}=\{w:|w|<1\}$ is the unit disk. Note that D_1^w (resp, D_2^w) lies in the upper (resp., lower) half plane and that $D_i^w\cap\mathbb{D}\neq\emptyset$.

Definition 4.3 (Property (R))**.** A function f on the closed unit disk $\overline{\mathbb{D}}$ satisfies Property (R) if f is of the form $f(w)=\sum_{n=0}^{\infty}a_n w^n$ for $|w|\leq 1$ with $\sum_{n=0}^{\infty}|a_n|<\infty$.

As surveyed in Appendix D.1, for each $n \in \mathbb{N}_0$, equation (4.4) has a 2-dimensional affine space of solutions given by adding to any particular solution the span of the Gauss hypergeometric functions $v^\alpha_{1,\chi,n}(w)$ and $w^{1-C_\chi} v^\alpha_{2,\chi,n}(w)$ satisfying Property (R), defined by

$$v^\alpha_{1,\chi,n}(w) := {}_2F_1(A_{\chi,n}, B_{\chi,n}, C_\chi, w), \tag{4.7}$$

$$v^\alpha_{2,\chi,n}(w) := {}_2F_1(1 + A_{\chi,n} - C_\chi, 1 + B_{\chi,n} - C_\chi, 2 - C_\chi, w), \tag{4.8}$$

with $A_{\chi,n}$, $B_{\chi,n}$, and C_χ from (4.6).

Corollary 4.4. *Suppose C_χ is not an integer. For $i \in \{1,2\}$, there exist functions $\{\phi^{\alpha,1}_{\chi,n,i}(w)\}_{n\geq 0}$ and $\{\phi^{\alpha,2}_{\chi,n,i}(w)\}_{n\geq 0}$ on the closed unit disk $\overline{\mathbb{D}}$ satisfying Property (R) such that $\phi^{\alpha,1}_{\chi,n,i}(w)$ and $w^{1-C_\chi}\phi^{\alpha,2}_{\chi,n,i}(w)$ are solutions to equations* (4.4) *and we have*

$$\phi^\alpha_{\chi,n}(w) = \phi^{\alpha,1}_{\chi,n,i}(w) + w^{1-C_\chi}\phi^{\alpha,2}_{\chi,n,i}(w) \quad \text{on } D^w_i \cap \mathbb{D},$$

where $w^{1-C_\chi} := e^{(1-C_\chi)\log w}$ with $\arg w \in (-\pi, \pi)$ so that w^{1-C_χ} has a discontinuity on $(-\infty, 0]$.

Proof. Fix $i \in \{1,2\}$. Choose $w_1 \neq w_2 \in D^w_i \cap \mathbb{D}$ such that

$$v^\alpha_{1,\chi,n}(w_1) w_2^{1-C_\chi} v^\alpha_{2,\chi,n}(w_2) - w_1^{1-C_\chi} v^\alpha_{2,\chi,n}(w_1) v^\alpha_{1,\chi,n}(w_2) \neq 0. \tag{4.9}$$

Now $\phi^\alpha_{\chi,0,i}$ is linear combination of $v^\alpha_{i,\chi,n}(w_1)$ and $w^{1-C_\chi}v^\alpha_{2,\chi,n}(w)$ uniquely specified by the values of $\phi^\alpha_{\chi,0,i}(w_1)$ and $\phi^\alpha_{\chi,0,i}(w_2)$. This gives the existence and uniqueness of $\phi^{\alpha,1}_{\chi,0,i}$ and $\phi^{\alpha,2}_{\chi,0,i}$. Note that $\widetilde{\wp}_l(w)$ are polynomials and hence entire functions for all $l \in \mathbb{N}$. The existence and uniqueness of $\phi^{\alpha,1}_{\chi,1,i}$ and $\phi^{\alpha,2}_{\chi,1,i}$ follows from Lemma D.5. Furthermore, the result for general n follows from inductively applying Lemma D.5. $\square$

Let $\mathbb{D}_1 := \{w : |w| < 1 \text{ and } \operatorname{Im} w > 0\}$ and $\mathbb{D}_2 := \{w : |w| < 1 \text{ and } \operatorname{Im} w < 0\}$. For $i = 1, 2$, define

$$\phi^\alpha_{\chi,n,i}(w) := \phi^{\alpha,1}_{\chi,n,i}(w) + w^{1-C_\chi}\phi^{\alpha,2}_{\chi,n,i}(w) \quad \text{on } \overline{\mathbb{D}}_i, \tag{4.10}$$

with w^{1-C_χ} defined in the same way as in Corollary 4.4. Then $\phi^\alpha_{\chi,n,i}$ is the analytic extension of $\phi^\alpha_{\chi,n}$ from $D^w_i \cap \mathbb{D}$ to $\overline{\mathbb{D}}_i$. By Lemma 4.1, $\phi^\alpha_{\chi,n}$ has a discontinuity on $[0,1)$, hence $\phi^\alpha_{\chi,n,1}$ and $\phi^\alpha_{\chi,n,2}$ do not agree on $[0,1)$. However, they are linked by the linear relations given in the next two lemmas.

Lemma 4.5. *Note that $0 \in [-1,0] \cap \overline{D^w_1} = [-1,0] \cap \overline{D^w_2}$. For each $n \in \mathbb{N}_0$ we have*

$$\phi^\alpha_{\chi,n,1}(1) = \phi^\alpha_{\chi,n,2}(1) \text{ and } \phi^\alpha_{\chi,n,2}(w) = e^{\pi\chi P - \pi \mathbf{i} l_\chi}\phi^\alpha_{\chi,n,1}(w) \text{ for } w \in [-1,0] \cap \overline{D^w_1}. \tag{4.11}$$

Proof. Define $f(w) := \phi^\alpha_{\chi,0}(1-w)$. By the symmetry of the hypergeometric equation under the change of variables $w \mapsto 1-w$, f solves the hypergeometric equation for parameters $(A,B,C) = (A_{\chi,0}, B_{\chi,0}, 1 + A_{\chi,0} + B_{\chi,0} - C_\chi)$. Since $C_\chi - A_{\chi,0} - B_{\chi,0} = \frac{1}{2}$, applying Lemma D.8 with $U := \{z \in \mathbb{D} : 1 - z \in D^w_i\}$, and $D := \mathbb{D} \setminus [0,1]$, we see that as $1 - w \in D^w$ tends to 1 so that w tends to 0, $\phi^\alpha_{\chi,0}(1-w)$ tends to a finite number, which we denote by $\phi^\alpha_{\chi,0}(1)$. Inductively applying Lemmas D.7 and D.8, we can define $\phi^\alpha_{\chi,n}(1)$ as the limit of $\phi^\alpha_{\chi,n}(1-w)$ as $w \in U$ tends to 0. On the other hand, we have that $\phi^\alpha_{\chi,n}(w) = \phi^\alpha_{\chi,n,1}(w)$ tends to $\phi^\alpha_{\chi,n,1}(1)$ as $w \to 1$ within $D^w_1 \cap \mathbb{D}$. Therefore $\phi^\alpha_{\chi,n,1}(1) = \phi^\alpha_{\chi,n}(1)$. Similarly $\phi^\alpha_{\chi,n,2}(1) = \phi^\alpha_{\chi,n}(1)$, hence $\phi^\alpha_{\chi,n,1}(1) = \phi^\alpha_{\chi,n,2}(1)$ as desired.

For the second identity, we first prove that

$$\psi^\alpha_{\chi,n}(u+1) = e^{\pi\chi P - \pi \mathbf{i} l_\chi}\psi^\alpha_{\chi,n}(u) \quad \text{for} \quad u \in \mathfrak{B}. \tag{4.12}$$

Since $\frac{\gamma\chi}{2}(-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}) = l_\chi$, Lemma B.7 implies that $\mathbb{E}\left[f_\nu(u+1)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right] = e^{-\pi \mathbf{i} l_\chi}\mathbb{E}\left[f_\nu(u)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right]$. Because of the $e^{\pi\chi P u}$ factor in $\hat{\psi}(u,\alpha)$, we have $\psi^\alpha_{\chi,n}(u+1) = e^{\pi\chi P - \pi \mathbf{i} l_\chi}\psi^\alpha_{\chi,n}(u)$. Now note that $\phi^\alpha_{\chi,n,1}(\sin^2(\pi \mathbf{i} t)) = \psi^\alpha_{\chi,n}(\mathbf{i}t)$ and $\phi^\alpha_{\chi,n,2}(\sin^2(\pi(1+\mathbf{i}t))) = \psi^\alpha_{\chi,n}(1+\mathbf{i}t)$ with $t > 0$ such that $\mathbf{i}t \in \mathfrak{B}$. By (4.12), we have $\psi^\alpha_{\chi,n}(\mathbf{i}t + 1) = e^{\pi\chi P - \pi \mathbf{i} l_\chi}\psi^\alpha_{\chi,n}(\mathbf{i}t)$, hence $\phi^\alpha_{\chi,n,2}(w) = e^{\pi\chi P - \pi \mathbf{i} l_\chi}\phi^\alpha_{\chi,n,1}(w)$ for $w \in [-1,0) \cap \overline{D^w_1}$. Taking the limit $w \to 0$ yields $\phi^\alpha_{\chi,n,2}(0) = e^{\pi\chi P - \pi \mathbf{i} l_\chi}\phi^\alpha_{\chi,n,1}(0)$. $\square$

Lemma 4.6. *In the setting of Corollary 4.4, for $w \in \overline{\mathbb{D}}$, we have that*

$$\phi^{\alpha,1}_{\chi,n,2}(w) = e^{\pi\chi P - \mathbf{i}\pi l_\chi}\phi^{\alpha,1}_{\chi,n,1}(w) \qquad \text{and} \qquad \phi^{\alpha,2}_{\chi,n,2}(w) = -e^{\pi\chi P + \mathbf{i}\pi l_\chi}\phi^{\alpha,2}_{\chi,n,1}(w). \tag{4.13}$$

Proof. By Lemma 4.5, we have $\phi^{\alpha,1}_{\chi,n,2}(0) = e^{\pi\chi P - \mathbf{i}\pi l_\chi}\phi^{\alpha,1}_{\chi,n,1}(0)$. Set $\phi_n = \phi^{\alpha,1}_{\chi,n,2} - e^{\pi\chi P - \mathbf{i}\pi l_\chi}\phi^{\alpha,1}_{\chi,n,1}$. Then ϕ_0 is a solution to the $n = 0$ case of (4.4) which satisfies Property (R) and has the value $\phi_0(0) = 0$. By the structure of the solution space of the hypergeometric equation, we must have $\phi_0 \equiv 0$. Since ϕ_1 is a solution to (4.4) with $n = 1$, we similarly get $\phi_1 \equiv 0$. Continuing via induction on n, we get $\phi_n \equiv 0$, hence $\phi^{\alpha,1}_{\chi,n,2}(w) = e^{\pi\chi P - \mathbf{i}\pi l_\chi}\phi^{\alpha,1}_{\chi,n,1}(w)$ for all n.

We choose $c > 0$ small enough such that $[-c, 0] \subset \overline{D^w_1}$. By Lemma 4.5, for $w \in [-c, 0]$ we have

$$\phi^{\alpha}_{\chi,n,2}(w) - \phi^{\alpha,1}_{\chi,n,2}(w) = e^{\pi\chi P - \mathbf{i}\pi l_\chi}(\phi^{\alpha}_{\chi,n,1}(w) - \phi^{\alpha,1}_{\chi,n,1}(w)).$$

On the other hand, by (4.10), since w^{1-C_χ} has branch cut at $(-\infty, 0)$, we have on $(-c, 0)$ that

$$\phi^{\alpha}_{\chi,n,1}(w) = \phi^{\alpha,1}_{\chi,n,1}(w) + e^{\pi(1-C_\chi)\mathbf{i}}|w|^{1-C_\chi}\phi^{\alpha,2}_{\chi,n,1}(w);$$
$$\phi^{\alpha}_{\chi,n,2}(w) = \phi^{\alpha,1}_{\chi,n,2}(w) + e^{-\pi(1-C_\chi)\mathbf{i}}|w|^{1-C_\chi}\phi^{\alpha,2}_{\chi,n,2}(w).$$

Putting these together, we have $\phi^{\alpha,2}_{\chi,n,2}(w) = e^{\pi\chi P - \mathbf{i}\pi l_\chi}e^{2(1-C_\chi)\pi\mathbf{i}}\phi^{\alpha,2}_{\chi,n,1}(w) = -e^{\pi\chi P + \mathbf{i}\pi l_\chi}\phi^{\alpha,2}_{\chi,n,1}(w)$ on $(-c, 0)$. Therefore, $\phi^{\alpha,2}_{\chi,n,2} = -e^{\pi\chi P + \mathbf{i}\pi l_\chi}\phi^{\alpha,2}_{\chi,n,1}$ on $\overline{\mathbb{D}}$ by their analyticity. □

By Lemma 3.6, there exists an open set in $\mathbb{C}^2$ containing $\{(\alpha, w) : \alpha \in (-\frac{4}{\gamma} + \chi, Q), w \in D^w\}$ on which $\phi^{\alpha}_{\chi,n}(w, q)$ has an analytic continuation. Proposition 4.2 allows us to extend the (α, w)-analyticity beyond this domain via the following lemma, which is used in Section 5.

Lemma 4.7. *In the setting of Corollary 4.4, the quantity $\phi^{\alpha,j}_{\chi,n,i}(w)$ is analytic in α on an open complex neighborhood of $\{\alpha : \alpha \in (-\frac{4}{\gamma} + \chi, Q)$ and $C_\chi \notin \mathbb{Z}\}$, for $i, j = 1, 2$, and $w \in \overline{\mathbb{D}}$.*

We prove Lemma 4.7 in Section 4.3 as a consequence of a more general Lemma 4.11. In Corollary 5.2, we extend the α-analyticity further to a complex neighborhood of $(-\frac{4}{\gamma} + \chi, 2Q - \chi)$ using OPE techniques.

4.3. **Construction of a particular solution.** We now construct a particular solution to (4.4) which will be used in the proof of Lemma 4.7 and Theorem 6.1. For $i = 1, 2$ and $n \geq 1$, recall $\mathbb{D}_i$ from (4.16) and let

$$g^{\alpha,j}_{\chi,n,i}(w) = \frac{l_\chi(l_\chi + 1)}{4\pi^2}\sum_{l=1}^{n}\widetilde{\wp}_l(w)\phi^{\alpha,j}_{\chi,n-l,i}(w) \quad \text{for } w \in \overline{\mathbb{D}}_i. \tag{4.14}$$

Then $g^{\alpha,j}_{\chi,n,i}(w)$ satisfies Property (R) by Corollary 4.4. By Lemma D.5, we can define the following functions, which will be particular solutions to (4.4).

Definition 4.8. For $n \geq 1$, and $i = 1, 2$, let $G^{\alpha,1}_{\chi,n,i}(w)$ be the unique function satisfying Property (R) such that

$$\left(\mathcal{H}_\chi - \left(\frac{1}{4}l_\chi^2 + \frac{1}{4}\chi^2(P^2 + 2n)\right)\right)G^{\alpha,1}_{\chi,n,i}(w) = g^{\alpha,1}_{\chi,n,i}(w)$$

and $G^{\alpha,2}_{\chi,n,i}(1) = 0$. Let $G^{\alpha,2}_{\chi,n,i}(w)$ be the unique function satisfying Property (R) such that

$$\left(\mathcal{H}_\chi - \left(\frac{1}{4}l_\chi^2 + \frac{1}{4}\chi^2(P^2 + 2n)\right)\right)w^{1-C_\chi}G^{\alpha,2}_{\chi,n,i}(w) = w^{1-C_\chi}g^{\alpha,2}_{\chi,n,i}(w)$$

and $G^{\alpha,2}_{\chi,n,i}(1) = 0$. Define $G^{\alpha}_{\chi,0,i}(w) = G^{\alpha,1}_{\chi,0,i}(w) = G^{\alpha,2}_{\chi,0,i}(w) = 0$, and let

$$G^{\alpha}_{\chi,n,i}(w) := G^{\alpha,1}_{\chi,n,i}(w) + w^{1-C_\chi}G^{\alpha,2}_{\chi,n,i}(w) \qquad \text{for } w \in \overline{\mathbb{D}}_i \text{ and } n \in \mathbb{N}_0.$$

Proposition 4.9. *For each $n \geq 0$, the function $G^{\alpha}_{\chi,n,i}(w)$ is a solution to (4.4). Moreover, they satisfy*

$$G^{\alpha}_{\chi,n,1}(1) = G^{\alpha}_{\chi,n,2}(1) = 0, \qquad G^{\alpha,1}_{\chi,n,2}(0) = e^{\pi\chi P - \mathbf{i}\pi l_\chi}G^{\alpha,1}_{\chi,n,1}(0), \qquad G^{\alpha,2}_{\chi,n,2}(0) = -e^{\pi\chi P + \mathbf{i}\pi l_\chi}G^{\alpha,2}_{\chi,n,1}(0).$$

Proof. By Definition 4.8, $G^{\alpha}_{\chi,n,i}(w)$ is a solution to equation (4.4) and $G^{\alpha}_{\chi,n,1}(1) = G^{\alpha}_{\chi,n,2}(1) = 0$. By Lemma 4.6, we have $g^{\alpha,1}_{\chi,n,2}(w) = e^{\pi\chi P - \mathbf{i}\pi l_\chi}g^{\alpha,1}_{\chi,n,1}(w)$ and $g^{\alpha,2}_{\chi,n,2}(w) = -e^{\pi\chi P + \mathbf{i}\pi l_\chi}g^{\alpha,2}_{\chi,n,1}(w)$, which implies that $G^{\alpha,1}_{\chi,n,2}(0) = e^{\pi\chi P - \mathbf{i}\pi l_\chi}G^{\alpha,1}_{\chi,n,1}(0)$ and $G^{\alpha,2}_{\chi,n,2}(0) = -e^{\pi\chi P + \mathbf{i}\pi l_\chi}G^{\alpha,2}_{\chi,n,1}(0)$. □

Now we state a generalization of Lemma 4.7, which will use the following generalization of Property (R).

Definition 4.10. Suppose $U \subset \mathbb{C}$ is an open set. We say that a function $g(w,\alpha)$ is (w,α)-regular on $\overline{\mathbb{D}} \times U$ if g can be written as $g(w,\alpha) = \sum_{n=0}^{\infty} a_n(\alpha) w^n$ satisfying two properties: (1) $a_n(\alpha)$ are analytic functions on U; and (2) $\sum_{n=0}^{\infty} |a_n(\alpha)| < \infty$ where the convergence holds uniformly on each compact subset of U.

We will repeatedly use two key facts about (w,α)-regularity. Firstly, if $f(\alpha)$ is analytic on a domain $U \subset \mathbb{C}$ and $g(w)$ satisfies Property (R) on $\overline{\mathbb{D}}$, then $f(\alpha)g(w)$ is (w,α)-regular on $\overline{\mathbb{D}} \times U$. Moreover, (w,α)-regularity is preserved by the solution to hypergeometric differential equation as stated in Lemma D.6.

Lemma 4.11. *For each $i,j \in \{1,2\}$ and $n \in \mathbb{N}_0$, there exists an open complex neighborhood $U = U_{i,j,n}$ of $\{(w,\alpha) : \alpha \in (-\frac{4}{\gamma} + \chi, Q)$ and $C_\chi \notin \mathbb{Z},\ w \in \mathbb{D}\}$ such that the functions $(w,\alpha) \mapsto G^{\alpha,j}_{\chi,n,i}(w)$ and $(w,\alpha) \mapsto \phi^{\alpha,j}_{\chi,n,i}(w)$ have extensions to $\overline{\mathbb{D}} \times U$ which are (w,α)-regular in the sense of Definition 4.10.*

Proof. Fix $i \in \{1,2\}$. Note that $G^{\alpha,j}_{\chi,0,i}(w) = 0$ trivially satisfies Lemma 4.11 and that the conclusion is vacuous for $\phi^{\alpha,j}_{\chi,0,i}(w)$. We now prove the statement for $\phi^{\alpha,j}_{\chi,n,i}$ and $G^{\alpha,j}_{\chi,n+1,i}$ by induction on n. Assume that for some $n \in \mathbb{N}_0$, the statement holds for $G^{\alpha,j}_{\chi,n,i}$ and $\phi^{\alpha,j}_{\chi,m,i}$ for each $m < n$.

Because any solution to an inhomogeneous hypergeometric equation is the sum of a particular solution and a solution to the homogeneous equation, by Definition 4.8 and Proposition 4.9, we may write

$$\phi^{\alpha}_{\chi,n,i}(w) = G^{\alpha}_{\chi,n,i}(w) + X^1_{\chi,n,i}(\alpha) v^{\alpha}_{1,\chi,n}(w) + X^2_{\chi,n,i}(\alpha) w^{1-C_\chi} v^{\alpha}_{2,\chi,n}(w), \tag{4.15}$$

for $X^1_{\chi,n,i}(\alpha)$ and $X^2_{\chi,n,i}(\alpha)$ independent of w. Therefore for $j = 1,2$, we have

$$\phi^{\alpha,j}_{\chi,n,i}(w) = G^{\alpha,j}_{\chi,n,i}(w) + X^j_{\chi,n,i}(\alpha) v^{\alpha}_{j,\chi,n}(w). \tag{4.16}$$

For $i = 1,2$ and $w \in D^w_i \cap \mathbb{D}$, recall from Corollary 4.4 that $\phi^{\alpha,1}_{\chi,n,i}(w) + w^{1-C_\chi}\phi^{\alpha,2}_{\chi,n,i}(w) = \phi^{\alpha}_{\chi,n,i}(w)$, which is analytic in α on a complex neighborhood of $(-\frac{4}{\gamma} + \chi, Q)$ by Lemma 3.6. Due to the analyticity of $G^{\alpha,j}_{\chi,n,i}(w)$ in α by induction hypothesis, we see that $F^\alpha(w) := X^1_{\chi,n,i}(\alpha) v^{\alpha}_{1,\chi,n}(w) + X^2_{\chi,n,i}(\alpha) w^{1-C_\chi} v^{\alpha}_{2,\chi,n}(w)$ is analytic in α on a complex neighborhood of $\{\alpha : \alpha \in (-\frac{4}{\gamma} + \chi, Q)$ and $C_\chi \notin \mathbb{Z}\}$ for $w \in D^w_i \cap \mathbb{D}$.

For each $\alpha_0 \in \mathbb{C}$, there exist $w_1, w_2 \in D^w_i \cap \mathbb{D}$ such that equation (4.9) holds with $\alpha = \alpha_0$ and $v^{\alpha}_{j,\chi,n}(w_k)$ is analytic at α_0 for $1 \le j,k \le 2$. Expressing $X^1_{\chi,n,i}(\alpha)$ and $X^2_{\chi,n,i}(\alpha)$ in terms of $F^\alpha(w_k)$ and $v^{\alpha}_{j,\chi,n}(w_k)$ with $1 \le j,k \le 2$ gives that $X^1_{\chi,n,i}(\alpha)$ and $X^2_{\chi,n,i}(\alpha)$ are analytic in α on a complex neighborhood U of $\{\alpha : \alpha \in (-\frac{4}{\gamma} + \chi, Q)$ and $C_\chi \notin \mathbb{Z}\}$. By Lemmas D.2 and D.3 on the regularity of $v^{\alpha}_{j,\chi,n}(\alpha)$, equation (4.16) yields that $\phi^{\alpha,j}_{\chi,n,i}$ is (w,α)-regular on $\overline{\mathbb{D}} \times U$.

Recall $g^{\alpha,j}_{\chi,n,i}(w)$ from (4.14). By the induction hypothesis, we see that $g^{\alpha,j}_{\chi,n,i}(w)$ is (w,α)-regular on $\overline{\mathbb{D}} \times U$. By Lemmas D.5 and D.6, we see that $G^{\alpha,j}_{\chi,n+1,i}$ is (w,α)-regular on $\overline{\mathbb{D}} \times U$. This concludes our induction. ☐

5. Operator product expansions for conformal blocks

In this section, we prove Theorem 5.1 which characterizes the q-series coefficients $\{\mathcal{A}_{\gamma,P,n}(\alpha)\}_{n\in\mathbb{N}_0}$ of $\mathcal{A}^q_{\gamma,P}(\alpha)$ from (2.11). First, we show that $\mathcal{A}_{\gamma,P,n}(\alpha)$ may be analytically continued in α to a complex neighborhood of $(-\frac{4}{\gamma}, 2Q)$. Second, we relate these functions to so-called *operator product expansions* of the (normalized) deformed conformal blocks $\phi^\alpha_\chi(u,q) := \sin(\pi u)^{l_\chi}\psi^\alpha_\chi(u,\tau)$, giving linear relations between the values of $\phi^{\alpha,j}_{\chi,n,j}(0)$ and $\mathcal{A}_{\gamma,P,n}(\alpha \pm \chi)$ for different values of n.

Fix $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$ and recall $l_\chi = \frac{\chi^2}{2} - \frac{\alpha\chi}{2}$ from (3.1). Define the functions

$$W^-_\chi(\alpha,\gamma) := \pi^{l_\chi}(2\pi e^{\mathbf{i}\pi})^{-\frac{1}{3}\left(2 + \frac{2\gamma l_\chi}{\chi} + \frac{4l_\chi}{\chi\gamma} + \frac{6l_\chi^2}{\chi^2}\right)}; \tag{5.1}$$

$$W^+_\chi(\alpha,\gamma) := -e^{2\mathbf{i}\pi l_\chi - 2\mathbf{i}\pi\chi^2}(2\pi e^{\mathbf{i}\pi})^{-\frac{1}{3}\left(\frac{\gamma l_\chi}{\chi} + \frac{2l_\chi}{\chi^2} - 8l_\chi + \frac{6l_\chi^2}{\chi^2}\right)}\pi^{-l_\chi - 1}\frac{1 - e^{2\pi\chi P - 2\mathbf{i}\pi l_\chi}}{\chi(Q-\alpha)}\left(\frac{4}{\gamma^2}\right)^{\mathbf{1}_{\chi=\frac{2}{\gamma}}} \frac{\Gamma(\frac{\alpha\chi}{2} - \frac{\chi^2}{2} + \frac{2\chi}{\gamma})\Gamma(1-\alpha\chi)\Gamma(\alpha\chi - \chi^2)}{\Gamma(\frac{\alpha\chi}{2} - \frac{\chi^2}{2})\Gamma(1 - \frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}} \tag{5.2}$$

and the quantities $\eta^{\pm}_{\chi,n}(\alpha)$ as coefficients of the q-series expansions

$$\Theta'_\tau(0)^{\frac{4}{3}\frac{l_\chi(l_\chi+1)}{\chi^2}+\frac{2}{3}l_\chi+\frac{2}{3}} = q^{\frac{1}{3}\frac{l_\chi(l_\chi+1)}{\chi^2}+\frac{1}{6}l_\chi+\frac{1}{6}}\sum_{n=0}^{\infty}\eta^{-}_{\chi,n}(\alpha)q^n; \tag{5.3}$$

$$\Theta'_\tau(0)^{\frac{4}{3}\frac{l_\chi(l_\chi+1)}{\chi^2}-\frac{2}{3}l_\chi} = q^{\frac{1}{3}\frac{l_\chi(l_\chi+1)}{\chi^2}-\frac{1}{6}l_\chi}\sum_{n=0}^{\infty}\eta^{+}_{\chi,n}(\alpha)q^n. \tag{5.4}$$

In terms of these quantities, we are ready to state Theorem 5.1, the main goal of this section. Moreover, Theorem 5.1 allows us to strengthen Lemma 4.11 into Corollary 5.2 giving analytic extensions of the functions $G^{\alpha,j}_{\chi,n,i}(w)$ and $\phi^{\alpha,j}_{\chi,n,i}(w)$.

Theorem 5.1. *For each $n\in\mathbb{N}_0$, the function $\mathcal{A}_{\gamma,P,n}(\alpha)$ can be analytically extended to a complex neighborhood of $(-\frac{4}{\gamma},2Q)$. Recalling $\phi^{\alpha,j}_{\chi,n,i}$ and C_χ from Corollary 4.4, for $\alpha\in(\chi,Q)$ and $C_\chi\notin\mathbb{Z}$ we have*

$$\phi^{\alpha,1}_{\chi,n,1}(0) = W^{-}_{\chi}(\alpha,\gamma)\left[\eta^{-}_{\chi,0}(\alpha)\mathcal{A}_{\gamma,P,n}(\alpha-\chi)+\sum_{m=0}^{n-1}\eta^{-}_{\chi,n-m}(\alpha)\mathcal{A}_{\gamma,P,m}(\alpha-\chi)\right]; \tag{5.5}$$

$$\phi^{\alpha,2}_{\chi,n,1}(0) = W^{+}_{\chi}(\alpha,\gamma)\left[\eta^{+}_{\chi,0}(\alpha)\mathcal{A}_{\gamma,P,n}(\alpha+\chi)+\sum_{m=0}^{n-1}\eta^{+}_{\chi,n-m}(\alpha)\mathcal{A}_{\gamma,P,m}(\alpha+\chi)\right], \tag{5.6}$$

where we interpret $\mathcal{A}_{\gamma,P,n}(\alpha)$ via the analytic extension above and use the convention that the value of the empty summation $\sum_{m=0}^{-1}$ is 0.

Corollary 5.2. *Fix $\gamma\in(0,2)$ and $\chi\in\{\frac{\gamma}{2},\frac{2}{\gamma}\}$. There exists a complex neighborhood V of $(-\frac{4}{\gamma}+\chi,2Q-\chi)$ such that $G^{\alpha,j}_{\chi,n,i}(w)$ and $\phi^{\alpha,j}_{\chi,n,i}(w)$ in Lemma 4.11 admit an extension on $\overline{\mathbb{D}}\times V$ which is (w,α)-regular (see Definition 4.10), for $i,j=1,2$ and $n\in\mathbb{N}_0$.*

Proof. By Theorem 5.1, there exists an open complex neighborhood V of $(-\frac{4}{\gamma}+\chi,2Q-\chi)$ on which $\phi^{\alpha,1}_{\chi,n,1}(0)$ and $\phi^{\alpha,2}_{\chi,n,1}(0)$ admit analytic extension in α. By Lemma 4.6, the same holds for $\phi^{\alpha,2}_{\chi,n,1}(0)$ and $\phi^{\alpha,2}_{\chi,n,2}(0)$.

Fix $i\in\{1,2\}$. Setting $w=0$ in (4.16) yields

$$X^{j}_{\chi,n,i}(\alpha) = \phi^{\alpha,j}_{\chi,n,i}(0) - G^{\alpha,j}_{\chi,n,i}(0). \tag{5.7}$$

For $n=0$, equation (5.7) implies that $X^{j}_{\chi,n,i}(\alpha)$ admits an analytic extension to V, hence $\phi^{\alpha,j}_{\chi,0,i}$ is (w,α)-regular on $\overline{\mathbb{D}}\times V$ by (4.15). Now by Lemma D.6 we see that $G^{\alpha,j}_{\chi,1,i}$ admits a (w,α)-regular extension on $\overline{\mathbb{D}}\times V$. This further implies that $X^{j}_{\chi,1,i}(\alpha)$ admits an analytic extension to V, hence $\phi^{\alpha,j}_{\chi,1,i}$ and $G^{\alpha,j}_{\chi,2,i}$ admit (w,α)-regular extension on $\overline{\mathbb{D}}\times V$. Now by induction in n we get that $\phi^{\alpha,j}_{\chi,n,i}$ and $G^{\alpha,j}_{\chi,n,i}$ admit (w,α)-regular extension on $\overline{\mathbb{D}}\times V$. □

The proof of Theorem 5.1 relies on the operator product expansion (OPE) for u-deformed conformal block. In Section 5.1, we state the OPE results Lemmas 5.3 and 5.4. Lemma 5.4 is itself a consequence of the asymptotic expansions given by Lemmas 5.5 and 5.6, whose proofs are deferred to Appendix E. We then complete the proof of Theorem 5.1 in Section 5.2. Theorem 5.1 and Corollary 5.2 will be the only ingredient from this section used in the proof of Theorem 2.13 in Section 6.

5.1. **Operator product expansion.** Throughout this section we assume that $\tau\in\mathbf{i}\mathbb{R}$ so that $q\in(0,1)$. Define the renormalized deformed block

$$\phi^{\alpha}_{\chi}(u,q) := \sin(\pi u)^{l_\chi}\psi^{\alpha}_{\chi}(u,\tau). \tag{5.8}$$

This section provides operator product expansions (OPEs) for $\phi^{\alpha}_{\chi}(u,q)$. Mathematically, these OPEs characterize the asymptotic behavior of $\phi^{\alpha}_{\chi}(u,q)$ as u tends to 0, which will differ based on the values of α and χ. The precise result will be stated in Lemmas 5.3 and 5.4 in terms of the function $\mathcal{A}^{q}_{\gamma,P}(\alpha)$ from (2.11) and an extension of this function to α in a complex neighborhood of $(Q,2Q)$ given in (5.14). This extension will be given in terms of the reflection coefficient of boundary Liouville CFT which we will introduce in (5.12). In the rest of this section, we introduce the reflection coefficient, use it to extend $\mathcal{A}^{q}_{\gamma,P}(\alpha)$, and then state and sketch the proofs of the OPEs in Lemmas 5.3 and 5.4, deferring some technical lemmas to Appendix E.

As introduced in Appendix C, let $Z_{\mathbb{H}}$ be a centered Gaussian process defined on the upper-half plane $\mathbb{H}$ with covariance given by

$$\mathbb{E}[Z_{\mathbb{H}}(x)Z_{\mathbb{H}}(y)] = 2\log\frac{|x|\vee|y|}{|x-y|} \quad \text{for } x,y\in\mathbb{H}. \tag{5.9}$$

For $\lambda>0$ consider the process

$$\mathcal{B}_s^\lambda := \begin{cases}\hat{B}_s-\lambda s & s\geq 0\\ \bar{B}_{-s}+\lambda s & s<0,\end{cases} \tag{5.10}$$

where $(\hat{B}_s-\lambda s)_{s\geq 0}$ and $(\bar{B}_s-\lambda s)_{s\geq 0}$ are two independent Brownian motions with negative drift conditioned to stay negative. Consider an independent coupling of $(\mathcal{B}^\lambda, Z_{\mathbb{H}})$ with $\lambda=\frac{Q-\alpha}{2}$, and let

$$\rho(\alpha,1,e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P}) := \frac{1}{2}\int_{-\infty}^{\infty} e^{\frac{\gamma}{2}\mathcal{B}_v^{\frac{Q-\alpha}{2}}}\left(e^{\frac{\gamma}{2}Z_{\mathbb{H}}(-e^{-v/2})}+e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P}e^{\frac{\gamma}{2}Z_{\mathbb{H}}(e^{-v/2})}\right)dv. \tag{5.11}$$

Then the function

$$\overline{R}(\alpha,1,e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P}) := \mathbb{E}\left[\left(\rho(\alpha,1,e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P})\right)^{\frac{2}{\gamma}(Q-\alpha)}\right] \tag{5.12}$$

is the *reflection coefficient* for boundary Liouville CFT, also known as the boundary two-point function. It was introduced in its most general form and computed in [RZ20]. An analogous function first appeared in the case of the Riemann sphere in [KRV19a] and a special case of $\overline{R}$ was computed in [RZ18]. This reflection coefficient is important because it appears in the first order asymptotics of the probability for a one-dimensional GMC measure to be large. This is also why it is natural for this function to appear in the OPE expansions. In [RZ20], the reflection coefficient was computed explicitly as

$$\overline{R}(\alpha,1,e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P}) = \frac{(2\pi)^{\frac{2}{\gamma}(Q-\alpha)-\frac{1}{2}}(\frac{2}{\gamma})^{\frac{\gamma}{2}(Q-\alpha)-\frac{1}{2}}}{(Q-\alpha)\Gamma(1-\frac{\gamma^2}{4})^{\frac{2}{\gamma}(Q-\alpha)}}\frac{\Gamma_{\frac{\gamma}{2}}(\alpha-\frac{\gamma}{2})e^{-\mathbf{i}\pi(\frac{\chi}{2}+\mathbf{i}P)(Q-\alpha)}}{\Gamma_{\frac{\gamma}{2}}(Q-\alpha)S_{\frac{\gamma}{2}}(\frac{\alpha}{2}+\frac{\chi}{2}+\mathbf{i}P)S_{\frac{\gamma}{2}}(\frac{\alpha}{2}-\frac{\chi}{2}-\mathbf{i}P)}, \tag{5.13}$$

where we have used the special functions $\Gamma_{\frac{\gamma}{2}}(x)$ and $S_{\frac{\gamma}{2}}(x)$ introduced in Appendix B.2.

We now extend the function $\alpha\mapsto\mathcal{A}^q_{\gamma,P}(\alpha)$ to a complex neighborhood of $(Q,2Q)$ by the expression

$$\begin{aligned}\mathcal{A}^q_{\gamma,P}(\alpha) := & -q^{\frac{1}{6}(1-\frac{\alpha}{\gamma}-Q(Q+\frac{\gamma}{2}-\alpha))}\eta(q)^{\frac{3\alpha\gamma}{2}+\frac{2\alpha}{\gamma}-2-\frac{3\alpha^2}{2}+(Q+\frac{\gamma}{2}-\alpha)(3\alpha-4Q)}\Theta_\tau'(0)^{(Q-\alpha)(\gamma-\alpha)}\\ &\times\frac{e^{\mathbf{i}\pi(\frac{\alpha\gamma}{2}-(\alpha-\frac{\gamma}{2}-Q)(\alpha-2Q))}(2\pi)^{(\alpha-\frac{\gamma}{2}-Q)(Q-\alpha)}}{(-\frac{\alpha}{\gamma}+1)(1-e^{\pi\gamma P-\mathbf{i}\pi\frac{\gamma^2}{2}+\mathbf{i}\pi\frac{\alpha\gamma}{2}})}\frac{\Gamma(-\frac{\gamma^2}{4})\Gamma(\frac{2\alpha}{\gamma}-1-\frac{4}{\gamma^2})\Gamma(1+\frac{4}{\gamma^2}-\frac{\alpha}{\gamma})}{\Gamma(\frac{\alpha\gamma}{2}-1-\frac{\gamma^2}{2})\Gamma(1+\frac{\gamma^2}{4}-\frac{\alpha\gamma}{2})\Gamma(\frac{\alpha}{\gamma}-1)}\\ &\times\overline{R}(\alpha-\frac{\gamma}{2},1,e^{-\mathbf{i}\pi\frac{\gamma^2}{4}+\pi\gamma P})\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\gamma}{2}(2Q-\alpha)}e^{\pi\gamma Px}dx\right)^{\frac{\alpha}{\gamma}-\frac{4}{\gamma^2}-1}\right].\end{aligned} \tag{5.14}$$

The GMC expectation in (5.14) is well-defined and analytic in α in a complex neighborhood of $(Q,2Q+\frac{4}{\gamma})$ thanks to the moment bounds given by Lemma C.4 and the analyticity provided by Lemma 3.6. The prefactor in front of the GMC expectation is an explicit meromorphic function of α with known poles; the exact formula (5.13) shows that it is analytic in α in a complex neighborhood of $\alpha\in(Q,2Q)$, making the entire expression of $\mathcal{A}^q_{\gamma,P}(\alpha)$ analytic in a complex neighborhood of $(Q,2Q)$.

Recall that the function $\alpha\mapsto\mathcal{A}^q_{\gamma,P}(\alpha)$ is defined on $(-\frac{4}{\gamma},Q)$ in (2.11). Since $(-\frac{4}{\gamma},Q)\cap(Q,2Q)=\emptyset$, a priori it is not clear whether the function defined in (5.14) has anything to do with the one in Lemma 2.9. However, the proof of Theorem 5.1 in Section 5.2 will show that the q-series coefficients $\mathcal{A}^q_{\gamma,P}(\alpha)$ admits an analytic extension on a complex neighborhood of $(-\frac{4}{\gamma},2Q)$ which is compatible with both (5.14) and (2.11).

We now use this definition to state the OPEs in Lemmas 5.3 and 5.4. Lemma 5.3 is an easy result which corresponds to direct evaluation of $\phi^\alpha_\chi(u,q)$ at $u=0$, while Lemma 5.4 concerns the next order asymptotics as $u\to 0$ and is more involved.

Lemma 5.3. *For $\alpha\in(-\frac{4}{\gamma}+\chi,Q)$, we have*

$$\phi^\alpha_\chi(0,q) = W^-_\chi(\alpha,\gamma)q^{\frac{P^2}{2}-\frac{1}{6}\frac{l_\chi(l_\chi+1)}{\chi^2}-\frac{1}{6}l_\chi-\frac{1}{6}}\Theta'(0)^{\frac{4}{3}\frac{l_\chi(l_\chi+1)}{\chi^2}+\frac{2}{3}l_\chi+\frac{2}{3}}\mathcal{A}^q_{\gamma,P}(\alpha-\chi). \tag{5.15}$$

Proof. By direct substitution and using equation (B.3) we have

$$\begin{aligned}
\phi_\chi^\alpha(0,q) &= q^{\frac{P^2}{2}+\frac{\gamma l_\chi}{12\chi}-\frac{1}{6}\frac{l_\chi^2}{\chi^2}}\pi^{l_\chi}\Theta'_\tau(0)^{-\frac{2l_\chi^2}{3\chi^2}-\frac{2l_\chi}{3}+\frac{4l_\chi}{3\gamma\chi}}\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}+\frac{\gamma\chi}{2}}e^{\pi\gamma P x}dx\right)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right]\\
&= q^{\frac{P^2}{2}+\frac{\gamma l_\chi}{12\chi}-\frac{1}{6}\frac{l_\chi^2}{\chi^2}}\pi^{l_\chi}\Theta'_\tau(0)^{-\frac{2l_\chi^2}{3\chi^2}-\frac{2l_\chi}{3}+\frac{4l_\chi}{3\gamma\chi}}q^{-\frac{\gamma l_\chi}{6\chi}-\frac{l_\chi}{3\chi\gamma}-\frac{1}{6}}\eta(q)^{\frac{2\gamma l_\chi}{\chi}+\frac{4l_\chi}{\chi\gamma}+2+\frac{6l_\chi^2}{\chi^2}}\mathcal{A}^q_{\gamma,P}(\alpha-\chi)\\
&= q^{\frac{P^2}{2}-\frac{1}{6}\frac{l_\chi(l_\chi+1)}{\chi^2}-\frac{1}{6}l_\chi-\frac{1}{6}}\pi^{l_\chi}\Theta'_\tau(0)^{-\frac{2l_\chi^2}{3\chi^2}-\frac{2l_\chi}{3}+\frac{4l_\chi}{3\gamma\chi}}\Theta'_\tau(0)^{\frac{2\gamma l_\chi}{3\chi}+\frac{4l_\chi}{3\chi\gamma}+\frac{2}{3}+\frac{2l_\chi^2}{\chi^2}}(2\pi e^{\mathbf{i}\pi})^{-\frac{2\gamma l_\chi}{3\chi}-\frac{4l_\chi}{3\chi\gamma}-\frac{2}{3}-\frac{2l_\chi^2}{\chi^2}}\mathcal{A}^q_{\gamma,P}(\alpha-\chi)\\
&= W_\chi^-(\alpha,\gamma)q^{\frac{P^2}{2}-\frac{1}{6}\frac{l_\chi(l_\chi+1)}{\chi^2}-\frac{1}{6}l_\chi-\frac{1}{6}}\Theta'(0)^{\frac{4}{3}\frac{l_\chi(l_\chi+1)}{\chi^2}+\frac{2}{3}l_\chi+\frac{2}{3}}\mathcal{A}^q_{\gamma,P}(\alpha-\chi).
\end{aligned}$$

□

Lemma 5.4. *Consider $u=\mathbf{i}t$ with $t\in(0,\frac{1}{2}\operatorname{Im}(\tau))$. There exists a small $\alpha_0>0$ such that when $\chi=\frac{\gamma}{2}$ and $\alpha\in(\frac{\gamma}{2},\frac{2}{\gamma})\cup(Q-\alpha_0,Q)$, or $\chi=\frac{2}{\gamma}$ and $\alpha\in(Q-\alpha_0,Q)$, we have*

$$\lim_{u\to0}\sin(\pi u)^{-2l_\chi-1}\Big(\phi_\chi^\alpha(u,q)-\phi_\chi^\alpha(0,q)\Big)=W_\chi^+(\alpha,\gamma)q^{\frac{P^2}{2}+\frac{l_\chi}{6}-\frac{1}{6}\frac{l_\chi(1+l_\chi)}{\chi^2}}\Theta'_\tau(0)^{\frac{4}{3}\frac{l_\chi(l_\chi+1)}{\chi^2}-\frac{2}{3}l_\chi}\mathcal{A}^q_{\gamma,P}(\alpha+\chi). \tag{5.16}$$

The proof of Lemma 5.4 will depend on Lemmas 5.5 and 5.6, which characterize the OPE in different domains for α. These lemmas concern the next order asymptotics as $u\to0$. To state them, we use the notation

$$l_0:=l_{\frac{\gamma}{2}}\qquad\text{and}\qquad\tilde{l}_0:=l_{\frac{2}{\gamma}} \tag{5.17}$$

and recall the definition of $\mathfrak{B}$ from Appendix B.5. For $\chi=\frac{\gamma}{2}$, $\alpha\in(\frac{\gamma}{2},\frac{2}{\gamma})$, the asymptotic expansion given in Lemma 5.5 is by direct computation. In the case $\chi\in\{\frac{\gamma}{2},\frac{2}{\gamma}\}$ and $\alpha\in(Q-\alpha_0,Q)$, performing the asymptotic expansion in Lemma 5.6 is more involved and requires an operation known as OPE with reflection. Both proofs are deferred to Appendix E, after which we give the proof of Lemma 5.4.

Lemma 5.5. *For $\alpha\in(\frac{\gamma}{2},\frac{2}{\gamma})$ so that $0<1+2l_0<1$ and $\alpha+\frac{\gamma}{2}<Q$, $u\in\mathfrak{B}$, we have*

$$\lim_{u\to0}\sin(\pi u)^{-2l_0-1}\Big(\phi^\alpha_{\frac{\gamma}{2}}(u,q)-\phi^\alpha_{\frac{\gamma}{2}}(0,q)\Big)=W^+_{\frac{\gamma}{2}}(\alpha,\gamma)q^{\frac{P^2}{2}+\frac{l_0}{6}-\frac{1}{6}\frac{l_0(1+l_0)}{\chi^2}}\Theta'_\tau(0)^{\frac{4}{3}\frac{l_0(l_0+1)}{\chi^2}-\frac{2}{3}l_0}\mathcal{A}^q_{\gamma,P}(\alpha+\frac{\gamma}{2}).$$

Lemma 5.6. *(OPE with reflection) Consider $u=\mathbf{i}t$ with $t\in(0,\frac{1}{2}\operatorname{Im}(\tau))$. Let $\chi=\frac{\gamma}{2}$ or $\frac{2}{\gamma}$. There exists small $\alpha_0>0$ such that for $\alpha\in(Q-\alpha_0,Q)$, we have the asymptotic expansion*

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}}\Theta_\tau(u+x)^{\frac{\chi\gamma}{2}}e^{\pi\gamma Px}dx\right)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right]-\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}+\frac{\chi\gamma}{2}}e^{\pi\gamma Px}dx\right)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right]\\
&=-u^{1+2l_\chi}(2\pi)^{(Q-\alpha)(\frac{\gamma}{3}-\frac{\chi}{3}+\frac{2}{3\gamma})}q^{\frac{1}{6}(Q-\alpha)(\chi+\frac{2}{\gamma}-2Q)}\Theta'_\tau(0)^{(Q-\alpha)(\frac{2\chi}{3}-\frac{4}{3\gamma}-\frac{2}{3\chi})}e^{\mathbf{i}\pi(Q-\alpha)(\frac{4}{3\gamma}-\frac{2\chi}{3}-\frac{4}{3\chi})}\\
&\times\frac{\Gamma(\frac{2\alpha}{\gamma}-\frac{4}{\gamma^2})\Gamma(\frac{2Q-\alpha-\chi}{\gamma})}{\Gamma(\frac{\alpha}{\gamma}-\frac{\chi}{\gamma})}\overline{R}(\alpha,1,e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P})\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\gamma}{2}(2Q-\alpha-\chi)}e^{\pi\gamma Px}dx\right)^{\frac{\alpha+\chi-2Q}{\gamma}}\right]+o(|u|^{1+2l_\chi}).
\end{aligned}$$

Given Lemmas 5.5 and 5.6, we may motivate the extension of $\mathcal{A}^q_{\gamma,P}(\alpha)$ in (5.14) as follows. For $\chi=\frac{\gamma}{2}$, we have two ways to perform the OPE, one without reflection for $\alpha\in(\frac{\gamma}{2},\frac{2}{\gamma})$ given by Lemma 5.5 and one with reflection for α close to Q given by Lemma 5.6. We simply define $\mathcal{A}^q_{\gamma,P}(\alpha)$ in (5.14) on $(Q,2Q)$ by unifying the two OPE expressions in the form presented by Lemma 5.4.

Proof of Lemma 5.4. For $\chi=\frac{\gamma}{2}$ and $\alpha\in(\frac{\gamma}{2},\frac{2}{\gamma})$, the claim is given by Lemma 5.5. In the case $\chi=\frac{\gamma}{2}$ and $\alpha\in(Q-\alpha_0,Q)$, the claim is implied by Lemma 5.6 and the definition of $\mathcal{A}^q_{\gamma,P}(\alpha+\frac{\gamma}{2})$.

We now check the case $\chi = \frac{2}{\gamma}$. The claim of Lemma 5.6 for $\chi = \frac{2}{\gamma}$ means that we have

(5.18)

$$\begin{aligned}
&\lim_{u\to 0} \sin(\pi u)^{-2\tilde{l}_0 - 1}\Big(\phi^{\alpha}_{\frac{2}{\gamma}}(u,q) - \phi^{\alpha}_{\frac{2}{\gamma}}(0,q)\Big)\\
&= -q^{\frac{P^2}{2} + \frac{\gamma^2}{24}\tilde{l}_0(1-\tilde{l}_0)}\Theta'_\tau(0)^{-\frac{\gamma^2}{6}\tilde{l}_0^2}\\
&\times \pi^{-1-\tilde{l}_0}(2\pi)^{\frac{\gamma}{3}(Q-\alpha)}q^{-\frac{\gamma}{6}(Q-\alpha)}\Theta'_\tau(0)^{-\frac{\gamma}{3}(Q-\alpha)}e^{-\mathbf{i}\pi\frac{2\gamma}{3}(Q-\alpha)}\\
&\times \frac{\Gamma(\frac{2\alpha}{\gamma} - \frac{4}{\gamma^2})\Gamma(\frac{2}{\gamma^2} + 1 - \frac{\alpha}{\gamma})}{\Gamma(\frac{\alpha}{\gamma} - \frac{2}{\gamma^2})}\overline{R}(\alpha, 1, e^{-\mathbf{i}\pi + \pi\gamma P})\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\gamma}{2}(2Q-\alpha-\frac{2}{\gamma})}e^{\pi\gamma P x}dx\right)^{\frac{\alpha}{\gamma} - \frac{2}{\gamma^2} - 1}\right].
\end{aligned}$$

By our definition of $\mathcal{A}^q_{\gamma,P}(\alpha + \frac{2}{\gamma})$, for $\alpha > \frac{\gamma}{2}$ we have

$$\begin{aligned}
(5.19)\quad \mathcal{A}^q_{\gamma,P}(\alpha + \frac{2}{\gamma}) &= -q^{-\frac{1}{6}(1+\frac{\alpha}{\gamma}+Q(\gamma-\alpha))}\eta(q)^{\frac{13\alpha\gamma}{2} - 1 - \frac{2\alpha}{\gamma} - \frac{2}{\gamma^2} - \frac{9\alpha^2}{2} - 2\gamma^2}\Theta'_\tau(0)^{(\frac{\gamma}{2}-\alpha)(\gamma - \frac{2}{\gamma} - \alpha)}\\
&\times \frac{e^{\mathbf{i}\pi(\frac{\alpha\gamma}{2} + 1 - (\alpha-\gamma)(\alpha-\gamma-\frac{2}{\gamma}))}(2\pi)^{(\alpha-\gamma)(\frac{\gamma}{2}-\alpha)}}{(1 - \frac{\alpha}{\gamma} - \frac{2}{\gamma^2})(1 + e^{\pi\gamma P - \mathbf{i}\pi\frac{\gamma^2}{2} + \mathbf{i}\pi\frac{\alpha\gamma}{2}})}\frac{\Gamma(-\frac{\gamma^2}{4})\Gamma(\frac{2\alpha}{\gamma} - 1)\Gamma(1 + \frac{2}{\gamma^2} - \frac{\alpha}{\gamma})}{\Gamma(\frac{\alpha\gamma}{2} - \frac{\gamma^2}{2})\Gamma(\frac{\gamma^2}{4} - \frac{\alpha\gamma}{2})\Gamma(\frac{\alpha}{\gamma} + \frac{2}{\gamma^2} - 1)}\\
&\times \overline{R}(\alpha + \frac{2}{\gamma} - \frac{\gamma}{2}, 1, e^{-\mathbf{i}\pi\frac{\gamma^2}{4} + \pi\gamma P})\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\gamma}{2}(2Q-\alpha-\frac{2}{\gamma})}e^{\pi\gamma P x}dx\right)^{\frac{\alpha}{\gamma} - \frac{2}{\gamma^2} - 1}\right].
\end{aligned}$$

To obtain the desired answer, by [RZ20, Theorem 1.7] we compute a ratio of reflection coefficients as

$$\begin{aligned}
\frac{\overline{R}(\alpha, 1, e^{-\mathbf{i}\pi + \pi\gamma P})}{\overline{R}(\alpha + \frac{2}{\gamma} - \frac{\gamma}{2}, 1, e^{-\mathbf{i}\pi\frac{\gamma^2}{4} + \pi\gamma P})} &= \frac{\overline{R}(\alpha, 1, e^{-\mathbf{i}\pi + \pi\gamma P})}{\overline{R}(\alpha + \frac{2}{\gamma}, 1, e^{\pi\gamma P})}\frac{\overline{R}(\alpha + \frac{2}{\gamma}, 1, e^{\pi\gamma P})}{\overline{R}(\alpha + \frac{2}{\gamma} - \frac{\gamma}{2}, 1, e^{-\mathbf{i}\pi\frac{\gamma^2}{4} + \pi\gamma P})}\\
&= \frac{2}{\gamma(Q-\alpha)}(2\pi)^{\frac{4}{\gamma^2} - 1}\frac{\Gamma(\frac{2\alpha}{\gamma})\Gamma(1 - \frac{2\alpha}{\gamma})}{\Gamma(1 - \frac{\gamma^2}{4})^{\frac{4}{\gamma^2} - 1}\Gamma(\frac{\gamma\alpha}{2} - \frac{\gamma^2}{2})\Gamma(1 - \frac{\gamma\alpha}{2} + \frac{\gamma^2}{4})}\frac{1 - e^{\frac{4\pi P}{\gamma} - \frac{4\mathbf{i}\pi}{\gamma^2} + \mathbf{i}\pi\frac{2\alpha}{\gamma}}}{1 + e^{\pi\gamma P - \frac{\mathbf{i}\pi\gamma^2}{2} + \mathbf{i}\pi\frac{\gamma\alpha}{2}}}.
\end{aligned}$$

Substituting (5.19) into (5.18) and simplifying yields the desired claim. □

5.2. **Proof of Theorem 5.1.** First, by Corollary 4.4, we have that $\phi^{\alpha}_{\chi,n,1}(0) = \phi^{\alpha,1}_{\chi,n,1}(0)$ and

$$(5.20)\qquad \phi^{\alpha,2}_{\chi,n,1}(0) = \lim_{w\to 0^-} w^{C_\chi - 1}(\phi^{\alpha}_{\chi,n,1}(w) - \phi^{\alpha,1}_{\chi,n,1}(w)) = \lim_{t\to 0^+}\sin(\pi\mathbf{i}t)^{-2l_\chi - 1}\Big(\phi^{\alpha}_{\chi,n}(\mathbf{i}t) - \phi^{\alpha}_{\chi,n}(0)\Big).$$

Taking a q-series expansion of the result of Lemma 5.3 and equation (5.3) then implies that (5.5) holds for $\alpha \in (-\frac{4}{\gamma} + \chi, Q)$. A similar q-series expansion using Lemma 5.4 and (5.4) implies that (5.6) holds for $\chi = \frac{\gamma}{2}$ and $\alpha \in (\frac{\gamma}{2}, \frac{2}{\gamma}) \cup (Q - \alpha_0, Q)$ or $\chi = \frac{2}{\gamma}$ and $\alpha \in (Q - \alpha_0, Q)$ for some small $\alpha_0 > 0$.

It remains to show that $\mathcal{A}_{\gamma,P,n}(\alpha)$ admits an analytic extension to a complex neighborhood of $(-\frac{4}{\gamma}, 2Q)$. We proceed by induction on $n \in \mathbb{N}_0$, suppose that such an extension exists for all $m < n$. We previously established (5.6) for $\chi = \frac{\gamma}{2}$ and $\alpha \in (\frac{\gamma}{2}, \frac{2}{\gamma}) \cup (Q - \alpha_0, Q)$, which we may rearrange as

$$(5.21)\qquad \mathcal{A}_{\gamma,P,n}(\alpha + \frac{\gamma}{2}) = [W^+_{\frac{\gamma}{2}}(\alpha,\gamma)\eta^+_{\frac{\gamma}{2},0}(\alpha)]^{-1}\phi^{\alpha,2}_{\frac{\gamma}{2},n,1}(0) - \sum_{m=0}^{n-1}\frac{\eta^+_{\frac{\gamma}{2},n-m}(\alpha)}{\eta^+_{\frac{\gamma}{2},0}(\alpha)}\mathcal{A}_{\gamma,P,m}(\alpha + \frac{\gamma}{2}).$$

By Lemma 4.7, $\phi^{\alpha,2}_{\frac{\gamma}{2},n,1}(0)$ is analytic in α on a complex neighborhood of $(-\frac{4}{\gamma} + \frac{\gamma}{2}, Q)$. Combined with the inductive hypothesis, the explicit expression for $W^+_\chi(\alpha,\gamma)$ from (5.2), and the fact that $\eta^+_{\chi,0}(\alpha) = (2\pi e^{\mathbf{i}\pi})^{\frac{4}{3}\frac{l_\chi(l_\chi+1)}{\chi^2} - \frac{2}{3}l_\chi}$, the right hand side of (5.21) provides an analytic extension of $\mathcal{A}_{\gamma,P,n}(\alpha)$ to a complex neighborhood of of $(-\frac{4}{\gamma} + \gamma, Q + \gamma)$. Recall that by Lemma 2.9 we know $\mathcal{A}_{\gamma,P,n}(\alpha)$ is analytic in a complex neighborhood of $(-\frac{4}{\gamma}, Q)$. Finally, by taking the q-expansion of our definition (5.14) of $\mathcal{A}^q_{\gamma,P}(\alpha)$ beyond $\alpha = Q$, we obtain that $\mathcal{A}_{\gamma,P,n}(\alpha)$ is also analytic in a complex neighborhood of $(Q, 2Q)$. Since the three intervals $(-\frac{4}{\gamma}, Q)$, $(-\frac{4}{\gamma} + \gamma, Q + \gamma)$, $(Q, 2Q)$ have an overlap, one obtains the desired claim that $\mathcal{A}_{\gamma,P,n}(\alpha)$ admits an analytic continuation to a complex neighborhood of $(-\frac{4}{\gamma}, 2Q)$.

6. Equivalence of the probabilistic conformal block and Nekrasov partition function

In this section, we prove Theorem 2.13 by showing that the q-series coefficients of $\widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha)$ and $\mathcal{Z}^\alpha_{\gamma,P}(q)$ both satisfy a system of shift equations relating their values at different α. A uniqueness result on solutions to such shift equations then implies the desired result. We now present the precise statement and the key idea of each step of the proof, deferring details of certain steps to later subsections.

Our first step is to establish the shift equations for $\mathcal{A}_{\gamma,P,0}(\alpha)$ and $\{\widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha)\}_{n\in\mathbb{N}}$, where $\{\mathcal{A}_{\gamma,P,n}(\alpha)\}$ and $\{\widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha)\}$ are defined in (2.12) and (2.13), respectively. We interpret $\mathcal{A}_{\gamma,P,n}(\alpha)$ according to its analytic continuation in α in Theorem 5.1 and $\widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha)$ according to the corresponding meromorphic extension via (2.13). To state the shift equations, recall $G^{\alpha,j}_{\chi,n,i}$ and $G^{\alpha}_{\chi,n,i}$ from Definition 4.8, where we interpret $G^{\alpha,j}_{\chi,n,i}$ via the analytic continuations given in Corollary 5.2. For $n \in \mathbb{N}_0$, define the quantities

$$V^{\alpha,1}_{\chi,n} := G^{\alpha,1}_{\chi,n,1}(0) \qquad \text{and} \qquad V^{\alpha,2}_{\chi,n} := G^{\alpha,2}_{\chi,n,1}(0).$$

Finally, recall $A_{\chi,n}$, $B_{\chi,n}$, C_χ from (4.6) and denote the *connection coefficients* from equation (D.2) by

$$\Gamma_{n,1} := \frac{\Gamma(C_\chi)\Gamma(C_\chi - A_{\chi,n} - B_{\chi,n})}{\Gamma(C_\chi - A_{\chi,n})\Gamma(C_\chi - B_{\chi,n})} \qquad \text{and} \qquad \Gamma_{n,2} := \frac{\Gamma(2 - C_\chi)\Gamma(C_\chi - A_{\chi,n} - B_{\chi,n})}{\Gamma(1 - A_{\chi,n})\Gamma(1 - B_{\chi,n})}. \tag{6.1}$$

We are now ready to state the shift equations for $\mathcal{A}_{\gamma,P,0}(\alpha)$ and $\{\widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha)\}_{n\in\mathbb{N}}$.

Theorem 6.1. *Recalling the quantities $W^\pm_\chi(\alpha,\gamma)$ from (5.1) and (5.2) and $\eta^\pm_{\chi,m}(\alpha)$ from (5.3) and (5.4), for $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$ and α in a complex neighborhood of $(-\frac{4}{\gamma} + \chi, 2Q - \chi)$, we have*

$$\mathcal{A}_{\gamma,P,0}(\alpha - \chi) = -\frac{W^+_\chi(\alpha,\gamma)}{W^-_\chi(\alpha,\gamma)} \frac{\Gamma_{0,2}}{\Gamma_{0,1}} \frac{1 + e^{\pi\chi P + \mathbf{i}\pi l_\chi}}{1 - e^{\pi\chi P - \mathbf{i}\pi l_\chi}} \frac{\eta^+_{\chi,0}(\alpha)}{\eta^-_{\chi,0}(\alpha)} \mathcal{A}_{\gamma,P,0}(\alpha + \chi). \tag{6.2}$$

Setting $\widetilde{V}^{\alpha,j}_{\chi,n} = V^{\alpha,j}_{\chi,n} W^-_\chi(\alpha,\gamma)^{-1}\eta^-_{\chi,0}(\alpha)^{-1}\mathcal{A}_{\gamma,P,0}(\alpha-\chi)^{-1}$, we have

$$\begin{aligned}\widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha - \chi) &+ \sum_{m=0}^{n-1} \frac{\eta^-_{\chi,n-m}(\alpha)}{\eta^-_{\chi,0}(\alpha)} \widetilde{\mathcal{A}}_{\gamma,P,m}(\alpha - \chi) \\ &= \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \frac{\Gamma_{0,1}}{\Gamma_{0,2}} \widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha + \chi) + \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \frac{\Gamma_{0,1}}{\Gamma_{0,2}} \sum_{m=0}^{n-1} \frac{\eta^+_{\chi,n-m}(\alpha)}{\eta^+_{\chi,0}(\alpha)} \widetilde{\mathcal{A}}_{\gamma,P,m}(\alpha+\chi) + \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \frac{1 + e^{\pi\chi P + \mathbf{i}\pi l_\chi}}{1 - e^{\pi\chi P - \mathbf{i}\pi l_\chi}} \widetilde{V}^{\alpha,2}_{\chi,n} + \widetilde{V}^{\alpha,1}_{\chi,n}.\end{aligned} \tag{6.3}$$

We prove Theorem 6.1 in Section 6.1 by combining the hypergeometric differential equations from Proposition 4.2 and the operator product expansions from Theorem 5.1.

Remark 6.2. By Lemma D.5 and the linearity of equations (4.4), we can see that $V^{\alpha,1}_{\chi,n}$ is a linear function of $\{\phi^{\alpha,1}_{\chi,m,1}(0)\}_{0\le m\le n}$. In the resulting expansion $V^{\alpha,1}_{\chi,n} = \sum_{m=0}^n c_{nm}(\alpha,\chi,P)\phi^{\alpha,1}_{\chi,m,1}(0)$, the linear coefficients c_{nm} are in principle explicit and meromorphic. By (6.36) below based on Theorem 5.1, $\widetilde{V}^{\alpha,1}_{\chi,n}$ is in turn an explicit linear combination of $\{\widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha-\chi)\}_{0\le m\le n}$. Similarly, $\widetilde{V}^{\alpha,2}_{\chi,n}$ is an explicit linear combination of $\{\widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha+\chi)\}_{0\le m\le n}$. Therefore, the shift equation (6.2) in Theorem 6.1 is a linear relation between $\{\widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha-\chi)\}_{0\le m\le n}$ and $\{\widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha+\chi)\}_{0\le m\le n}$. Although this viewpoint is conceptually appealing, we will not employ it in our proofs, as the coefficients $c_{nm}(\alpha,\chi,P)$ are rather tedious to work with.

Our second step is to prove an appropriate uniqueness result for our shift equations. To this end, we may place equation (6.3) into the following setting. For $n \in \mathbb{N}$, consider the following shift equation

$$X_n(\alpha - \chi) = Y_n(\chi,\alpha)X_n(\alpha+\chi) + Z_n(\chi,\alpha) \tag{6.4}$$

on unknown functions $X_n(\alpha)$ for $Y_n(\chi,\alpha) := \frac{\Gamma_{n,2}\Gamma_{0,1}}{\Gamma_{n,1}\Gamma_{0,2}}$ and

$$\begin{aligned}Z_n(\chi,\alpha) := -\sum_{m=0}^{n-1} \frac{\eta^-_{\chi,n-m}(\alpha)}{\eta^-_{\chi,0}(\alpha)} \widetilde{\mathcal{A}}_{\gamma,P,m}(\alpha-\chi) &+ \frac{\Gamma_{n,2}\Gamma_{0,1}}{\Gamma_{n,1}\Gamma_{0,2}} \sum_{m=0}^{n-1} \frac{\eta^+_{\chi,n-m}(\alpha)}{\eta^+_{\chi,0}(\alpha)} \widetilde{\mathcal{A}}_{\gamma,P,m}(\alpha+\chi) \\ &+ \frac{\Gamma_{n,2}}{\Gamma_{n,1}} \frac{1 + e^{\pi\chi P + \mathbf{i}\pi l_\chi}}{1 - e^{\pi\chi P - \mathbf{i}\pi l_\chi}} \widetilde{V}^{\alpha,2}_{\chi,n} + \widetilde{V}^{\alpha,1}_{\chi,n}.\end{aligned} \tag{6.5}$$

By (6.3), the shift equation (6.4) holds with $\widetilde{\mathcal{A}}_{\gamma,P,n}$ in place of X_n for each $n \in \mathbb{N}$.

We now show that the shift equations (6.4) have a unique solution up to constant factor.

Proposition 6.3. *Fix $\gamma \in (0,2)$ with γ^2 irrational and $P \in \mathbb{R}$. For $n \in \mathbb{N}$, let $X_n^1(\alpha)$ and $X_n^2(\alpha)$ be meromorphic functions on a complex neighborhood V of $(-\frac{4}{\gamma}, 2Q)$. Suppose (6.4) with $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$ and $\alpha \in (-\frac{4}{\gamma}+\chi, 2Q-\chi)$ holds with X_n^i in place of X_n for $i = 1, 2$ and that $X_n^1(\alpha_0) = X_n^2(\alpha_0)$ for some $\alpha_0 \in (-\frac{4}{\gamma}+\chi, 2Q-\chi)$. Then, $X_n^1(\alpha) = X_n^2(\alpha)$ for all $\alpha \in V$.*

Proof. Define $\Delta_n(\alpha) := X_n^1(\alpha) - X_n^2(\alpha)$. Subtracting the given equations for $i = 1, 2$, we obtain that

$$\Delta_n(\alpha-\chi) = Y_n(\chi,\alpha)\Delta_n(\alpha+\chi) \quad \text{for } \chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\} \text{ and } \alpha \in (-\frac{4}{\gamma}+\chi, 2Q-\chi). \tag{6.6}$$

Since Γ has poles at $\{0,-1,-2,...\}$ and no zeros, for $P \in \mathbb{R}$ and $n \in \mathbb{N}$ the explicit expression

$$Y_n(\chi,\alpha) = \frac{\Gamma(\frac{1}{2}-\frac{1}{2}l_\chi+\mathbf{i}\frac{\chi}{2}\sqrt{P^2+2n})\Gamma(\frac{1}{2}-\frac{1}{2}l_\chi-\mathbf{i}\frac{\chi}{2}\sqrt{P^2+2n})}{\Gamma(1+\frac{1}{2}l_\chi+\mathbf{i}\frac{\chi}{2}\sqrt{P^2+2n})\Gamma(1+\frac{1}{2}l_\chi-\mathbf{i}\frac{\chi}{2}\sqrt{P^2+2n})} \frac{\Gamma(1+\frac{1}{2}l_\chi+\mathbf{i}\frac{\chi}{2}P)\Gamma(1+\frac{1}{2}l_\chi-\mathbf{i}\frac{\chi}{2}P)}{\Gamma(\frac{1}{2}-\frac{1}{2}l_\chi+\mathbf{i}\frac{\chi}{2}P)\Gamma(\frac{1}{2}-\frac{1}{2}l_\chi-\mathbf{i}\frac{\chi}{2}P)},$$

yields that $Y_n(\chi,\alpha)$ is meromorphic in $\alpha \in \mathbb{C}$ without real zeros or poles. Because the interval $(-\frac{4}{\gamma}, 2Q)$ has length bigger than γ, the function $\Delta_n(\alpha)$ admits a meromorphic extension to a complex neighborhood U of $\mathbb{R}$, which we still denote by $\Delta_n(\alpha)$, such that $\Delta_n(\alpha-\frac{\gamma}{2}) = Y_n(\chi,\alpha)\Delta_n(\alpha+\frac{\gamma}{2})$ for each $\alpha \in U$. Since (6.6) holds for $\alpha \in (-\frac{4}{\gamma}+\chi, 2Q-\chi)$, we have $\Delta_n(\alpha-\frac{2}{\gamma}) = Y_n(\chi,\alpha)\Delta_n(\alpha+\frac{2}{\gamma})$ for each $\alpha \in U$.

Since $\Delta_n(\alpha_0) = 0$ and $Y_n(\chi,\alpha) \neq 0$ for $\alpha \in \mathbb{R}$, we have that $\Delta_n(\alpha) = 0$ for any α which can be reached from α_0 by a sequence of additions or subtractions of γ or $\frac{4}{\gamma}$. Note that $\alpha_0 + \mathbb{Z}\gamma + \mathbb{Z}\frac{4}{\gamma}$ is dense in $\mathbb{R}$ because $\gamma^2 \notin \mathbb{Q}$. Since Δ_n is meromorphic on U, we must have $\Delta_n(\alpha) = 0$ for all $\alpha \in U$. □

Our third step is to obtain an explicit expression for $\mathcal{A}_{\gamma,P,0}(\alpha)$.

Proposition 6.4. *For $\gamma \in (0,2)$, $\alpha \in (-\frac{4}{\gamma}, Q)$, and $P \in \mathbb{R}$, we have*

$$\mathcal{A}_{\gamma,P,0}(\alpha) = e^{\frac{\mathbf{i}\pi\alpha^2}{2}}\left(\frac{\gamma}{2}\right)^{\frac{\gamma\alpha}{4}} e^{-\frac{\pi\alpha P}{2}}\Gamma(1-\frac{\gamma^2}{4})^{\frac{\alpha}{\gamma}} \frac{\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}+\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}-\mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+\mathbf{i}P)}{\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma})\Gamma_{\frac{\gamma}{2}}(Q-\mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q+\mathbf{i}P)\Gamma_{\frac{\gamma}{2}}(Q-\alpha)}. \tag{6.7}$$

We will prove Proposition 6.4 in Section 6.2. Note that the equation (6.2) for $\mathcal{A}_{\gamma,P,0}(\alpha)$ is also of the form (6.4), where for $n = 0$ we take $Z_0 = 0$, and Y_0 is a different explicit expression. We will check that the right hand side of (6.7) satisfies the same shift equation as $\mathcal{A}_{\gamma,P,0}(\alpha)$ and use the uniqueness of the shift equation to prove Proposition 6.4. We remark that Proposition 6.4 allows us to explicitly compute the normalization Z of Definition 2.6 thanks to (2.15).

Our fourth step is to establish Zamolodchikov's recursion when $-\frac{\alpha}{\gamma} \in \mathbb{N}$ in Theorem 6.5, which we will prove in Section 6.4. The key idea of its proof will be outlined after we first explain how it leads to a proof of Theorem 2.13.

Theorem 6.5. *Suppose $-\frac{\alpha}{\gamma} \in \mathbb{N}$ for $\gamma \in (0,2)$ and $\alpha \in (-\frac{4}{\gamma}, Q)$, and $q \in (0,1)$. Then $\widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha)$ admits a meromorphic continuation in P to all of $\mathbb{C}$ under which*

$$\widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha) = \sum_{n,m=1}^{\infty} q^{2nm}\frac{R_{\gamma,m,n}(\alpha)}{P^2-P_{m,n}^2}\widetilde{\mathcal{A}}^q_{\gamma,P_{-m,n}}(\alpha) + [q^{-\frac{1}{12}}\eta(q)]^{\alpha(Q-\frac{\alpha}{2})-2}, \tag{6.8}$$

where $R_{\gamma,m,n}(\alpha)$ and $P_{m,n}$ are defined in (2.21) and (2.20).

Our fifth step is to show that Theorem 2.13 holds for $-\frac{\alpha}{\gamma} \in \mathbb{N}$ using Theorem 6.5.

Theorem 6.6. *Suppose $-\frac{\alpha}{\gamma} \in \mathbb{N}$ for $\gamma \in (0,2)$ and $\alpha \in (-\frac{4}{\gamma}, Q)$, and $q \in (0,1)$. Let $\widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha)$ be defined under the meromorphic extension to $P \in \mathbb{C}$ from Theorem 6.5. Then $\mathcal{Z}^\alpha_{\gamma,P}(q) = \widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha)$ as formal q-series.*

Proof. By Theorem 6.5, (2.19), and (2.22), when $N \in \mathbb{N}$, the formal q-series expansions for both $\mathcal{Z}^\alpha_{\gamma,P}(q)$ and the meromorphic continuation of $\widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha)$ solve the recursion (6.8). Denoting their difference by

$$\Delta^q_{\gamma,P}(\alpha) = \sum_{n=0}^{\infty}\Delta_{\gamma,P,n}(\alpha)q^n,$$

we find by subtraction that

$$\sum_{n=0}^{\infty} \Delta_{\gamma,P,n}(\alpha) q^n = \sum_{n,m=1}^{\infty} q^{2nm} \frac{R_{\gamma,m,n}(\alpha)}{P^2 - P_{m,n}^2} \sum_{k=0}^{\infty} \Delta_{\gamma,P_{-m,n},k}(\alpha) q^k.$$

Equating q-series coefficients of both sides expresses $\Delta_{\gamma,P,n}(\alpha)$ as a linear combination of $\Delta_{\gamma,P,m}(\alpha)$ with $m < n$. By the form of the right hand side, we find that $\Delta_{\gamma,P,0}(\alpha) = 0$, hence an induction shows that $\Delta_{\gamma,P,n}(\alpha) = 0$ as needed. □

Our final step is to put everything together and prove Theorem 2.13 in Section 6.5. We will prove Theorem 2.13 by combining Theorem 6.6 with a detailed analytic analysis of the shift equation (6.4) from Theorem 6.1 and Proposition 6.3. The key observation is that by Theorem 6.6 the q-series coefficients $\mathcal{Z}_{\gamma,P,n}(\alpha)$ of $\mathcal{Z}^{\alpha}_{\gamma,P}(q)$ satisfy the shift equation (6.4) in the case when $\chi = \frac{\gamma}{2}$ and $-\frac{\alpha}{\gamma} \in \mathbb{N}$. In Section 6.5, we then view χ as a variable and show inductively that $Z_n(\chi, \alpha)$ from (6.4) admits a meromorphic extension to a neighborhood of $\chi \in [0, \infty)$. The same holds for $\mathcal{Z}_{2\chi,P,n}(\alpha)$ due to its explicit expression (2.17), meaning the shift equation (6.4) for $\mathcal{Z}_{2\chi,P,n}(\alpha)$ is an equality of meromorphic functions which holds on the set $\{\chi \mid -\frac{\alpha}{2\chi} \in \mathbb{N}\}$. Using a well-known fact about meromorphic functions (Lemma 6.7), we see that (6.4) holds for $\mathcal{Z}_{\gamma,P,n}(\alpha)$ whenever each quantity in (6.4) is well defined. Finally, the uniqueness of Proposition 6.3 implies as desired that $\mathcal{Z}_{\gamma,P,n}(\alpha) = \widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha)$, which proves Theorem 2.13.

Lemma 6.7. *If f and g are meromorphic functions on a domain $U \subset \mathbb{C}$ with $f(z_k) = g(z_k)$ for some $z_k \in U$ with an accumulation point in U, then $f = g$ on all of U.*

We now explain the idea for the proof of Theorem 6.5. The starting point is the following observation.

Lemma 6.8. *If $N = -\frac{\alpha}{\gamma} \in \mathbb{N}$ for $\gamma \in (0, 2)$, $\alpha \in (-\frac{4}{\gamma}, Q)$, and $q \in (0, 1)$, the function $\mathcal{A}^q_{\gamma,P}(\alpha)$ is given by*

$$\mathcal{A}^q_{\gamma,P}(\alpha) := q^{\frac{\alpha^2}{24} - \frac{\alpha}{12}Q + \frac{1}{6}} \eta(q)^{\frac{5}{4}\alpha\gamma + \frac{2\alpha}{\gamma} - \frac{5}{4}\alpha^2 - 2} \left(\int_0^1\right)^N \prod_{1 \leq i < j \leq N} |\Theta_\tau(x_i - x_j)|^{-\frac{\gamma^2}{2}} \prod_{i=1}^N \Theta_\tau(x_i)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x_i} \prod_{i=1}^N dx_i. \tag{6.9}$$

Proof. Since $-\frac{\alpha}{\gamma} = N \in \mathbb{N}$, we can write

$$\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x} dx\right)^{-\frac{\alpha}{\gamma}}\right] = \left(\int_0^1\right)^N \prod_{i=1}^N \Theta_\tau(x_i)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x_i} \prod_{1 \leq i < j \leq N} e^{\frac{\gamma^2}{4}\mathbb{E}[Y_\tau(x_i)Y_\tau(x_j)]} \prod_{i=1}^N dx_i.$$

Using the explicit formula for $\mathbb{E}[Y_\tau(x_i)Y_\tau(x_j)]$ from (2.4), we get Lemma 6.8. □

The N-fold integral from (6.9) is an example of a Dotsenko-Fateev integral. Using Lemma 6.8, we prove the following key identity relating values of $\mathcal{A}^q_{\gamma,P}(\alpha)$ at $P_{m,n}$ and $P_{-m,n}$.

Proposition 6.9. *If $N = -\frac{\alpha}{\gamma} \in \mathbb{N}$ for $\gamma \in (0, 2)$ and $\alpha \in (-\frac{4}{\gamma}, Q)$, then viewing $P \mapsto \mathcal{A}^q_{\gamma,P}(\alpha)$ as an entire function, we have*

$$\mathcal{A}^q_{\gamma,P_{m,n}}(\alpha) = q^{2nm} e^{-\frac{\mathrm{i}\pi\alpha\gamma m}{2}} \mathcal{A}^q_{\gamma,P_{-m,n}}(\alpha) \quad \textit{for } (m, n) \in \mathbb{Z}^2. \tag{6.10}$$

Proposition 6.9 is proved in Section 6.3. Using (6.9) we show that $\mathcal{A}^q_{\gamma,P}(\alpha)$ is analytic in $P \in \mathbb{C}$. Therefore, if $\frac{-\alpha}{\gamma} = N \in \mathbb{N}$, for $n \in \mathbb{N}_0$, the function $P \mapsto \mathcal{A}_{\gamma,P,n}(\alpha)$ admits analytic extension to $P \in \mathbb{C}$. Thus $P \mapsto \widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha)$ admits meromorphic extension to $P \in \mathbb{C}$, as asserted in Theorem 6.5. Applying a *rotation-of-contour* trick to this integral and using the quasi-periodicity of the theta function will give Proposition 6.9.

In Section 6.4 we use Propositions 6.4 and 6.9 to understand the pole structure of the meromorphic extension of $\widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha)$, showing that it has simple poles at $\{\pm P_{m,n} : n \in \mathbb{N} \text{ and } 1 \leq m \leq N\}$, and moreover the residues at these poles are given by $R_{\gamma,m,n}(\alpha)$ in (6.8). Finally, we show that

$$\lim_{P \to \infty} \widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha) = [q^{-\frac{1}{12}}\eta(q)]^{\alpha(Q - \frac{\alpha}{2}) - 2},$$

which gives the last term in (6.8). Combining these three ingredients gives Theorem 6.5.

The remaining five subsections of this section are devoted to proving Theorem 6.1, Proposition 6.4, Proposition 6.9, Theorem 6.5, and Theorem 2.13, respectively.

6.1. **Proof of Theorem 6.1.** For $i,j\in\{1,2\}$ and $n\in\mathbb{N}_0$, recall $\phi^{\alpha,j}_{\chi,n,i}(w)$ and $\phi^{\alpha}_{\chi,n,i}(w)$ from Corollary 4.4, where we use the analytic extension of $\phi^{\alpha,j}_{\chi,n,i}(w)$ in α from Corollary 5.2. Since $\phi^{\alpha}_{\chi,n,i}(w)-G^{\alpha}_{\chi,n,i}(w)$ is a special solution to the homogeneous variant of (4.3), the discussion of the linear solution space around 0 and 1 of such differential equations in Appendix D.1 implies that for some $X^j_{\chi,n,i}(\alpha),Y^j_{\chi,n,i}(\alpha)$ we have

$$\begin{aligned}\phi^{\alpha}_{\chi,n,i}(w)&=G^{\alpha}_{\chi,n,i}(w)+X^1_{\chi,n,i}(\alpha)\,{}_2F_1(A_{\chi,n},B_{\chi,n},C_\chi;w)\\&\quad+X^2_{\chi,n,i}(\alpha)\,w^{1-C_\chi}{}_2F_1(1+A_{\chi,n}-C_\chi,1+B_{\chi,n}-C_\chi,2-C_\chi;w)\\ \phi^{\alpha}_{\chi,n,i}(w)&=G^{\alpha}_{\chi,n,i}(w)+Y^1_{\chi,n,i}(\alpha)\,{}_2F_1(A_{\chi,n},B_{\chi,n},1+A_{\chi,n}+B_{\chi,n}-C_\chi;1-w)\\&\quad+Y^2_{\chi,n,i}(\alpha)\,(1-w)^{C_\chi-A_{\chi,n}-B_{\chi,n}}{}_2F_1(C_\chi-A_{\chi,n},C_\chi-B_{\chi,n},1+C_\chi-A_{\chi,n}-B_{\chi,n};1-w).\end{aligned}$$

Together, these equations imply for $i\in\{1,2\}$ that

$$\begin{aligned}\phi^{\alpha,1}_{\chi,n,i}(w)&=G^{\alpha,1}_{\chi,n,i}(w)+X^1_{\chi,n,i}(\alpha)\,{}_2F_1(A_{\chi,n},B_{\chi,n},C_\chi;w);\\ \phi^{\alpha,2}_{\chi,n,i}(w)&=G^{\alpha,2}_{\chi,n,i}(w)+X^2_{\chi,n,i}(\alpha)\,{}_2F_1(1+A_{\chi,n}-C_\chi,1+B_{\chi,n}-C_\chi,2-C_\chi;w),\end{aligned}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

By the connection coefficients in (6.1) and the connection equation (D.2), we have for $i\in\{1,2\}$ that

$$Y^1_{\chi,n,i}(\alpha)=\Gamma_{n,1}X^1_{\chi,n,i}(\alpha)+\Gamma_{n,2}X^2_{\chi,n,i}(\alpha).$$

Because $\phi^{\alpha}_{\chi,n,1}(1)=\phi^{\alpha}_{\chi,n,2}(1)$, $G^{\alpha}_{\chi,n,1}(1)=G^{\alpha}_{\chi,n,2}(1)=0$, and $C_\chi-A_{\chi,n}-B_{\chi,n}=\frac{1}{2}$, this implies that

$$X^1_{\chi,n,1}(\alpha)-X^1_{\chi,n,2}(\alpha)=-\frac{\Gamma_{n,2}}{\Gamma_{n,1}}(X^2_{\chi,n,1}(\alpha)-X^2_{\chi,n,2}(\alpha)).\tag{6.11}$$

In addition, by Lemma 4.6 and Theorem 5.1, we have

$$\begin{aligned}X^1_{\chi,n,1}(\alpha)+G^{\alpha,1}_{\chi,n,1}(0)&=\phi^{\alpha,1}_{\chi,n,1}(0)=W^-_\chi(\alpha,\gamma)\left[\eta^-_{\chi,0}(\alpha)\mathcal{A}_{\gamma,P,n}(\alpha-\chi)+\sum_{m=0}^{n-1}\eta^-_{\chi,n-m}(\alpha)\mathcal{A}_{\gamma,P,m}(\alpha-\chi)\right]\\ X^2_{\chi,n,1}(\alpha)+G^{\alpha,2}_{\chi,n,1}(0)&=\phi^{\alpha,2}_{\chi,n,1}(0)=W^+_\chi(\alpha,\gamma)\left[\eta^+_{\chi,0}(\alpha)\mathcal{A}_{\gamma,P,n}(\alpha+\chi)+\sum_{m=0}^{n-1}\eta^+_{\chi,n-m}(\alpha)\mathcal{A}_{\gamma,P,m}(\alpha+\chi)\right]\\ X^1_{\chi,n,2}(\alpha)+G^{\alpha,1}_{\chi,n,2}(0)&=\phi^{\alpha,1}_{\chi,n,2}(0)=e^{\pi\chi P-\mathbf{i}\pi l_\chi}(X^1_{\chi,n,1}(\alpha)+G^{\alpha,1}_{\chi,n,1}(0))\\ X^2_{\chi,n,2}(\alpha)+G^{\alpha,2}_{\chi,n,2}(0)&=\phi^{\alpha,2}_{\chi,n,2}(0)=-e^{\pi\chi P+\mathbf{i}\pi l_\chi}(X^2_{\chi,n,1}(\alpha)+G^{\alpha,2}_{\chi,n,1}(0))\end{aligned}$$

for $W^{\pm}_\chi(\alpha,\gamma)$ defined in (5.1) and (5.2). Combining (6.11), the last two equalities, and Proposition 4.9, we find that

$$(1-e^{\pi\chi P-\mathbf{i}\pi l_\chi})X^1_{\chi,n,1}(\alpha)=-\frac{\Gamma_{n,2}}{\Gamma_{n,1}}(1+e^{\pi\chi P+\mathbf{i}\pi l_\chi})X^2_{\chi,n,1}(\alpha).$$

Finally, substituting Theorem 5.1 into the first equality, we find that

$$\begin{aligned}&\eta^-_{\chi,0}(\alpha)\mathcal{A}_{\gamma,P,n}(\alpha-\chi)=W^-_\chi(\alpha,\gamma)^{-1}(X^1_{\chi,n,1}(\alpha)+G^{\alpha,1}_{\chi,n,1}(0))-\sum_{m=0}^{n-1}\eta^-_{\chi,n-m}(\alpha)\mathcal{A}_{\gamma,P,m}(\alpha-\chi)\\ &=-W^-_\chi(\alpha,\gamma)^{-1}\frac{\Gamma_{n,2}}{\Gamma_{n,1}}\frac{1+e^{\pi\chi P+\mathbf{i}\pi l_\chi}}{1-e^{\pi\chi P-\mathbf{i}\pi l_\chi}}X^2_{\chi,n,1}(\alpha)+W^-_\chi(\alpha,\gamma)^{-1}G^{\alpha,1}_{\chi,n,1}(0)-\sum_{m=0}^{n-1}\eta^-_{\chi,n-m}(\alpha)\mathcal{A}_{\gamma,P,m}(\alpha-\chi)\\ &=-W^+_\chi(\alpha,\gamma)W^-_\chi(\alpha,\gamma)^{-1}\frac{\Gamma_{n,2}}{\Gamma_{n,1}}\frac{1+e^{\pi\chi P+\mathbf{i}\pi l_\chi}}{1-e^{\pi\chi P-\mathbf{i}\pi l_\chi}}\eta^+_{\chi,0}(\alpha)\mathcal{A}_{\gamma,P,n}(\alpha+\chi)\\ &\quad-W^+_\chi(\alpha,\gamma)W^-_\chi(\alpha,\gamma)^{-1}\frac{\Gamma_{n,2}}{\Gamma_{n,1}}\frac{1+e^{\pi\chi P+\mathbf{i}\pi l_\chi}}{1-e^{\pi\chi P-\mathbf{i}\pi l_\chi}}\sum_{m=0}^{n-1}\eta^+_{\chi,n-m}(\alpha)\mathcal{A}_{\gamma,P,m}(\alpha+\chi)\\ &\quad+W^-_\chi(\alpha,\gamma)^{-1}\frac{\Gamma_{n,2}}{\Gamma_{n,1}}\frac{1+e^{\pi\chi P+\mathbf{i}\pi l_\chi}}{1-e^{\pi\chi P-\mathbf{i}\pi l_\chi}}V^{\alpha,2}_{\chi,n}+W^-_\chi(\alpha,\gamma)^{-1}V^{\alpha,1}_{\chi,n}-\sum_{m=0}^{n-1}\eta^-_{\chi,n-m}(\alpha)\mathcal{A}_{\gamma,P,m}(\alpha-\chi).\end{aligned}$$

Specializing the above equation to $n=0$ yields (6.2). For $n\geq 1$, dividing both sides of the equation by $W^-_\chi(\alpha,\gamma)\eta^-_{\chi,0}(\alpha)\mathcal{A}_{\gamma,P,0}(\alpha-\chi)$ and applying (6.2) yields (6.3), completing the proof.

Remark 6.10. We illustrate our arguments by deducing some integral identities on hypergeometric functions from our shift equations. These will not be used in the remainder of the paper. For $n = 2$, the shift equation (6.3) for $\chi = \frac{\gamma}{2}$ becomes

$$\widetilde{\mathcal{A}}_{\gamma,P,2}(\alpha - \frac{\gamma}{2}) + \frac{\eta^-_{\frac{\gamma}{2},2}(\alpha)}{\eta^-_{\frac{\gamma}{2},0}(\alpha)} = \frac{\Gamma_{2,2}\Gamma_{0,1}}{\Gamma_{2,1}\Gamma_{0,2}}\widetilde{\mathcal{A}}_{\gamma,P,2}(\alpha + \frac{\gamma}{2}) + \frac{\Gamma_{2,2}\Gamma_{0,1}}{\Gamma_{2,1}\Gamma_{0,2}}\frac{\eta^+_{\frac{\gamma}{2},2}(\alpha)}{\eta^+_{\frac{\gamma}{2},0}(\alpha)} + X$$

for

$$X := \eta^-_{\frac{\gamma}{2},0}(\alpha)^{-1}\mathcal{A}_{\gamma,P,0}(\alpha - \frac{\gamma}{2})^{-1}\left(W^-_{\frac{\gamma}{2}}(\alpha,\gamma)^{-1}\frac{\Gamma_{n,2}}{\Gamma_{n,1}}\frac{1+e^{\pi\frac{\gamma}{2}P+\mathbf{i}\pi l_0}}{1-e^{\pi\frac{\gamma}{2}P-\mathbf{i}\pi l_0}}V^{\alpha,2}_{\frac{\gamma}{2},n} + W^-_{\frac{\gamma}{2}}(\alpha,\gamma)^{-1}V^{\alpha,1}_{\frac{\gamma}{2},n}\right).$$

By Proposition 4.2 and Theorem 5.1, we find that

$$\begin{aligned}\phi^{\alpha}_{\frac{\gamma}{2},0}(w) &= W^-_{\frac{\gamma}{2}}(\alpha,\gamma)\eta^-_{\frac{\gamma}{2},0}(\alpha)\mathcal{A}_{\gamma,P,0}(\alpha - \frac{\gamma}{2})\,{}_2F_1(A_{\frac{\gamma}{2},0}, B_{\frac{\gamma}{2},0}, C_{\frac{\gamma}{2}}; w)\\ &\quad + W^+_{\frac{\gamma}{2}}(\alpha,\gamma)\eta^+_0(\alpha)\mathcal{A}_{\gamma,P,0}(\alpha + \frac{\gamma}{2})w^{1-C_{\frac{\gamma}{2}}}\,{}_2F_1(1 + A_{\frac{\gamma}{2},0} - C_{\frac{\gamma}{2}}, 1 + B_{\frac{\gamma}{2},0} - C_{\frac{\gamma}{2}}, 2 - C_{\frac{\gamma}{2}}; w).\end{aligned}$$

Computing $V^{\alpha,j}_{\frac{\gamma}{2},2}$ using the fact from (B.5) that $\widetilde{\wp}_2(w) = 16\pi^2 w$, we find that

$$\begin{aligned}X = \frac{4l_0(l_0+1)}{1-C_{\frac{\gamma}{2}}}\int_0^1 &\frac{t}{t^{1-C_{\frac{\gamma}{2}}}(1-t)^{C_{\frac{\gamma}{2}}-A_{\frac{\gamma}{2},2}-B_{\frac{\gamma}{2},2}}}\\ &\left(t^{1-C_{\frac{\gamma}{2}}}\,{}_2F_1(1 + A_{\frac{\gamma}{2},2} - C_{\frac{\gamma}{2}}, 1 + B_{\frac{\gamma}{2},2} - C_{\frac{\gamma}{2}}, 2 - C_{\frac{\gamma}{2}}, t) - \frac{\Gamma_{2,2}}{\Gamma_{2,1}}\,{}_2F_1(A_{\frac{\gamma}{2},2}, B_{\frac{\gamma}{2},2}, C_{\frac{\gamma}{2}}, t)\right)\\ &\left({}_2F_1(A_{\frac{\gamma}{2},0}, B_{\frac{\gamma}{2},0}, C_{\frac{\gamma}{2}}; t) - \frac{\Gamma_{0,1}}{\Gamma_{0,2}}t^{1-C_{\frac{\gamma}{2}}}\,{}_2F_1(1 + A_{\frac{\gamma}{2},0} - C_{\frac{\gamma}{2}}, 1 + B_{\frac{\gamma}{2},0} - C_{\frac{\gamma}{2}}, 2 - C_{\frac{\gamma}{2}}; t)\right)dt.\end{aligned} \tag{6.12}$$

Theorem 2.13 yields $\widetilde{\mathcal{A}}_2(\alpha) = \mathcal{Z}_2(P,\alpha,\gamma) = -\alpha(Q - \frac{\alpha}{2}) + 2 + 4\frac{\frac{\alpha^2}{4}(Q-\frac{\alpha}{2})^2 - \frac{\alpha}{2}(Q-\frac{\alpha}{2})}{2Q^2+2P^2}$. Using the explicit expressions for $\eta^{\pm}$ from (5.3) and (5.4), we find that (6.3) for $n = 2$ and $\chi = \frac{\gamma}{2}$ implies that

$$X = \frac{8l_0(l_0+1)}{\gamma^2}\left[-1 + \frac{(4l_0+\gamma^2)(4l_0+\gamma^2+4)}{4\gamma^2(Q^2+P^2)} + \frac{\Gamma_{2,2}\Gamma_{0,1}}{\Gamma_{2,1}\Gamma_{0,2}}\Big(1 - \frac{(4l_0-\gamma^2)(4l_0+4-\gamma^2)}{4\gamma^2(Q^2+P^2)}\Big)\right],$$

which we verified numerically in Mathematica for a few generic values of α, γ, P. We do not know a direct method to evaluate the integral expression for X from (6.12).

6.2. **Proof of Proposition 6.4.** We will check that equation (6.2) can be written as

$$\mathcal{A}_{\gamma,P,0}(\alpha - \chi) = Y_0(\alpha,\chi)\mathcal{A}_{\gamma,P,0}(\alpha + \chi), \tag{6.13}$$

where $\mathcal{A}_{\gamma,P,0}(\alpha)$ is meromorphically extended in α as in Theorem 5.1 and

$$Y_0(\alpha,\chi) := e^{4\mathbf{i}\pi l_\chi - 2\mathbf{i}\pi\chi^2}e^{\pi\chi P}\Gamma(1-\frac{\gamma^2}{4})^{-\frac{2\chi}{\gamma}}\frac{\Gamma(\frac{2\chi}{\gamma} - l_\chi)\Gamma(1+2l_\chi-\chi^2)\Gamma(1+2l_\chi)}{\Gamma(1+l_\chi)\Gamma(1+l_\chi-\mathbf{i}\chi P)\Gamma(1+l_\chi+\mathbf{i}\chi P)}(\frac{4}{\gamma^2})^{\mathbf{1}_{\chi=\frac{2}{\gamma}}}. \tag{6.14}$$

We deduce Proposition 6.4 from (6.13) as follows. Let $A(\alpha)$ be the claimed expression for $\mathcal{A}_{\gamma,P,0}(\alpha)$ given by the right-hand side of (6.7). By using (B.11), we find that $A(\alpha - \chi) = Y_0(\alpha,\chi)A(\alpha+\chi)$. Recalling that $l_\chi = \frac{\chi^2}{2} - \frac{\alpha\chi}{2}$ from (3.1), we observe that $Y_0(\chi,\alpha)$ is meromorphic in $\alpha \in \mathbb{C}$ with countably many zeros and poles, so we can find $\alpha_0 \in (-\frac{4}{\gamma} + \chi, 2Q - \chi)$ such that $Y_0(\chi,\alpha)$ has no zeros or poles in the set $\alpha_0 + \mathbb{Z}\gamma + \mathbb{Z}\frac{4}{\gamma}$. Let $\Delta_0(\alpha) := \mathcal{A}_{\gamma,P,0}(\alpha) - cA(\alpha)$ where c is such that $\Delta_0(\alpha_0) = 0$. The same argument as in the proof of Proposition 6.3 implies that $\Delta_0(\alpha) = 0$ if $\gamma^2 \notin \mathbb{Q}$. Continuity in γ implies that $\Delta_0(\alpha) = 0$ for all $\gamma \in (0,2)$. Since $\mathcal{A}_{\gamma,P,0}(0) = A(0)$ by direct computation, we must have $c = 1$, meaning that $\mathcal{A}_{\gamma,P,0}(\alpha) = A(\alpha)$ for all $\gamma \in (0,2)$ and where all $\alpha \in (-\frac{4}{\gamma}, Q)$.

It remains to prove (6.13). By (6.2), it suffices to show that for $Y_0(\alpha,\chi)$ defined in (6.14), we have

$$-\frac{W^+_\chi(\alpha,\gamma)}{W^-_\chi(\alpha,\gamma)}\frac{\Gamma_{0,2}}{\Gamma_{0,1}}\frac{1+e^{\pi\chi P+\mathbf{i}\pi l_\chi}}{1-e^{\pi\chi P-\mathbf{i}\pi l_\chi}}\frac{\eta^+_{\chi,0}(\alpha)}{\eta^-_{\chi,0}(\alpha)} = Y_0(\alpha,\chi).$$

By (5.1) and (5.2) we have

$$-\frac{W_\chi^+(\alpha,\gamma)}{W_\chi^-(\alpha,\gamma)} = \frac{e^{2\mathbf{i}\pi l_\chi - 2\mathbf{i}\pi\chi^2}(2\pi e^{\mathbf{i}\pi})^{-\frac{1}{3}\left(\frac{\gamma l_\chi}{\chi}+\frac{2l_\chi}{\chi^2}-8l_\chi+\frac{6l_\chi^2}{\chi^2}\right)}\pi^{-2l_\chi-1}\frac{1-e^{2\pi\chi P-2\mathbf{i}\pi l_\chi}}{\chi(Q-\alpha)}\frac{\Gamma(\frac{\alpha\chi}{2}-\frac{\chi^2}{2}+\frac{2\chi}{\gamma})\Gamma(1-\alpha\chi)\Gamma(\alpha\chi-\chi^2)}{\Gamma(\frac{\alpha\chi}{2}-\frac{\chi^2}{2})\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}}}{(2\pi e^{\mathbf{i}\pi})^{-\frac{1}{3}\left(2+\frac{2\gamma l_\chi}{\chi}+\frac{4l_\chi}{\chi\gamma}+\frac{6l_\chi^2}{\chi^2}\right)}}\left(\frac{4}{\gamma^2}\right)^{\mathbf{1}_{\chi=\frac{2}{\gamma}}}.$$

By the reflection and duplication formulas for the gamma function (see (B.7) and (B.8)), we have

$$\begin{aligned}\frac{\Gamma_{0,2}}{\Gamma_{0,1}} &= \frac{\Gamma(2-C_\chi)\Gamma(C_\chi-A_{\chi,0})\Gamma(C_\chi-B_{\chi,0})}{\Gamma(C_\chi)\Gamma(1-A_{\chi,0})\Gamma(1-B_{\chi,0})}\\ &= \frac{\Gamma(\frac{3}{2}+l_\chi)\Gamma(\frac{1}{2}-\frac{1}{2}l_\chi-\mathbf{i}\frac{\chi P}{2})\Gamma(\frac{1}{2}-\frac{1}{2}l_\chi+\mathbf{i}\frac{\chi P}{2})}{\Gamma(\frac{1}{2}-l_\chi)\Gamma(1+\frac{1}{2}l_\chi-\mathbf{i}\frac{\chi P}{2})\Gamma(1+\frac{1}{2}l_\chi+\mathbf{i}\frac{\chi P}{2})} = \frac{2^{2l_\chi}}{\pi^3}\frac{\Gamma(\frac{3}{2}+l_\chi)\cos(\frac{\pi}{2}l_\chi-\mathbf{i}\pi\frac{\chi P}{2})\cos(\frac{\pi}{2}l_\chi+\mathbf{i}\pi\frac{\chi P}{2})}{\Gamma(\frac{1}{2}-l_\chi)\Gamma(1+l_\chi-\mathbf{i}\chi P)\Gamma(1+l_\chi+\mathbf{i}\chi P)}.\end{aligned}$$

By (5.3) and (5.4) we have $\frac{\eta^+_{\chi,0}(\alpha)}{\eta^-_{\chi,0}(\alpha)} = (2\pi e^{\mathbf{i}\pi})^{-\frac{4}{3}l_\chi-\frac{2}{3}}$. Putting these together, we find that

$$-\frac{W_\chi^+(\alpha,\gamma)}{W_\chi^-(\alpha,\gamma)}\frac{\Gamma_{0,2}}{\Gamma_{0,1}}\frac{1+e^{\pi\chi P+\mathbf{i}\pi l_\chi}}{1-e^{\pi\chi P-\mathbf{i}\pi l_\chi}}\frac{\eta^+_{\chi,0}(\alpha)}{\eta^-_{\chi,0}(\alpha)} = U^1_{\chi,\gamma,P}(\alpha)U^2_{\chi,\gamma,P}(\alpha)$$

for

$$\begin{aligned}U^1_{\chi,\gamma,P}(\alpha) &= e^{2\mathbf{i}\pi l_\chi-2\mathbf{i}\pi\chi^2}(2\pi e^{\mathbf{i}\pi})^{-\frac{1}{3}\left(\frac{\gamma l_\chi}{\chi}+\frac{2l_\chi}{\chi^2}-8l_\chi+\frac{6l_\chi^2}{\chi^2}\right)}\pi^{-2l_\chi-1}(2\pi e^{\mathbf{i}\pi})^{\frac{1}{3}\left(2+\frac{2\gamma l_\chi}{\chi}+\frac{4l_\chi}{\chi\gamma}+\frac{6l_\chi^2}{\chi^2}\right)}(2\pi e^{\mathbf{i}\pi})^{-\frac{4}{3}l_\chi-\frac{2}{3}}\\ &= \pi^{-1}2^{2l_\chi}e^{4\mathbf{i}\pi l_\chi-2\mathbf{i}\pi\chi^2}\\ U^2_{\chi,\gamma,P}(\alpha) &= \frac{(1+e^{\pi\chi P-\mathbf{i}\pi l_\chi})(1+e^{\pi\chi P+\mathbf{i}\pi l_\chi})}{\cos(\frac{\pi}{2}l_\chi-\mathbf{i}\pi\frac{\chi P}{2})\cos(\frac{\pi}{2}l_\chi+\mathbf{i}\pi\frac{\chi P}{2})}\frac{\Gamma(\frac{2\chi}{\gamma}-l_\chi)\Gamma(1+2l_\chi-\chi^2)\Gamma(-2l_\chi)}{(1+2l_\chi)\Gamma(-l_\chi)\Gamma(1-\frac{\gamma^2}{4})^{\frac{2\chi}{\gamma}}}\\ &\quad\frac{2^{2l_\chi}\pi\Gamma(\frac{3}{2}+l_\chi)}{\Gamma(\frac{1}{2}-l_\chi)\Gamma(1+l_\chi-\mathbf{i}\chi P)\Gamma(1+l_\chi+\mathbf{i}\chi P)}\left(\frac{4}{\gamma^2}\right)^{\mathbf{1}_{\chi=\frac{2}{\gamma}}}\\ &= e^{\pi\chi P}2^{-2l_\chi}\pi\Gamma(1-\frac{\gamma^2}{4})^{-\frac{2\chi}{\gamma}}\frac{\Gamma(\frac{2\chi}{\gamma}-l_\chi)\Gamma(1+2l_\chi-\chi^2)\Gamma(1+2l_\chi)}{\Gamma(1+l_\chi)\Gamma(1+l_\chi-\mathbf{i}\chi P)\Gamma(1+l_\chi+\mathbf{i}\chi P)}\left(\frac{4}{\gamma^2}\right)^{\mathbf{1}_{\chi=\frac{2}{\gamma}}}.\end{aligned}$$

Since $Y_0(\alpha,\chi) = U^1_{\chi,\gamma,P}(\alpha)U^2_{\chi,\gamma,P}(\alpha)$, we conclude the proof.

6.3. **Proof of Proposition 6.9.** Proposition 6.9 follows from Proposition 6.11 below and the fact that

$$\mathcal{A}^q_{\gamma,P_{m,n}}(\alpha) = \prod_{k=1}^m \frac{\mathcal{A}^q_{\gamma,P_{m-2k+2,n}}(\alpha)}{\mathcal{A}^q_{\gamma,P_{m-2k,n}}(\alpha)}\mathcal{A}^q_{\gamma,P_{-m,n}}(\alpha).$$

We now prove Proposition 6.11.

Proposition 6.11. *If $N = -\frac{\alpha}{\gamma} \in \mathbb{N}$, for $\gamma \in (0,2)$ and $\alpha \in (-\frac{4}{\gamma}, Q)$, viewing $P \mapsto \mathcal{A}^q_{\gamma,P}(\alpha)$ as an entire function, we have*

$$\mathcal{A}^q_{\gamma,P_{m,n}}(\alpha) = q^{2n+(m-1)\frac{\gamma^2}{2}}e^{-\frac{\mathbf{i}\pi\alpha\gamma}{2}}\mathcal{A}^q_{\gamma,P_{m-2,n}}(\alpha) \qquad \text{for } (m,n)\in\mathbb{Z}^2. \tag{6.15}$$

Proof. Recall the domain $D = \{x+\tau y : x\in(0,1) \text{ or } y\in(0,1)\}$ from (B.17) and define on D the functions

$$\begin{aligned}g_{m,n}(u) &:= q^{\frac{\alpha^2}{24}-\frac{\alpha}{12}Q+\frac{1}{6}}\eta(q)^{\frac{5}{4}\alpha\gamma+\frac{2\alpha}{\gamma}-\frac{5}{4}\alpha^2-2}\Theta_\tau(u)^{-\frac{\alpha\gamma}{2}+\frac{m\gamma^2}{4}+\frac{\alpha\gamma}{4}}\Theta_\tau(1-u)^{-\frac{m\gamma^2}{4}-\frac{\alpha\gamma}{4}}e^{\pi\gamma P_{m,n}u};\\ f(P,u) &:= \left(\int_0^1\right)^{N-1}\prod_{1\le i<j\le N-1}|\Theta_\tau(x_i-x_j)|^{-\frac{\gamma^2}{2}}\\ &\qquad\prod_{i=1}^{N-1}\Theta_\tau(x_i-u)^{-\frac{\gamma^2}{4}}\Theta_\tau(u-x_i)^{-\frac{\gamma^2}{4}}\Theta_\tau(x_i)^{-\frac{\alpha\gamma}{2}}e^{\pi\gamma P x_i}\prod_{i=1}^{N-1}dx_i,\end{aligned}$$

where we interpret fractional powers of Θ via Appendix B.4.

By the Dotsenko-Fateev integral expression (6.9), for $N \in \mathbb{N}$ we have

$$\mathcal{A}^q_{\gamma,P_{m,n}}(\alpha) = e^{\mathbf{i}\pi(N-1)\frac{\gamma^2}{4}} \int_0^1 g_{m,n}(u) f(P_{m,n}, u) du. \tag{6.16}$$

Define the fundamental domain $\mathbb{T}_0$ to be the parallelogram bounded by $0, 1, \tau, 1+\tau$. We see that both $f(P,u)$ and $g_{m,n}(u)$ are holomorphic in u on the interior of $\mathbb{T}_0$, so integrating along a contour limiting to the boundary of $\mathbb{T}_0$, we conclude that

$$\int_0^1 g_{m,n}(u) f(P,u) du + \int_1^{1+\tau} g_{m,n}(u) f(P,u) du - \int_0^{\tau} g_{m,n}(u) f(P,u) du - \int_{\tau}^{1+\tau} g_{m,n}(u) f(P,u) = 0. \tag{6.17}$$

By (B.19), we have $f(P, u+1) = f(P,u)$ if $u \in \{x + y\tau : x \in \mathbb{R}, y \in (0,1)\}$. Moreover, since $\pi\gamma P_{m,n} - \frac{\mathbf{i}\pi m\gamma^2}{2} = 2\pi\mathbf{i}n$, we find

$$g_{m,n}(u+1) = e^{\pi\gamma P_{m,n} + \mathbf{i}\pi(\frac{\alpha\gamma}{2} - \frac{m\gamma^2}{2} - \frac{\alpha\gamma}{2})} g_{m,n}(u) = g_{m,n}(u), \qquad \text{if } u \in \{x + y\tau : y \in (0,1)\}.$$

Therefore $\int_0^\tau g_{m,n}(u) f(P,u) du = \int_1^{1+\tau} g_{m,n}(u) f(P,u) du$, and thus (6.17) implies that

$$\int_0^1 g_{m,n}(u) f(P,u) du = \int_\tau^{1+\tau} g_{m,n}(u) f(P,u) du = \int_0^1 g_{m,n}(u+\tau) f(P, u+\tau) du.$$

By a direct computation using (B.20), we find that if $u \in \{x + y\tau : x \in (0,1), y \in \mathbb{R}\}$, then

$$\begin{aligned} g_{m,n}(u+\tau) &= e^{\pi P_{m,n}\gamma\tau} e^{-2\pi\mathbf{i}(-\frac{\alpha\gamma}{2} + \frac{m\gamma^2}{4} + \frac{\alpha\gamma}{4})(u - \frac{1}{2} + \frac{\tau}{2})} e^{-2\pi\mathbf{i}(-\frac{m\gamma^2}{4} - \frac{\alpha\gamma}{4})(u - \frac{1}{2} + \frac{\tau}{2})} g_{m,n}(u) \\ &= e^{\pi P_{m,n}\gamma\tau} e^{\mathbf{i}\pi\alpha\gamma(u - \frac{1}{2} + \frac{\tau}{2})} g_{m,n}(u) \\ f(P, u+\tau) &= e^{(N-1)(\mathbf{i}\pi\gamma^2 u + \frac{\mathbf{i}\pi\gamma^2\tau}{2})} f(P - \mathbf{i}\gamma, u). \end{aligned}$$

Combining these, we find that if $u \in \{x + y\tau : x \in (0,1), y \in \mathbb{R}\}$, then

$$\begin{aligned} g_{m,n}(u+\tau) f(P_{m,n}, u+\tau) &= e^{\pi P_{m,n}\gamma\tau} e^{\mathbf{i}\pi\alpha\gamma(u - \frac{1}{2} + \frac{\tau}{2})} e^{(N-1)(\mathbf{i}\pi\gamma^2 u + \frac{\mathbf{i}\pi\gamma^2\tau}{2})} g_{m,n}(u) f(P_{m,n} - \mathbf{i}\gamma, u) \\ &= q^{2n + (m-1)\frac{\gamma^2}{2}} e^{-\frac{\mathbf{i}\pi\alpha\gamma}{2}} g_{m-2,n}(u) f(P_{m-2,n}, u). \end{aligned}$$

Integrating both sides over $[0,1]$ and recalling (6.16), we obtain (6.15). ☐

6.4. **Proof of Theorem 6.5.** The function $\mathcal{A}^q_{\gamma,P}(\alpha)$ may be analytically extended to $P \in \mathbb{C}$ via the Dotsenko-Fateev integral (6.9) when $-\frac{\alpha}{\gamma} \in \mathbb{N}$, hence the function $P \to \widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha) = \frac{\mathcal{A}^q_{\gamma,P}(\alpha)}{\mathcal{A}_{\gamma,P,0}(\alpha)}$ meromorphically extends to $P \in \mathbb{C}$. In this subsection, we first state four lemmas characterizing the pole structure of $\widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha)$ and then prove Theorem 6.5 assuming these lemmas. We then devote most of the subsection to the proof of these lemmas.

Lemma 6.12. *For $\gamma \in (0,2)$, $\alpha \in (-\frac{4}{\gamma}, 0)$, $q \in (0,1)$, and $N = -\frac{\alpha}{\gamma} \in \mathbb{N}$, we have $\widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha) = \widetilde{\mathcal{A}}^q_{\gamma,-P}(\alpha)$ for all $P \in \mathbb{C}$.*

Lemma 6.13. *For $\gamma \in (0,2)$, $\alpha \in (-\frac{4}{\gamma}, 0)$, $q \in (0,1)$, and $N = -\frac{\alpha}{\gamma} \in \mathbb{N}$, the function $P \mapsto \mathcal{A}_{\gamma,P,0}(\alpha)^{-1}$ is meromorphic with poles only at $P = \pm P_{m,n}$ for $n \in \mathbb{N}$ and $1 \leq m \leq N$. Moreover, the pole at $P_{m,n}$ is simple and has residue $\operatorname{Res}_{P=P_{m,n}} \widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha)$ given by*

$$\operatorname*{Res}_{P=P_{m,n}} \widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha) = q^{2nm} \frac{1}{2P_{m,n}} R_{\gamma,m,n}(\alpha) \widetilde{\mathcal{A}}^q_{\gamma,P_{-m,n}}(\alpha) \quad \text{for } n \in \mathbb{N} \text{ and } 1 \leq m \leq N. \tag{6.18}$$

Lemma 6.14. *For $\gamma \in (0,2)$, $\alpha \in (-\frac{4}{\gamma}, 0)$, $q \in (0,1)$, and $N = -\frac{\alpha}{\gamma} \in \mathbb{N}$, $\widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha)$ is a rational function in P for each $n \in \mathbb{N}$.*

Lemma 6.15. *For $\gamma \in (0,2)$, $\alpha \in (-\frac{4}{\gamma}, 0)$, $q \in (0,1)$, and $N = -\frac{\alpha}{\gamma} \in \mathbb{N}$, the limit $\lim_{\mathbb{R} \ni P \to -\infty} \widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha)$ exists and equals the coefficient a_n in the expansion $\sum_{n=0}^\infty a_n q^n$ of the analytic function $[q^{-\frac{1}{12}}\eta(q)]^{\alpha(Q - \frac{\alpha}{2}) - 2}$ near $q = 0$.*

Assuming these lemmas, we now prove Theorem 6.5. Notice that $\widetilde{\mathcal{A}}^q_{\gamma,P,k}(\alpha)$ is a rational function in P by Lemma 6.14. The poles of this function must be located at $P = \pm P_{m,n}$ by Lemma 6.13, hence by Lemmas 6.12 and 6.15, we find that

$$\widetilde{\mathcal{A}}_{\gamma,P,k}(\alpha) = \sum_{n,m=1}^{\infty} \frac{2P_{m,n} \operatorname*{Res}_{P=P_{m,n}} \widetilde{\mathcal{A}}_{\gamma,P,k}(\alpha)}{P^2 - P_{m,n}^2} + a_k,$$

where the sum $\sum_{n,m=1}^{\infty}$ above contains only finitely many non-zero summands. Lemma 6.13 implies that $\operatorname*{Res}_{P=P_{m,n}} \widetilde{\mathcal{A}}_{\gamma,P,k}(\alpha)$ is non-zero only if $k \geq 2mn$. Applying Cauchy Residue Theorem around $P = P_{m,n}$, we have $\operatorname{Res}_{P=P_{m,n}} \widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha) = \sum_{k=0}^{\infty} \operatorname*{Res}_{P=P_{m,n}} \widetilde{\mathcal{A}}_{\gamma,P,k}(\alpha) q^k$ as a convergent series. Therefore (6.18) yields that

$$\widetilde{\mathcal{A}}_{\gamma,P,k}(\alpha) = \sum_{n,m\in\mathbb{N}, 2mn\leq k} \frac{R_{\gamma,m,n}(\alpha)}{P^2 - P_{m,n}^2} \widetilde{\mathcal{A}}_{\gamma,P_{-m,n},k-2mn}(\alpha) + a_k,$$

which implies that the q^k coefficients of both sides of (6.8) are equal. This implies that (6.8) holds as an equality of formal q-power series, yielding Theorem 6.5.

In the rest of this subsection, we prove Lemmas 6.12–6.15 in order.

Proof of Lemma 6.12. Recall the definition of $\mathcal{A}^q_{\gamma,P}(\alpha)$ from (2.11). For $P \in \mathbb{R}$, we have

$$\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P x} dx\right)^{-\frac{\alpha}{\gamma}}\right] = \mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(1-x)} \Theta_\tau(1-x)^{-\frac{\alpha\gamma}{2}} e^{\pi\gamma P(1-x)} dx\right)^{-\frac{\alpha}{\gamma}}\right].$$

Since $\{Y_\tau(1-x)\}_{0\leq x\leq 1}$ and $\{Y_\tau(x)\}_{0\leq x\leq 1}$ are equal in law and $\Theta_\tau(1-x) = \Theta_\tau(x)$, for $P \in \mathbb{R}$ we see that $\mathcal{A}^q_{\gamma,P}(\alpha) = e^{-\pi P\alpha} \mathcal{A}^q_{\gamma,-P}(\alpha)$. This implies $\mathcal{A}_{\gamma,P,0}(\alpha) = C\mathcal{A}_{\gamma,-P,0}(\alpha)$ and hence $\widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha) = \widetilde{\mathcal{A}}^q_{\gamma,-P}(\alpha)$ for $P \in \mathbb{R}$. Because $\mathcal{A}^q_{\gamma,P}(\alpha)$ is meromorphic in P for $-\frac{\alpha}{\gamma} = N \in \mathbb{N}$, the same holds for all $P \in \mathbb{C}$. □

Proof of Lemma 6.13. Specializing Proposition 6.4 to $N = -\frac{\alpha}{\gamma} \in \mathbb{N}$, we get

$$\mathcal{A}_{\gamma,P,0}(\alpha) = e^{\frac{\mathbf{i}\pi\gamma^2 N^2}{2}} \frac{e^{\frac{\pi\gamma PN}{2}}}{\Gamma(1-\frac{\gamma^2}{4})^N} \prod_{j=1}^{N} \frac{\Gamma(1-\frac{j\gamma^2}{4})\Gamma(1+\frac{(2N-j+1)\gamma^2}{4})}{\Gamma(1+\frac{j\gamma^2}{4}+\frac{\mathbf{i}\gamma P}{2})\Gamma(1+\frac{j\gamma^2}{4}-\frac{\mathbf{i}\gamma P}{2})}. \tag{6.19}$$

Recall that the Γ function has simple poles at $\{0,-1,-2,\ldots\}$ and has no zeros. Since $\frac{\mathbf{i}\gamma P_{m,n}}{2} = -\frac{m\gamma^2}{4} - n$, equation (6.19) yields that $\mathcal{A}_{\gamma,P,0}(\alpha)$ has simple zeros at $P = \pm P_{m,n}$ for $n \in \mathbb{N}$ and $1 \leq m \leq N$. This and the fact that $\mathcal{A}_{\gamma,P,n}(\alpha)$ is analytic in P yield the claimed pole structure of $\widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha)$.

We now compute the residue of $\mathcal{A}_{\gamma,P,0}(\alpha)^{-1}$ at each of its poles. Define the function

$$f(P) := \prod_{j=1}^{N} \Gamma(1+\frac{j\gamma^2}{4}+\frac{\mathbf{i}\gamma P}{2})\Gamma(1+\frac{j\gamma^2}{4}-\frac{\mathbf{i}\gamma P}{2}) \tag{6.20}$$

so that for some C independent of P we have

$$\mathcal{A}_{\gamma,P,0}(\alpha)^{-1} = Ce^{-\frac{\pi\gamma PN}{2}} f(P). \tag{6.21}$$

For $n \in \mathbb{N}$ and $1 \leq m \leq N$, we have

$$\begin{aligned}\operatorname*{Res}_{P=P_{m,n}} f(P) &= \prod_{j=1}^{N} \Gamma(1+\frac{j\gamma^2}{4}-\frac{\mathbf{i}\gamma P_{m,n}}{2}) \prod_{j=1,j\neq m}^{N} \Gamma(1+\frac{j\gamma^2}{4}+\frac{\mathbf{i}\gamma P_{m,n}}{2}) \operatorname*{Res}_{P=P_{m,n}} \Gamma(1+\frac{m\gamma^2}{4}+\frac{\mathbf{i}\gamma P}{2})\\ &= \frac{2}{\mathbf{i}\gamma}\frac{(-1)^{n-1}}{(n-1)!} \prod_{j=1}^{N} \Gamma(1+\frac{j\gamma^2}{4}+m\frac{\gamma^2}{4}+n) \prod_{j=1,j\neq m}^{N} \Gamma(1+\frac{j\gamma^2}{4}-m\frac{\gamma^2}{4}-n),\end{aligned}$$

where we note that $\frac{\mathbf{i}\gamma P_{m,n}}{2} = -\frac{m\gamma^2}{4} - n$ and

$$\operatorname*{Res}_{P=P_{m,n}} \Gamma(1+\frac{m\gamma^2}{4}+\frac{\mathbf{i}\gamma P}{2}) = \frac{2}{\mathbf{i}\gamma} \operatorname*{Res}_{x=1-n} \Gamma(x) = \frac{2}{\mathbf{i}\gamma}\frac{(-1)^{n-1}}{(n-1)!}.$$

Recall $R_{\gamma,m,n}(\alpha)$ defined in (2.20). Using $\Gamma(x+n) = \Gamma(x-n)\prod_{l=-n}^{n-1}(x+l)$, we now compute

$$\begin{aligned}\frac{\mathrm{Res}_{P=P_{m,n}} f(P)}{f(P_{-m,n})} &= \frac{2}{\mathbf{i}\gamma}\frac{(-1)^{n-1}}{(n-1)!}\frac{\prod_{j=1}^{N}\prod_{l=-n}^{n-1}(1+\frac{j\gamma^2}{4}+\frac{m\gamma^2}{4}+l)}{n!\prod_{j=1,j\neq m}^{N}\prod_{l=-n}^{n-1}(1+\frac{j\gamma^2}{4}-\frac{m\gamma^2}{4}+l)}\\ &= \frac{2}{\mathbf{i}\gamma}\frac{(-1)^{n-1}}{(n-1)!}\frac{\prod_{j=-m}^{m-1}\prod_{l=-n}^{n-1}(1+\frac{\gamma^2}{4}+\frac{N\gamma^2}{4}+\frac{j\gamma^2}{4}+l)}{n!\prod_{j=1-m,j\neq 0}^{m}\prod_{l=-n}^{n-1}(1+\frac{j\gamma^2}{4}+l)},\end{aligned} \tag{6.22}$$

where we used $\frac{\prod_{j=1}^{N}(1+\frac{j\gamma^2}{4}+\frac{m\gamma^2}{4}+l)}{\prod_{j=1,j\neq m}^{N}(1+\frac{j\gamma^2}{4}-\frac{m\gamma^2}{4}+l)} = \frac{\prod_{j=-m}^{m-1}(1+\frac{\gamma^2}{4}+\frac{N\gamma^2}{4}+\frac{j\gamma^2}{4}+l)}{\prod_{j=1-m,j\neq 0}^{m}(1+\frac{j\gamma^2}{4}+l)}$ for each m, l, N. Note that

$$\prod_{j=-m}^{m-1}\prod_{l=-n}^{n-1}(1+\frac{\gamma^2}{4}+\frac{N\gamma^2}{4}+\frac{j\gamma^2}{4}+l) = (\frac{\gamma}{2})^{4mn}\prod_{j=-m}^{m-1}\prod_{l=-n}^{n-1}(Q-\frac{\alpha}{2}+\frac{j\gamma}{2}+\frac{2l}{\gamma}). \tag{6.23}$$

Since $(-1)^{n-1}(n-1)!n! = \prod_{l=-n,l\neq -1}^{n-1}(1+l)$, we have

$$(-1)^{n-1}(n-1)!n!\prod_{j=1-m,j\neq 0}^{m}\prod_{l=-n}^{n-1}(1+\frac{j\gamma^2}{4}+l) = (\frac{m\gamma^2}{4}+n)\prod_{(j,l)\in S_{m,n}}(1+\frac{j\gamma^2}{4}+l), \tag{6.24}$$

where we recall the definition of $S_{m,n}$ from below (2.20). Combining (2.20), (6.22), (6.23), and (6.24) yields

$$\frac{\mathrm{Res}_{P=P_{m,n}} f(P)}{f(P_{-m,n})} = \frac{1}{2P_{m,n}}R_{\gamma,m,n}(\alpha).$$

Since $e^{-\frac{\pi\gamma(P_{m,n}-P_{-m,n})N}{2}} = e^{\frac{\mathbf{i}\pi\alpha\gamma m}{2}}$ for $N = -\frac{\alpha}{\gamma}$, by (6.21), for $n\in\mathbb{N}$ and $1\le m\le N$ we have

$$\operatorname*{Res}_{P=P_{m,n}}\mathcal{A}_{\gamma,P,0}(\alpha)^{-1} = e^{\frac{\mathbf{i}\pi\alpha\gamma m}{2}}\frac{1}{2P_{m,n}}R_{\gamma,m,n}(\alpha)\mathcal{A}_{\gamma,P_{-m,n},0}(\alpha)^{-1}.$$

Combining this with $\mathcal{A}^q_{\gamma,P_{m,n}}(\alpha) = q^{2nm}e^{-\frac{\mathbf{i}\pi\alpha\gamma m}{2}}\mathcal{A}^q_{\gamma,P_{-m,n}}(\alpha)$ from Proposition 6.9, we obtain (6.18). □

Proof of Lemma 6.14. By (6.18), we must have $\mathrm{Res}_{P=P_{m,n}}\widetilde{\mathcal{A}}_{\gamma,P,k}(\alpha) = 0$ for $n,k\in\mathbb{N}$ and $1\le m\le N$ such that $2mn > k$. By Lemma 6.12, $\mathrm{Res}_{P=-P_{m,n}}\widetilde{\mathcal{A}}_{\gamma,P,k}(\alpha) = 0$ as well. Because all poles are located at $P = \pm P_{m,n}$ for some m,n by Lemma 6.13, the meromorphic function $P\mapsto\widetilde{\mathcal{A}}_{\gamma,P,k}$ has finitely many poles.

We now show that $\widetilde{\mathcal{A}}_{\gamma,P,k}(\alpha)$ has polynomial growth at ∞ and is therefore rational. Let

$$r := \min\{|P_{m,n}-P_{m',n'}| : 1\le m,m'\le N,\quad n,n'\in\mathbb{N},\quad \text{and } (m,n)\neq(m',n')\},$$

which is positive. For $1\le m\le N$ and $n\in\mathbb{N}$, let ${\mathcal{B}^+}_{m,n}$ (resp. $\mathcal{B}^-_{m,n}$) be the ball around $P_{m,n}$ (resp. $-P_{m,n}$) with radius $\frac{r}{3}$, so that $\mathcal{B}^{\pm}_{m,n}\cap\mathcal{B}^{\pm}_{m',n'} = \emptyset$ for $(m,n)\neq(m',n')$. Define $\mathbb{C}^\circ := \mathbb{C}\setminus\cup_{1\le m\le N,n\ge 1}({\mathcal{B}^+}_{m,n}\cup\mathcal{B}^-_{m,n})$. Recall from (6.21) that $\mathcal{A}_{\gamma,P,0}(\alpha)^{-1} = C(\alpha,\gamma)e^{-\pi N\gamma P/2}f(P)$ for $f(P)$ in (6.20) and some explicit function $C(\alpha,\gamma)$. We claim that there exists $K\in\mathbb{N}$ such that

$$M := \sup_{P\in\mathbb{C}^\circ}|P|^{-K}e^{\pi N\gamma|\,\mathrm{Re}(P)|/2}|f(P)| < \infty. \tag{6.25}$$

Given (6.25), we can prove Lemma 6.14 as follows. For $\mathrm{Re}\,P\le 0$ and $P\in\mathbb{C}^\circ$ we have

$$|P|^{-K}|\mathcal{A}_{\gamma,P,0}(\alpha)^{-1}| = C(\alpha,\gamma)|P|^{-K}e^{-\pi N\gamma\mathrm{Re}(P)/2}|f(P)| \le MC(\alpha,\gamma).$$

On the other hand, since $|e^{\pi\gamma Px}|\le 1$ for $\mathrm{Re}\,P\le 0$, we have $|\mathcal{A}^q_{\gamma,P}(\alpha)| < \infty$, where the bound only depends on $\alpha,\gamma,|q|$ and is uniform in $\mathrm{Re}\,P\le 0$. Applying Cauchy's theorem in q to extract q-series coefficients, for each $k\in\mathbb{N}$, we get $C_k(\alpha,\gamma) := \sup_{\mathrm{Re}\,P\le 0}|\mathcal{A}_{\gamma,P,k}(\alpha)| < \infty$. Since $\widetilde{\mathcal{A}}_{\gamma,P,k}(\alpha) = \widetilde{\mathcal{A}}_{\gamma,-P,k}(\alpha)$, we further get $\sup_{P\in\mathbb{C}^\circ}|P|^{-K}|\widetilde{\mathcal{A}}_{\gamma,P,k}(\alpha)| < C_k$ with $C_k = MC(\alpha,\gamma)C_k(\alpha,\gamma)$. Note that $P^{-K}\widetilde{\mathcal{A}}_{\gamma,P,k}(\alpha)$ is analytic for large enough $|P|$ because it has finitely many poles. By the maximal modulus theorem, $|P^{-K}\widetilde{\mathcal{A}}_{\gamma,P,k}(\alpha)|\le C_k$ for $|P|$ sufficiently large. We conclude that $P^{-K}\widetilde{\mathcal{A}}_{\gamma,P,k}(\alpha)$ is a rational function and hence $\widetilde{\mathcal{A}}_{\gamma,P,k}(\alpha)$ is as well.

It remains to prove (6.25). Note that $\Gamma(1+\frac{j\gamma^2}{4}+\frac{\mathbf{i}\gamma P}{2}) = \pi\sin(\pi(1+\frac{j\gamma^2}{4}+\frac{\mathbf{i}\gamma P}{2}))^{-1}\Gamma(-\frac{j\gamma^2}{4}-\frac{\mathbf{i}\gamma P}{2})^{-1}$ by (B.7). Moreover, we have $\max_{P\in\mathbb{C}^\circ}e^{\pi\gamma|\,\mathrm{Re}(P)|/2}|\sin(\pi(1+\frac{j\gamma^2}{4}+\frac{\mathbf{i}\gamma P}{2}))^{-1}| < \infty$ by our choice of $\mathbb{C}^\circ$. Since

$f(P) = f(-P)$, to prove (6.25) it suffices to show that for each $1 \leq j \leq N$, there exist some $C_j > 0$ and $K_j \in \mathbb{N}$ such that

$$\left| \frac{\Gamma(1 + \frac{j\gamma^2}{4} - \frac{\mathbf{i}\gamma P}{2})}{\Gamma(-\frac{j\gamma^2}{4} - \frac{\mathbf{i}\gamma P}{2})} \right| \leq C_j |P|^{K_j} \quad \text{for } \operatorname{Im} P \geq 0. \tag{6.26}$$

Because $\frac{\Gamma(1+\frac{j\gamma^2}{4}-\frac{\mathbf{i}\gamma P}{2})}{\Gamma(-\frac{j\gamma^2}{4}-\frac{\mathbf{i}\gamma P}{2})}$ is analytic in P for $\operatorname{Im} P \geq 0$, it suffices to check that it is polynomially bounded for $|P|$ large. By Stirling's approximation (B.9), $\Gamma(z) \sim \sqrt{\frac{2\pi}{z}} e^{-z} z^z (1 + O(|z|^{-1}))$ as $|z| \to \infty$ with $\operatorname{Re} z \geq 0$. Under the assumption that $\operatorname{Im} P \geq 0$, Stirling's approximation applies to $\Gamma(1 + \frac{j\gamma^2}{4} - \frac{\mathbf{i}\gamma P}{2})$ and $\Gamma(-\frac{j\gamma^2}{4} - \frac{\mathbf{i}\gamma P}{2})$ as $|P|$ grows large and yields that (6.26) holds if $K_j > 1 + \frac{j\gamma^2}{4} - (-\frac{j\gamma^2}{4}) = \frac{j\gamma^2}{2} + 1$, which implies (6.25) and concludes the proof. □

Proof of Lemma 6.15. Throughout the proof, we assume $P < 0$ and $N = \frac{-\alpha}{\gamma} \in \mathbb{N}$. Recall from (2.27) that $\widetilde{\mathcal{A}}^q_{\gamma,P}(\alpha) = (q^{-\frac{1}{12}}\eta(q))^{\alpha(Q-\frac{\alpha}{2})-2} \frac{\hat{\mathcal{A}}^q_{\gamma,P}(\alpha)}{\hat{\mathcal{A}}^0_{\gamma,P}(\alpha)}$ with

$$\hat{\mathcal{A}}^q_{\gamma,P}(\alpha) = \mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}(F_\tau(x) - F_\tau(0))} (2\sin(\pi x))^{-\alpha\gamma/2} e^{\pi\gamma P x} e^{\frac{\gamma}{2} Y_\infty(x)} dx \right)^{-\frac{\alpha}{\gamma}} \right] \tag{6.27}$$

as in Lemma 2.16 (our expression for $\hat{\mathcal{A}}^q_{\gamma,P}(\alpha)$ is a rewriting of its definition below (2.25)). For general $\alpha \in (-\frac{4}{\gamma}, Q)$, the equation (6.27) only holds for $q \in (0, r_\alpha)$ with r_α as in Lemma 2.9 (a). But under our assumption that $-\frac{\alpha}{\gamma} \in \mathbb{N}$, the right side of (6.27) is analytic in q, so (6.27) holds as long as $|q| < r_\alpha$.

For a fixed $r \in (0, r_\alpha)$, we claim that

$$\lim_{\mathbb{R} \ni P \to -\infty} \frac{\hat{\mathcal{A}}^q_{\gamma,P}(\alpha)}{\hat{\mathcal{A}}^0_{\gamma,P}(\alpha)} = 1 \text{ uniformly for } q \in \mathbb{C} \text{ with } |q| = r. \tag{6.28}$$

This implies the result by applying Cauchy's theorem in q on a circle of radius r to show that the n-th q-series coefficient of $\frac{\hat{\mathcal{A}}^q_{\gamma,P}(\alpha)}{\hat{\mathcal{A}}^0_{\gamma,P}(\alpha)}$ is 1 for $n = 0$ and 0 for $n \geq 1$.

It remains to establish (6.28). To lighten the notation, we define the measure

$$\mu_q(dx) := e^{\frac{\gamma}{2}(F_\tau(x) - F_\tau(0))} (2\sin(\pi x))^{-\alpha\gamma/2} e^{\pi\gamma P x} e^{\frac{\gamma}{2} Y_\infty(x)} dx \quad \text{on } [0,1].$$

Then $\hat{\mathcal{A}}^q_{\gamma,P}(\alpha) = \mathbb{E}[\mu_q([0,1])^N]$ for $|q| < r_\alpha$. For $\varepsilon \in (0,1)$, we have

$$\mathbb{E}[\mu_q([0,1])^N] = \sum_{i=0}^N \binom{N}{i} \mathbb{E}[\mu_q([0,\varepsilon])^i \mu_q([\varepsilon,1])^{N-i}]. \tag{6.29}$$

For $I \subset [0,1]$, let $M_r(I) := \sup_{x \in I, |q|=r} e^{\frac{\gamma}{2}|F_\tau(x) - F_\tau(0)|}$. Then $|\mu_q(I)| \leq M_r(I)\mu_0(I)$. By Holder's inequality and the independence of F_τ and Y_∞, we find

$$\begin{aligned} \left| \mathbb{E}[\mu_q([0,\varepsilon])^i \mu_q([\varepsilon,1])^{N-i}] \right| &\leq \mathbb{E}[M_r([0,1])^N] \mathbb{E}[\mu_0([0,\varepsilon])^i \mu_0([\varepsilon,1])^{N-i}] \\ &\leq \mathbb{E}[M_r([0,1])^N] \mathbb{E}[\mu_0([0,\varepsilon])^N] \cdot \left(\frac{\mathbb{E}[\mu_0([\varepsilon,1])^N]}{\mathbb{E}[\mu_0([0,\varepsilon])^N]} \right)^{1-\frac{i}{N}}. \end{aligned} \tag{6.30}$$

For $P < 0$, note that $\mathbb{E}[\mu_0([\varepsilon,1])^N] \leq e^{-\pi\gamma N|P|\varepsilon} \mathbb{E}\left[\left(\int_\varepsilon^1 (2\sin(\pi x))^{-\alpha\gamma/2} e^{\frac{\gamma}{2} Y_\infty(x)} dx \right)^N \right]$ and

$$\mathbb{E}[\mu_0([0,\varepsilon])^N] \geq e^{-\pi\gamma N|P|\varepsilon/2} \mathbb{E}\left[\left(\int_0^{\varepsilon/2} (2\sin(\pi x))^{-\alpha\gamma/2} e^{\frac{\gamma}{2} Y_\infty(x)} dx \right)^N \right].$$

Since $\mathbb{E}[M_r([0,1])^N] < \infty$ and $\mathbb{E}[\mu_0([0,1])^N] < \infty$, by (6.30) there exists a constant $C = C(\varepsilon, r)$ such that

$$\left| \mathbb{E}[\mu_q([0,\varepsilon])^i \mu_q([\varepsilon,1])^{N-i}] \right| \leq \mathbb{E}[\mu_0([0,1])^N] C(\varepsilon, r) e^{-\frac{1}{2}\pi\varepsilon\gamma|P|} \quad \text{for } i < N, |q| = r \text{ and } P < 0. \tag{6.31}$$

Applying (6.29) and (6.31), we have for a possibly enlarged $C(\varepsilon, r)$ that

$$\mathbb{E}[\mu_q([0,1])^N] = \mathbb{E}[\mu_q([0,\varepsilon])^N] + \mathbb{E}[\mu_0([0,1])^N] C(\varepsilon, r) e^{-\frac{1}{2}\pi\varepsilon\gamma|P|} \quad \text{for } |q| = r \text{ and } P < 0. \tag{6.32}$$

A similar argument shows that $\mathbb{E}[\mu_0([0,1])^N] = \mathbb{E}[\mu_0([0,\varepsilon])^N](1 + C(\varepsilon)e^{-\frac{1}{2}\pi\varepsilon\gamma|P|})$ for some $C(\varepsilon) > 0$.

Set $m_\varepsilon := \sup_{x\in I, |q|=r} \left| e^{\frac{\gamma}{2}(F_\tau(x) - F_\tau(0))} - 1 \right|$ so that $|\mu_q([0,\varepsilon]) - \mu_0([0,\varepsilon])| \le m_\varepsilon \mu_0([0,\varepsilon])$. Then

$$\begin{aligned} \left| \mu_q([0,\varepsilon])^N - \mu_0([0,\varepsilon])^N \right| &\le |\mu_q([0,\varepsilon]) - \mu_0([0,\varepsilon])| \times N M_r([0,1])^N \mu_0([0,\varepsilon])^{N-1} \\ &\le m_\varepsilon \times N M_r([0,1])^N \mu_0([0,\varepsilon])^N. \end{aligned}$$

By the dominated convergence theorem and the continuity of $F_\tau(x)$ at $x = 0$, we have $\lim_{\varepsilon\to 0} \mathbb{E}[m_\varepsilon M_r([0,1])^N] = 0$, hence $\lim_{\varepsilon \to 0} \frac{\mathbb{E}[\mu_q([0,\varepsilon])^N]}{\mathbb{E}[\mu_0([0,\varepsilon])^N]} = 1$ uniformly in $|q| = r$ and $P < 0$. Combined with (6.32), we get (6.28). □

6.5. **Proof of Theorem 2.13.** We will consider the different quantities as functions of χ. We say that a function $f(\chi)$ is χ-*good* if it admits meromorphic extension to a complex neighborhood of $[0, \infty)$. We say that a function $f(w, \chi)$ is (w, χ)-*good* if there exist a complex neighborhood of U of $[0, \infty)$, and sequences $\{z_k \in U\}_{k\in\mathbb{N}}$ and $\{m_k \in \mathbb{N}\}_{k\in\mathbb{N}}$ such that for each $w \in \overline{\mathbb{D}}$, $f(w, \chi)$ is meromorphic in χ on U with poles at z_k with multiplicity m_k, and moreover f admits an extension to $\overline{\mathbb{D}} \times (U \setminus \{z_k\}_{k\in\mathbb{N}})$ where f is (w, χ)-regular in the sense of Definition 4.10, with χ in place of α.

For the proof, for $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$ and $j \in \{1, 2\}$ we define the normalized expressions

$$\widetilde{G}^{\alpha,j}_{\chi,n,1}(w) := \frac{G^{\alpha,j}_{\chi,n,i}(w)}{W^-_\chi(\alpha,\gamma)\eta^-_{\chi,0}(\alpha)\mathcal{A}_{\gamma,P,0}(\alpha-\chi)} \quad \text{and} \quad \widetilde{\phi}^{\alpha,j}_{\chi,n,1}(w) := \frac{\phi^{\alpha,j}_{\chi,n,1}(w)}{W^-_\chi(\alpha,\gamma)\eta^-_{\chi,0}(\alpha)\mathcal{A}_{\gamma,P,0}(\alpha-\chi)}. \tag{6.33}$$

which are similar to $\widetilde{V}^{\alpha,j}_{\chi,n}$ defined in Theorem 6.1.

We will now induct on n to prove the following statements $(\mathrm{a})_n$ and $(\mathrm{b})_n$ indexed by $n \in \mathbb{N}_0$. Theorem 2.13 then follows from statement $(\mathrm{a})_n$.

- $(\mathrm{a})_n$: $\mathcal{Z}_{\gamma,P,n}(\alpha) = \widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha)$ for $\alpha \in (-\frac{4}{\gamma}, 2Q)$, $\gamma \in (0,2)$ and $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$.
- $(\mathrm{b})_n$: $(w,\chi) \mapsto \widetilde{\phi}^{\alpha,j}_{\chi,n,1}(w)$ is (w,χ)-good for each $\alpha \in \mathbb{R}$ and $j \in \{1,2\}$.

Normalizing (4.16) implies that for some $\widetilde{X}^j_{\chi,n}(\alpha)$ not depending on w, we have

$$\widetilde{\phi}^{\alpha,j}_{\chi,n,1}(w) = \widetilde{G}^{\alpha,j}_{\chi,n,1}(w) + \widetilde{X}^j_{\chi,n}(\alpha) v^\alpha_{j,\chi,n}(w) \text{ for } n \in \mathbb{N}_0 \text{ and } j = 1,2. \tag{6.34}$$

For $n = 0$, since $\mathcal{Z}_{\gamma,P,0}(\alpha) = \widetilde{\mathcal{A}}_{\gamma,P,0}(\alpha) = 1$, statement $(\mathrm{a})_0$ holds. Since $\widetilde{G}^{\alpha,j}_{\chi,0,i}(w) = 0$, we have $\widetilde{X}^1_{\chi,n}(\alpha) = \widetilde{\phi}^{\alpha,1}_{\chi,n,1}(0) = 1$ and $\widetilde{X}^2_{\chi,n}(\alpha) = \widetilde{\phi}^{\alpha,1}_{\chi,n,2}(0) = -1$. By the expression for $v^\alpha_{j,\chi,n}(w)$ in (4.7) and (4.8), statement $(\mathrm{b})_0$ holds.

For $n \in \mathbb{N}$, we assume by induction that statements $(\mathrm{a})_m$ and $(\mathrm{b})_m$ hold for $m = 0, \dots, n-1$. We first prove statement $(\mathrm{a})_n$. First fix $\alpha < 0$. By our induction hypothesis, for $m < n$, we have that $\widetilde{\mathcal{A}}_{\gamma,P,m}(\alpha) = \mathcal{Z}_{\gamma,P,m}(\alpha)$ is a rational function in $Q = \chi + \chi^{-1}$ by its explicit expression, and $\widetilde{V}^{\alpha,j}_{\chi,m} = \widetilde{G}^{\alpha,j}_{\chi,m,1}(0)$ is χ-good. By the explicit expressions for l_χ, $\Gamma_{n,1}$, and $\Gamma_{n,2}$, these quantities are meromorphic in $\chi \in \mathbb{C}$. Moreover, (B.3) and the definition of $\eta^\pm_{\chi,n}(\alpha)$ yield that

$$\prod_{k=1}^\infty (1-q^{2k})^{4\frac{l_\chi(l_\chi+1)}{\chi^2} + 2l_\chi + 2} = \sum_{n=1}^\infty \frac{\eta^-_{\chi,n}(\alpha)}{\eta^-_{\chi,0}(\alpha)} q^n \text{ and } \prod_{k=1}^\infty (1-q^{2k})^{4\frac{l_\chi(l_\chi+1)}{\chi^2} - 2l_\chi} = \sum_{n=1}^\infty \frac{\eta^+_{\chi,n}(\alpha)}{\eta^+_{\chi,0}(\alpha)} q^n$$

so that $\frac{\eta^\pm_{\chi,n}(\alpha)}{\eta^\pm_{\chi,0}(\alpha)}$ are rational functions in χ for $n \in \mathbb{N}_0$. Putting all these facts together and using the explicit expression (6.5), we deduce that the function $\chi \mapsto Z_n(\chi, \alpha)$ is χ-good.

For $n \in \mathbb{N}$, let $\chi_k = \frac{-\alpha}{2k}$. Then for $k \in \mathbb{N}$ large enough we have $\alpha > -\frac{4}{\gamma_k} + \frac{\gamma_k}{2}$ with $\gamma_k = 2\chi_k$. For such k, Theorems 6.1 and 6.6 yield that

$$\mathcal{Z}_{\gamma,P,n}(\alpha - \chi) = Y_n(\chi,\alpha)\mathcal{Z}_{\gamma,P,n}(\alpha+\chi) + Z_n(\chi,\alpha) \tag{6.35}$$

for $\chi = \chi_k$. Since $\lim_{k\to\infty} \chi_k = 0$ and both sides of (6.35) are χ-good, by Lemma 6.7 the equation (6.35) must hold for all $\chi \in [0, \infty)$ after the meromorphic extension in χ.

Now, fix $\gamma \in (0,2)$ and $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$. By the previous paragraph (6.35) holds for $\alpha \in (-\frac{4}{\gamma} + \chi, 0)$. By Theorem 5.1 and Corollary 5.2, both sides of (6.35) can be viewed as meromorphic functions in α in a complex neighborhood of $(-\frac{4}{\gamma}+\chi, 2Q-\chi)$. Therefore (6.35) holds for all $\gamma \in (0,2)$, $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$, and $(-\frac{4}{\gamma}+\chi, 2Q-\chi)$. If $\gamma^2 \notin \mathbb{Q}$, Theorem 6.1 and Proposition 6.3 imply statement $(\mathrm{a})_n$; continuity in γ of both $\mathcal{Z}_{\gamma,P,n}(\alpha)$ and $\widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha)$ implies $(\mathrm{a})_n$ for all $\gamma \in (0,2)$. This concludes the proof of $(\mathrm{a})_n$.

It remains to prove statement $(\mathrm{b})_n$; we use an argument parallel to the proof of Corollary 5.2 with χ in place of α. By Theorem 5.1 and (6.2), we have

$$\widetilde{\phi}^{\alpha,1}_{\chi,n,1}(0) = \left[\widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha-\chi) + \sum_{m=0}^{n-1} \frac{\eta^-_{\chi,n-m}(\alpha)}{\eta^-_{\chi,0}(\alpha)} \widetilde{\mathcal{A}}_{\gamma,P,m}(\alpha-\chi)\right]; \tag{6.36}$$

$$\widetilde{\phi}^{\alpha,2}_{\chi,n,1}(0) = -\left[\widetilde{\mathcal{A}}_{\gamma,P,n}(\alpha+\chi) + \sum_{m=0}^{n-1} \frac{\eta^+_{\chi,n-m}(\alpha)}{\eta^+_{\chi,0}(\alpha)} \widetilde{\mathcal{A}}_{\gamma,P,m}(\alpha+\chi)\right] \frac{\Gamma_{0,1}}{\Gamma_{0,2}} \frac{1-e^{\pi\chi P - \mathbf{i}\pi l_\chi}}{1+e^{\pi\chi P + \mathbf{i}\pi l_\chi}}. \tag{6.37}$$

Since statement $(\mathrm{a})_m$ holds for all $m \leq n$, equations (6.36) and (6.37) imply that $\widetilde{\phi}^{\alpha,j}_{\chi,n,1}(0)$ is χ-good for $j=1,2$. We also need to understand $\widetilde{G}^{\alpha,1}_{\chi,n,1}(w)$. By Definition 4.8 and (6.33),

$$\left(\mathcal{H}_\chi - \left(\frac{1}{4}l_\chi^2 + \frac{1}{4}\chi^2(P^2+2n)\right)\right) \widetilde{G}^{\alpha,1}_{\chi,n,1}(w) = \widetilde{g}^{\alpha,1}_{\chi,n,1}(w)$$

where $\widetilde{g}^{\alpha,1}_{\chi,n,1}(w)$ is defined as $g^{\alpha,1}_{\chi,n,1}(w)$ in (4.14) with $\widetilde{\phi}^{\alpha,j}_{\chi,n-l,1}(w)$ in place of $\phi^{\alpha,j}_{\chi,n-l,1}(w)$. Since statement $(\mathrm{b})_m$ holds for $m<n$, we have that $\widetilde{g}^{\alpha,1}_{\chi,n,1}(w)$ is (w,χ)-good. By Lemma D.6, we see that $\widetilde{G}^{\alpha,1}_{\chi,n,1}(w)$ is (w,χ)-good. The same argument shows that $\widetilde{G}^{\alpha,2}_{\chi,n,1}(w)$ is (w,χ)-good as well. Therefore $\widetilde{V}^{\alpha,j}_{\chi,n} = \widetilde{G}^{\alpha,j}_{\chi,n,1}(0)$ is χ-good for $j=1,2$. By (6.34) we have $\widetilde{X}^j_{\chi,n}(\alpha) = \widetilde{\phi}^{\alpha,j}_{\chi,n,1}(0) - \widetilde{V}^{\alpha,j}_{\chi,n}$, which is χ-good. Therefore $\widetilde{X}^j_{\chi,n}(\alpha) v^\alpha_{j,\chi,n}(w)$ is (w,χ)-good. Again by (6.34) we get statement $(\mathrm{b})_n$.

Appendix A. Conformal blocks in mathematical physics

This appendix provides an overview of conformal blocks as they appear in the study of two-dimensional conformal field theories in mathematical physics. We do not attempt to give an exhaustive list of references for this vast space, but refer the reader to the surveys [Kac98, DFMS97, Rib14] for a guide to the literature.

Recall that a two-dimensional conformal field theory (CFT) is a quantum field theory whose symmetry group has Lie algebra containing the Virasoro algebra Vir, which for a central charge c is the associative algebra with generators $\{L_n\}_{n\in\mathbb{Z}}$ and $\mathbf{1}$ and relations

$$\mathbf{1}L_n = L_n\mathbf{1} \qquad \text{and} \qquad L_nL_m - L_mL_n = (n-m)L_{n+m} + \frac{c}{12}(n-1)n(n+1)\delta_{n+m,0}\mathbf{1}. \tag{A.1}$$

The spectrum $\mathcal{S}$ of a two-dimensional CFT is given by a representation of a tensor product $\mathsf{Vir}\times\overline{\mathsf{Vir}}$ of two copies of the Virasoro algebra corresponding to the factorization into so-called chiral and anti-chiral sectors. Letting the chiral and anti-chiral Virasoro algebras Vir and $\overline{\mathsf{Vir}}$ have generators $\{L_n\}_{n\in\mathbb{Z}}$ and $\{\overline{L}_n\}_{n\in\mathbb{Z}}$, respectively, the central elements L_0 and $\overline{L}_0$ are simultaneously diagonalizable on $\mathcal{S}$.

Under the philosophy of conformal field theory, the spectrum decomposes into irreducible highest-weight representations of $\mathsf{Vir}\times\overline{\mathsf{Vir}}$, with each representation appearing with multiplicity depending on the specific CFT. Liouville CFT is a one-parameter family of CFTs that can be parameterized by either the central charge c of Vir and $\overline{\mathsf{Vir}}$, the background charge Q, or the coupling constant $b=\frac{\gamma}{2}$. These parameters are related by $c=1+6Q^2$ and $Q=2/\gamma+\gamma/2$.

The irreducible representations appearing in the spectrum of Liouville CFT are parametrized by a *momentum* parameter $P\in\mathbb{R}$. For $\Delta := \frac{1}{4}(Q^2+P^2)$, they are the *Verma modules* $M_{\Delta,c}$ and they give rise to the direct integral decomposition

$$\mathcal{S}_{\text{Liouville}} = \int_0^\infty M_{\Delta,c}\otimes\overline{M}_{\Delta,c}\,dP. \tag{A.2}$$

For parameters Δ, c, the Verma module $M_{\Delta,c}$ is the representation of Vir characterized as follows. There exists a vector $v_{\Delta,c}\in M_{\Delta,c}$ satisfying

$$L_n v_{\Delta,c} = 0 \text{ for } n>0 \qquad \text{and} \qquad L_0 v_{\Delta,c} = \Delta v_{\Delta,c}. \tag{A.3}$$

Moreover, the set of vectors

$$\{L_{-n_1}\cdots L_{-n_k}v_{\Delta,c} \mid n_1\geq\cdots\geq n_k\geq 1\} \tag{A.4}$$

forms a basis of $M_{\Delta,c}$. The action of Vir on $M_{\Delta,c}$ is given by commuting the action of a generator L_n on a basis vector in (A.4) to create a linear combination of other basis vectors using the relations (A.1). The resulting representation is infinite dimensional, but the eigendecomposition of the action of L_0 gives it a grading with finite dimensional graded pieces. Each eigenvalue of L_0 lies in $\Delta + \mathbb{Z}_{\geq 0}$ and is called a *weight*, and each eigenspace is called a weight space. The vector $v_{\Delta,c}$ is called a *highest weight* vector.

Based on the decomposition of the spectrum, correlation functions for any CFT can be written as combinations of quantities called conformal blocks which correspond to highest weight irreducible representations of Vir and are independent of the specific CFT. We now give a physical definition of the 1-point toric conformal block which appears in this paper, beginning by defining the so-called primary fields. For each $\alpha \in \mathbb{R}$, define $\Delta_\alpha := \frac{\alpha}{2}(Q - \frac{\alpha}{2})$. The *primary field* $\phi_{\Delta_\alpha}(z)$ of *conformal dimension* Δ_α is a formal series

$$\phi_{\Delta_\alpha}(z) = z^{-\Delta_\alpha} \sum_{n=0}^{\infty} \phi_{\Delta_\alpha, n} z^n,$$

where each $\phi_{\Delta_\alpha,n}$ is an operator $\phi_{\Delta_\alpha,n} : M_{\Delta,c} \to M_{\Delta,c}$. It is uniquely determined by the constraint

$$L_n \phi_{\Delta_\alpha}(z) - \phi_{\Delta_\alpha}(z) L_n = z^n \Big(z\partial_z + (n+1)\Delta_\alpha \Big) \phi_{\Delta_\alpha}(z) \qquad \text{for all } n \in \mathbb{Z} \tag{A.5}$$

and the normalization

$$\phi_{\Delta_\alpha}(z) v_{\Delta,c} = z^{-\Delta_\alpha} v_{\Delta,c} + (\text{l.o.t.}), \tag{A.6}$$

where we recall that $v_{\Delta,c} \in M_{\Delta,c}$ is the highest weight vector and (l.o.t.) denotes terms of lower weight under L_0.

For a modular parameter $q = e^{\mathbf{i}\pi\tau}$, the 1-point toric conformal block $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ of *intermediate dimension* $\Delta = \frac{1}{4}(Q^2 + P^2)$ and conformal dimension Δ_α is defined physically as the formal series in q given by

$$\mathcal{F}^{\alpha}_{\gamma,P}(q) := \operatorname{Tr}|_{M_{\Delta,c}} \Big(q^{-2\Delta + 2L_0} \phi_{\Delta_\alpha}(1) \Big). \tag{A.7}$$

In this expression, we evaluate the formal series for the primary field $\phi_{\Delta_\alpha}(z)$ at $z = 1$ by noting that (A.5) implies $\phi_{\Delta_\alpha}(z)$ preserves weight spaces of $M_{\Delta,c}$ and hence the trace in (A.7) is the sum of finite-dimensional traces over each weight space weighted by the corresponding eigenvalues of $-\Delta + L_0$ (which are non-negative integers). The q-series expansion of $\mathcal{F}^{\alpha}_{\gamma,P}(q)$ may be determined by computing the diagonal matrix elements of $\phi_{\Delta_\alpha}(z)$ in the basis (A.4) using (A.5) and summing over them. As a concrete example, to compute the first 2 terms of $\mathcal{F}^{\alpha}_{\gamma,P}(q)$, applying (A.5) for $n = 0$ constrains $\phi_{\Delta_\alpha}(z) v_{\Delta,c}$ to take the form

$$\phi_{\Delta_\alpha}(z) v_{\Delta,c} = z^{-\Delta_\alpha} v_{\Delta,c} + A z^{-\Delta_\alpha + 1} L_{-1} v_{\Delta,c} + (\text{l.o.t.}). \tag{A.8}$$

Applying L_1 to both sides, using (A.5) for $n = 1$, and equating the coefficient of $v_{\Delta,c}$ yields that $A = \frac{\Delta_\alpha}{2\Delta}$. We conclude by (A.5) for $n = -1$ that

$$\phi_{\Delta_\alpha}(z) L_{-1} v_{\Delta,c} = L_{-1} \phi_{\Delta_\alpha}(z) v_{\Delta,c} - \partial_z \phi_{\Delta_\alpha}(z) v_{\Delta,c} = \Big(1 + \frac{\Delta_\alpha}{2\Delta}(\Delta_\alpha - 1) \Big) L_{-1} v_{\Delta,c} + (\text{l.o.t.}).$$

Therefore, summing the diagonal matrix elements corresponding to $v_{\Delta,c}$ and $L_{-1} v_{\Delta,c}$ yields the expansion

$$\mathcal{F}^{\alpha}_{\gamma,P}(q) = 1 + \frac{2\Delta + \Delta_\alpha^2 - \Delta_\alpha}{2\Delta} q^2 + O(q^4).$$

In [FL10], a more detailed version of this analysis was used to show that this definition of the conformal block satisfies Proposition 2.11. From the representation theory of the Virasoro algebra, it is known that for $m, n \geq 0$ there exists a non-commutative polynomial $P_{m,n}(L_{-1}, L_{-2}, \ldots)$ for which the vector

$$\chi_{m,n} = P_{m,n}(L_{-1}, L_{-2}, \ldots) v_{\Delta,c}$$

satisfies $L_k \chi_{m,n} = 0$ for all $k > 0$ if and only if $\Delta = \Delta_{\alpha_{m,n}}$ for $\alpha_{m,n} = -m\frac{\gamma}{2} - n\frac{2}{\gamma}$. In this case, we say that $\chi_{m,n}$ is a *singular vector*. Mapping $v_{\Delta_{m,-n},c}$ to $\chi_{m,n}$ yields an injective mapping of Verma modules

$$M_{\Delta_{\alpha_{m,-n}},c} \to M_{\Delta_{\alpha_{m,n}},c}. \tag{A.9}$$

We refer the interested reader to [KR87, Chapter 8] for more details on these singular vectors. If $\chi_{m,n}$ appears in the ansatz (A.8) with prefactor $X z^{-\Delta_\alpha + mn} \chi_{m,n}$ for an unknown coefficient X, we may determine

X by applying $P_{m,n}(L_1, L_2, \ldots)$ to both sides of (A.8). Because $P_{m,n}(L_1, L_2, \ldots)\chi_{m,n} = 0$ for $\Delta = \Delta_{\alpha_{m,n}}$) and all coefficients are rational functions in Δ, we find that

$$P_{m,n}(L_1, L_2, \ldots)\chi_{m,n} = (\Delta - \Delta_{\alpha_{m,n}})\psi_{m,n}$$

for some vector $\psi_{m,n} \in M_{\Delta,c}$. Equating coefficients of $v_{\Delta,c}$ on both sides using (A.5) shows that X has a pole at $\Delta = \Delta_{\alpha_{m,n}}$. A more detailed analysis of the residue at this pole using (A.9) yields the exact formula of Proposition 2.11. We refer the interested reader to [FL10, Section 1] for more details.

Appendix B. Conventions and facts on special functions

This appendix collects the conventions and facts on the special functions we use in the main text. We direct the interested reader to [DLMF, Chapters 20 and 23] and [Bar04] for more details.

B.1. **Jacobi theta function and Weierstrass's elliptic function.** Throughout, we fix a modular parameter $\tau \in \mathbb{H}$ and set $q = e^{\mathbf{i}\pi\tau}$. The *Jacobi theta function* is defined for $u \in \mathbb{C}$ by

$$\Theta_\tau(u) := -2q^{1/4}\sin(\pi u)\prod_{k=1}^{\infty}(1-q^{2k})(1-2\cos(2\pi u)q^{2k}+q^{4k}). \tag{B.1}$$

The *Dedekind eta function* is defined by $\eta(\tau) := e^{\frac{\mathbf{i}\pi\tau}{12}}\prod_{k=1}^{\infty}(1-e^{2k\mathbf{i}\pi\tau})$. Another parametrization which we use throughout the text is

$$\eta(q) = q^{\frac{1}{12}}\prod_{k=1}^{\infty}(1-q^{2k}). \tag{B.2}$$

We will use the following elementary fact in Section 2.

Lemma B.1. *The function $\log(q^{-\frac{1}{12}}\eta(q))$ is analytic on $\mathbb{D}$, hence for each $\beta \in \mathbb{R}$, $[q^{-\frac{1}{12}}\eta(q)]^{\beta}$ defines a power series in q convergent for $|q| < 1$.*

Proof. This follows from the absolute summability of $\sum_{k=1}^{\infty}\log(1-q^{2k})$ for $|q| < 1$. □

Although the expression of (B.2) is a multi-valued function in q, we interpret it as a single-valued function in τ. In terms of these expressions, we have

$$\Theta_\tau'(0) = -2\pi q^{1/4}\prod_{k=1}^{\infty}(1-q^{2k})^3 = -2\pi\eta(q)^3. \tag{B.3}$$

Weierstrass's elliptic function $\wp$ is defined in terms of $\Theta_\tau(u)$ by

$$\wp(u) := \frac{\Theta_\tau'(u)^2}{\Theta_\tau(u)^2} - \frac{\Theta_\tau''(u)}{\Theta_\tau(u)} + \frac{1}{3}\frac{\Theta_\tau'''(0)}{\Theta_\tau'(0)}. \tag{B.4}$$

It admits the following expansion (see e.g. [DLMF, Equation (23.8.1)])

$$\wp(u) = \frac{\pi^2}{\sin^2(\pi u)} - 8\pi^2\sum_{n=1}^{\infty}\frac{nq^{2n}}{1-q^{2n}}\cos(2\pi n u) - \frac{\pi^2}{3} + 8\pi^2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1-q^{2n})^2}, \tag{B.5}$$

which implies that $\wp(u)$ further admits a q-expansion

$$\wp(u) = \sum_{n=0}^{\infty}\wp_n(u)q^n, \quad \text{where } \wp_n(u) \equiv 0 \text{ for odd } n. \tag{B.6}$$

By (B.5), for $n \geq 1$, there exists a unique polynomial $\widetilde{\wp}_n(w)$ such that $\widetilde{\wp}_n(w) = \wp_n(u)$ for $w = \sin^2(\pi u)$. For example, $\widetilde{\wp}_1(w) = 0$, $\widetilde{\wp}_2(w) = 16\pi^2 w$.

B.2. **Gamma function and double gamma function.** The gamma function is defined by $\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt$ for $\operatorname{Re} z > 0$. In particular, $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$. It has a meromorphic extension to $\mathbb{C}$ where it has simple poles at $\{0, -1, -2, \cdots\}$. Besides the basic shift equation $\Gamma(z+1) = z\Gamma(z)$, the gamma function also satisfies Euler's reflection formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \qquad \text{for } z \notin \mathbb{Z} \tag{B.7}$$

and the Legendre duplication formula

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma(z+\frac{1}{2}). \tag{B.8}$$

By (B.7), $\Gamma(z)$ has no zeros and $\Gamma(z)^{-1}$ is an entire function with simple zeros at $\{0, -1, -2, \cdots\}$. The $z \to \infty$ asymptotics of $\Gamma(z)$ is governed by the Stirling's approximation. For each $\delta \in (0, \pi)$, we have

$$\Gamma(z) \sim \sqrt{\frac{2\pi}{z}} e^{-z} z^z (1 + O(|z|^{-1})) \quad \text{for } |z| > 1 \text{ and } \arg z \in (\delta - \pi, \pi - \delta), \tag{B.9}$$

where the error term $O(|z|^{-1})$ depends on δ.

Finally we introduce the *double gamma function* $\Gamma_{\frac{\gamma}{2}}(z)$. For $\operatorname{Re}(z) > 0$, it is defined by

$$\log \Gamma_{\frac{\gamma}{2}}(z) := \int_0^\infty \frac{dt}{t}\left[\frac{e^{-zt} - e^{-\frac{Qt}{2}}}{(1 - e^{-\frac{\gamma t}{2}})(1 - e^{-\frac{2t}{\gamma}})} - \frac{(\frac{Q}{2} - z)^2}{2} e^{-t} + \frac{z - \frac{Q}{2}}{t}\right]. \tag{B.10}$$

Like the usual gamma function, $\Gamma_{\frac{\gamma}{2}}(z)$ admits meromorphic extension to all of $\mathbb{C}$. It has no zeros and simple poles at the points of the set $\{-\frac{\gamma n}{2} - \frac{2m}{\gamma} \mid n, m \in \mathbb{N}\}$ and satisfies for $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$ the functional equation

$$\Gamma_{\frac{\gamma}{2}}(z + \chi) = \sqrt{2\pi}\frac{\chi^{\chi z - \frac{1}{2}}}{\Gamma(\chi z)}\Gamma_{\frac{\gamma}{2}}(z). \tag{B.11}$$

For $\gamma^2 \notin \mathbb{Q}$, $\Gamma_{\frac{\gamma}{2}}(z)$ is completely specified by this functional equation and the special value $\Gamma_{\frac{\gamma}{2}}(\frac{Q}{2}) = 1$. The other values of γ can be recovered by continuity. We also introduce the function $S_{\frac{\gamma}{2}}(z)$ given by

$$S_{\frac{\gamma}{2}}(z) := \frac{\Gamma_{\frac{\gamma}{2}}(z)}{\Gamma_{\frac{\gamma}{2}}(Q - z)}. \tag{B.12}$$

B.3. **Identities on $\Theta_\tau(u)$.** In Section 3 we use the following identities on the theta function $\Theta_\tau(u)$. Here we use the notations $\Theta'_\tau(u) := \partial_u \Theta_\tau(u)$ and $\Theta''_\tau(u) := \partial_{uu}\Theta_\tau(u)$, where ∂_τ and ∂_u are holomorphic derivatives.

$$\mathbf{i}\pi\partial_\tau\Theta_\tau(u) = \frac{1}{4}\Theta''_\tau(u). \tag{B.13}$$

$$\frac{\Theta''_\tau(a-b)}{\Theta_\tau(a-b)} + \frac{\Theta''_\tau(a)}{\Theta_\tau(a)} + \frac{\Theta''_\tau(b)}{\Theta_\tau(b)} - 2\frac{\Theta'_\tau(a-b)}{\Theta_\tau(a-b)}\Big(\frac{\Theta'_\tau(a)}{\Theta_\tau(a)} - \frac{\Theta'_\tau(b)}{\Theta_\tau(b)}\Big) - 2\frac{\Theta'_\tau(a)}{\Theta_\tau(a)}\frac{\Theta'_\tau(b)}{\Theta_\tau(b)} - \frac{\Theta'''_\tau(0)}{\Theta'_\tau(0)} = 0. \tag{B.14}$$

$$\Theta_\tau(u + \tau/2) = -\mathbf{i}e^{-\mathbf{i}\pi u}q^{-\frac{1}{3}}\eta(q)\prod_{n=1}^\infty (1 - q^{2n-1}e^{2\pi i u})(1 - q^{2n-1}e^{-2\pi i u}). \tag{B.15}$$

(B.13) comes from [WW02, Section 2.14], (B.14) is stated in [FLNO09, Equation (A.10)] and may be derived by applying the operator $\partial_y\partial_u - \frac{2\mathbf{i}}{\pi}\partial_\tau$ to [WW02, Exercise 21.13], and (B.15) comes from direct substitution.

We use the following form of the log-derivative of $\Theta_\tau(u)$ from [DLMF, Equation (20.5.10)] in Section B.4.

$$\frac{\Theta'_\tau(u)}{\Theta_\tau(u)} = \pi\frac{\cos(\pi u)}{\sin(\pi u)} + 4\pi\sum_{n=1}^\infty \frac{q^{2n}}{1 - q^{2n}}\sin(2\pi n u). \tag{B.16}$$

B.4. **Fractional powers of $\Theta_\tau(u)$.** We first recall the following fact.

Lemma B.2. *Suppose f is analytic on a simply connected domain D such that $f(z) \neq 0$ for each $z \in D$. Then there exists an analytic function g on D such that $f = e^g$.*

The zero set of Θ_τ is given by the lattice $\{m+n\tau : m,n \in \mathbb{Z}\}$. In our paper we need to consider fractional powers of Θ_τ on $D \cup (\mathbb{R} \setminus \mathbb{Z})$ where

$$D := \{x + \tau y : x \in (0,1) \text{ or } y \in (0,1)\} \subset \mathbb{C}. \tag{B.17}$$

We must fix a convention for $\log \Theta_\tau$ on D in order to define fractional powers of Θ_τ.

Recall (B.1). Let $\phi_\tau(u) = -2q^{1/4}\prod_{k=1}^\infty (1-q^{2k})(1-2\cos(2\pi u)q^{2k}+q^{4k})$ so that $\Theta_\tau(u) = \sin(\pi u)\phi_\tau(u)$. If $u \in D$ or $u \in \mathbb{R}\setminus\mathbb{Z}$, since $\sin(\pi u) \neq 0$ and $\Theta_\tau(u) \neq 0$, we have $\phi_\tau(u) \neq 0$. Similarly, for $u \in \mathbb{Z}$, it is easy to check that $\phi_\tau(u) \neq 0$. Therefore there exists a simply connected domain $\mathcal{D} \subset \mathbb{H}\times\mathbb{C}$ containing $\mathbb{H}\times(D\cup\mathbb{R})$ such that $\phi_\tau(u) \neq 0$ for $(\tau, u) \in \mathcal{D}$. Note that $\phi_\tau(u)$ is negative real when $q \in (0,1)$ and $u \in \mathbb{R}$. By the two-variable variant of Lemma B.2, a unique bi-holomorphic function $\log \phi_\tau(u)$ can be defined on $\mathcal{D}$ such that $e^{\log \phi_\tau(u)} = \phi_\tau(u)$ and moreover, $\operatorname{Im}(\log(\phi_\tau(u))) = \pi$ for $\tau \in \mathbf{i}\mathbb{R}_{>0}$ and $u \in \mathbb{R}$.

Since the zero set of $\sin(\pi u)$ is $\mathbb{Z}$, a unique analytic function $\log\sin(\pi u)$ can be defined on D such that $e^{\log\sin(\pi u)} = \sin(\pi u)$ and $\lim_{u\to\frac12}\log\sin(\pi u) = 0$. Now we let

$$\log\Theta_\tau(u) := \log\sin(\pi u) + \log\phi_\tau(u) \qquad \text{for } u \in D. \tag{B.18}$$

One can check that for each $k \in \mathbb{Z}$, $\lim_{t\to 0^+}\operatorname{Im}(\log\sin(u+it)) = -k\pi$ for $u \in (k\pi,(k+1)\pi)$. Since $\log\phi_\tau$ is continuous at $u \in \mathbb{R}$, we can extend the definition of $\log\Theta_\tau$ in (B.18) to $\mathbb{R}\setminus\mathbb{Z}$ by requiring $\operatorname{Im}(\log\Theta_\tau(u)) = \lim_{t\to0^+}\operatorname{Im}(\log\Theta_\tau(u+it))$ for $u \in \mathbb{R}\setminus\mathbb{Z}$.

Throughout the paper we use the following convention for fractional powers of Θ_τ.

Definition B.3. For $u \in D\cup(\mathbb{R}\setminus\mathbb{Z})$ and $c \in \mathbb{R}$, $\Theta_\tau(u)^c = e^{c\log\Theta_\tau(u)}$ with $\log\Theta_\tau(u)$ defined above.

Under Definition B.3, for $c \in \mathbb{R}$, we have by [DLMF, Chapter 20.2] that

$$\Theta_\tau(u+1)^c = e^{-\mathbf{i}\pi c}\Theta_\tau(u)^c \qquad \text{if } u \in \{x+y\tau : y \in (0,1)\}; \tag{B.19}$$

$$\Theta_\tau(u+\tau)^c = e^{-2\pi\mathbf{i}c(u-\frac12+\frac{\tau}{2})}\Theta_\tau(u)^c \qquad \text{if } u \in \{x+y\tau : x \in (0,1)\}. \tag{B.20}$$

Moreover, since $\operatorname{Im}(\log(\phi_\tau(u))) = \pi$ for $\tau \in \mathbf{i}\mathbb{R}_{>0}$ and $u \in (0,\pi)$, we have

$$\Theta_\tau(x)^c = e^{-\mathbf{i}c\pi}|\Theta_\tau(x)|^{-\alpha\gamma/2} \qquad \text{for } x \in (0,1) \text{ and } q \in (0,1). \tag{B.21}$$

B.5. **On the definition of the u-deformed block.** Let $\mathfrak{B} := \{z : 0 < \operatorname{Im}(z) < \frac34\operatorname{Im}(\tau)\}$ so that $\mathfrak{B} \subset D$ with D from (B.17). Fix $c > 0$ and a finite measure ν supported on $[0,1]$. Let

$$f_\nu(u) := \int_0^1 \Theta(u+x)^c \nu(dx) \text{ for } u \in \overline{\mathfrak{B}} := \mathfrak{B}\cup\partial\mathfrak{B}, \tag{B.22}$$

where $\Theta(u+x)^c$ is given by Definition B.3. In Section 3, we define the u-deformed block in terms of $\mathbb{E}\left[f_\nu(u)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}\right]$ with a special choice of ν in (3.2) depending on the GMC measure $e^{\frac{\gamma}{2}Y_\tau(x)}dx$. In order to make sense of $\log f_\nu$ and hence $f_\nu(u)^{-\frac{\alpha}{\gamma}+\frac{\chi}{\gamma}}$, we first record the following two lemmas.

Lemma B.4. *There exists $q_0 > 0$ such that if $q \in (0,q_0)$, then $\operatorname{Im}(\log\Theta_\tau)' < 0$ on $\mathfrak{B}$.*

Proof. Note that $(\log\Theta_\tau)' = \Theta_\tau'/\Theta_\tau$. Since $\operatorname{Re}(z) = \frac{e^{-4\pi\operatorname{Im}(z)}-1}{|e^{2\pi\mathbf{i}z}-1|^2}$ and $\operatorname{Im}(\sin z) = \cos(\operatorname{Re}(z))(e^{\operatorname{Im}(z)}-e^{-\operatorname{Im}(z)})$, by (B.16) we have

$$\begin{aligned}\operatorname{Im}\left(\frac{\Theta_\tau'(z)}{\Theta_\tau(z)}\right) &= \pi\operatorname{Re}\left(\frac{e^{\mathbf{i}\pi z}+e^{-\mathbf{i}\pi z}}{e^{\mathbf{i}\pi z}-e^{-\mathbf{i}\pi z}}\right) + 2\pi\sum_{n=1}^\infty \frac{q^{2n}}{1-q^{2n}}(e^{2\pi n\operatorname{Im}(z)}-e^{-2\pi n\operatorname{Im}(z)})\cos(2\pi n\operatorname{Re}(z))\\ &= -\pi\frac{1-e^{-4\pi\operatorname{Im}(z)}}{|e^{2\mathbf{i}\pi z}-1|^2} + 2\pi\sum_{n=1}^\infty\frac{q^{2n}}{1-q^{2n}}(e^{2\pi n\operatorname{Im}(z)}-e^{-2\pi n\operatorname{Im}(z)})\cos(2\pi n\operatorname{Re}(z)).\end{aligned}$$

Since $\operatorname{Im}(z) > 0$, we have $\pi\frac{1-e^{-4\pi\operatorname{Im}(z)}}{|e^{2\mathbf{i}\pi z}-1|^2} > \frac{\pi}{4}(1-e^{-4\pi\operatorname{Im}(z)})$. Note that

$$2\pi\sum_{n=1}^\infty\frac{q^{2n}}{1-q^{2n}}(e^{2\pi n\operatorname{Im}(z)}-e^{-2\pi n\operatorname{Im}(z)})\cos(2\pi n\operatorname{Re}(z)) < \frac{2\pi}{1-q^2}\left(\frac{q^2e^{2\pi\operatorname{Im}(z)}}{1-q^2e^{2\pi\operatorname{Im}(z)}} - \frac{q^2e^{-2\pi\operatorname{Im}(z)}}{1-q^2e^{-2\pi\operatorname{Im}(z)}}\right).$$

Set $h(x) = \frac{q^2 x}{1-q^2 x}$. Since $h'(x) = \frac{q^2}{(1-q^2x)^2} \le \frac{q^2}{(1-q^2 e^{2\pi \operatorname{Im}(z)})^2}$ for $x \in [e^{-2\pi \operatorname{Im}(z)}, e^{2\pi \operatorname{Im}(z)}]$, we have

$$\begin{aligned}\frac{2\pi}{1-q^2}\left(\frac{q^2 e^{2\pi \operatorname{Im}(z)}}{1-q^2 e^{2\pi \operatorname{Im}(z)}} - \frac{q^2 e^{-2\pi \operatorname{Im}(z)}}{1-q^2 e^{-2\pi \operatorname{Im}(z)}}\right) &< \frac{2\pi}{1-q^2}\frac{q^2}{(1-q^2 e^{2\pi \operatorname{Im}(z)})^2}\left(e^{2\pi \operatorname{Im}(z)} - e^{-2\pi \operatorname{Im}(z)}\right)\\ &= \frac{2\pi}{1-q^2}\frac{q^2 e^{2\pi \operatorname{Im}(z)}}{(1-q^2 e^{2\pi \operatorname{Im}(z)})^2}\left(1 - e^{-4\pi \operatorname{Im}(z)}\right)\\ &< \frac{2\pi}{1-q_0^2}\frac{q_0}{(1-q_0)^2}\left(1 - e^{-4\pi \operatorname{Im}(z)}\right).\end{aligned}$$

When $\operatorname{Im} z < \frac{3}{4}\operatorname{Im}\tau$, we have $q^2 e^{2\pi \operatorname{Im} z} < q^{\frac{1}{2}}$. By the monotonicity of $\frac{x}{(1-x)^2}$ on $(0,1)$, we have

$$\frac{2\pi}{1-q^2}\frac{q^2 e^{2\pi \operatorname{Im}(z)}}{(1-q^2 e^{2\pi \operatorname{Im}(z)})^2}\left(1 - e^{-4\pi \operatorname{Im}(z)}\right) < \frac{2\pi}{1-q^2}\frac{q^{\frac{1}{2}}}{(1-q^{\frac{1}{2}})^2}\left(1 - e^{-4\pi \operatorname{Im}(z)}\right).$$

Since $\lim_{q\to 0} \frac{2\pi}{1-q^2}\frac{q^{\frac{1}{2}}}{(1-q^{\frac{1}{2}})^2} = 0$, we have the existence of q_0 with the desired property. □

Remark B.5. Our q_0 in Lemma B.4 may not be optimal, but its existence is all we need in this paper.

By (B.19), we have $\Theta_\tau(u+1) = -\Theta_\tau(u)$. The next lemma concerns the range of $\Theta_\tau(u+x)$ for $x \in (0,1)$.

Lemma B.6. *Fix $q \in (0, q_0)$ and $u \in \mathfrak{B}$. Let the straight line between $\Theta_\tau(u)$ and $\Theta_\tau(u+1) = -\Theta_\tau(u)$ divide the complex plane into two open half planes $\mathbb{H}_u^-$ and $\mathbb{H}_u^+$, where $\mathbb{H}_u^-$ contains a small clockwise rotation of $\Theta_\tau(u)$ viewed as a vector. For $x \in (0,1)$, we have $\Theta_\tau(u+x) \in \mathbb{H}_u^-$.*

Proof. Let $f(x) = \operatorname{Im} g(u+x)$ for $x \in \mathbb{R}$. By Lemma B.4, $f(1) - f(0) = -\pi$ and $f'(x) < 0$ for $x \in \mathbb{R}$. Therefore by the definition of $\mathbb{H}_u^-$, for $x \in (0,1)$, we have $\Theta_\tau(u+x) \in \mathbb{H}_u^-$. □

The following lemma allows us to make sense of $\log f_\nu$ in Definition B.8.

Lemma B.7. *Fix $q \in (0, q_0)$ and $c > 0$. Then $f_\nu(u)$ from* (B.22) *is analytic on $\mathfrak{B}$ and continuous on $\overline{\mathfrak{B}}$. Moreover, $f_\nu(u+1) = e^{-c\pi \mathbf{i}} f_\nu(u)$ and $f_\nu(u) \neq 0$ for $u \in \mathfrak{B}$. Finally, $f_\nu(1) > 0$.*

Proof. Since Θ_τ^c is bounded on $\overline{\mathcal{B}}$ and continuous except at integers, we see that f_ν is continuous on $\overline{\mathfrak{B}}$. By (B.19), $f_\nu(u+1) = e^{-c\pi \mathbf{i}} f_\nu(u)$ for $u \in \overline{\mathfrak{B}}$. Note that $\Theta_\tau(u)^{-c} f_\nu(u) = \int_0^1 \Theta_\tau(u+x)^c \Theta_\tau(u)^{-c}\nu(dx)$. By Lemma B.6, $\operatorname{Im}(\Theta_\tau(u+x)^c\Theta_\tau(u)^{-c}) < 0$ for $u \in \mathfrak{B}$. Therefore $\operatorname{Im}(\Theta_\tau(u)^{-c} f_\nu(u)) < 0$, hence $f_\nu(u) \neq 0$. Since the support of ν is $[0,1]$, $\Theta_\tau(1+x) > 0$ for $x \in (0,1)$, we have $f_\nu(1) > 0$. □

Recalling that $f_\nu(1) > 0$ from Lemma B.7, in Definition B.8 we now define $f_\nu(u)^\beta$ for use in Section 3.

Definition B.8. Fix $q \in (0, q_0)$ as in Lemma B.7. Let $\log f_\nu$ be the function on $\mathfrak{B}$ such that

$$f_\nu = e^{\log f_\nu} \text{ on } \mathfrak{B} \quad \text{and} \quad \lim_{z\to 1} \operatorname{Im}[\log f_\nu(z)] = 0.$$

For each $\beta \in \mathbb{R}$, define $\left(\int_0^1 \Theta(u+x)^c \nu(dx)\right)^\beta := e^{\beta \log f_\nu(u)}$.

Appendix C. Background on log-correlated fields and Gaussian multiplicative chaos

Let us first provide a general definition of log-correlated fields.

Definition C.1. A centered Gaussian process X on a domain $U \subset \mathbb{R}^d$ is called a log-correlated field if it admits a covariance kernel of the form

$$\mathbb{E}[X(x)X(y)] = c \log \frac{1}{|x-y|} + g(x,y), \tag{C.1}$$

where c is a positive constant and $g : U \times U \mapsto \mathbb{R}$ is a continuous function.

Due to the singularity of the log kernel, these fields cannot be defined as pointwise functions but only as random generalized functions (distributions). Given a random variable $\mathcal{X}$, it will be convenient to use the notation $\mathbb{E}[\mathcal{X} X(x)]$ to designate the distribution defined for all suitable test functions h by

$\int dx h(x)\mathbb{E}[\mathcal{X}X(x)] := \mathbb{E}\left[\mathcal{X}\left(\int dx h(x)X(x)\right)\right]$. In a similar fashion, the covariance kernel (C.1) should be understood as meaning that for all test functions h_1, h_2, one has

$$\mathbb{E}\left[\left(\int dx h_1(x)X(x)\right)\left(\int dy h_2(y)X(y)\right)\right] = \int\int dx dy h_1(x)h_2(y)\mathbb{E}[X(x)X(y)].$$

Consider a field X as in Definition C.1 with $d = 1, c = 2$, and fix $\gamma \in (0, 2)$. For a large class of continuous regularizations $\{X_n\}$ of X, $e^{\frac{\gamma}{2}X_n - \frac{\gamma^2}{8}\mathbb{E}[X_n(x)^2]}dx$ converges in probability to the unique GMC measure $e^{\frac{\gamma}{2}X}dx$ associated with X, see, e.g. [Ber17]. Definition 2.4 is a special case of such limiting procedures.

Consider the log-correlated field $X_{\mathbb{H}}$ on the upper half plane $\mathbb{H}$ whose covariance is given by

$$\mathbb{E}[X_{\mathbb{H}}(x)X_{\mathbb{H}}(y)] = \log\frac{1}{|x-y||x-\bar{y}|} - \log|x+\mathbf{i}|^2 - \log|y+\mathbf{i}|^2 + 2\log 2 \qquad \text{for } x, y \in \mathbb{H}. \tag{C.2}$$

The field $X_{\mathbb{H}}$ is an example of a *free boundary Gaussian free field* (GFF) on $\mathbb{H}$. We can restrict $X_{\mathbb{H}}$ to $\mathbb{R}$, giving a field $\{X_{\mathbb{H}}(x)\}_{x\in\mathbb{R}}$ whose covariance kernel is still given by (C.2) with $x, y \in \mathbb{R}$. Let $\phi(x) := -\mathbf{i}\frac{e^{2\pi\mathbf{i}x}-1}{e^{2\pi\mathbf{i}x}+1} \in \mathbb{R}$ for $x \in [0, 1]$. One can check that $X_{\mathbb{H}}(\phi(x))$ has the law of Y_∞ in Lemma 2.1. To understand this fact geometrically, we extend the map ϕ by $\phi(x) = -\mathbf{i}\frac{e^{2\pi\mathbf{i}x}-1}{e^{2\pi\mathbf{i}x}+1}$ for $x \in \mathbb{R}_{>0} \times [0, 1]$, viewing $\mathbb{R}_{>0} \times [0, 1]$ as a subset of $\mathbb{C}$. Then ϕ conformally maps the half cylinder $\mathcal{C}_+$ obtained by gluing the two vertical boundaries of $[0, 1] \times \mathbb{R}_{>0}$ to $\mathbb{H}$. By the conformal invariance of free boundary GFF, $\{X_{\mathbb{H}}(\phi(\cdot))\}$ on the half cylinder is a free boundary GFF normalized such that the average over $[0, 1]$ equals zero. Therefore Y_∞ can be thought of as the boundary restriction of a GFF on the half cylinder. Similarly, the field Y_τ can be understood as the restriction of a GFF on the torus with modular parameter τ to the interval $[0, 1]$ (see the definition of the GFF on the torus in [Bav19, Equation (2.5)]).

In a few technical GMC arguments involving Y_∞, it is convenient to transform to the upper half plane as the corresponding statements are worked out for $X_{\mathbb{H}}$ in the literature. For this, we need the following fact.

Lemma C.2 (Coordinate change). *Let $X_{\mathbb{H}}$ and ϕ be defined as above and $Y_\infty(x) := X_{\mathbb{H}}(\phi(x))$ for $x \in [0, 1]$. Then the measure $|(\phi^{-1}(y)')|e^{\frac{\gamma}{2}X_{\mathbb{H}}(y)}dy$ on $\mathbb{R}$ is the pushforward of the measure $e^{\frac{\gamma}{2}Y_\infty(x)}dx$ under ϕ.*

For $x \in \mathbb{H}$, let $\overline{X}(x)$ be the average of $X_{\mathbb{H}}$ over the semi-circle $\{z \in \mathbb{H} : |z| = |x|\}$. Let $Z_{\mathbb{H}} := X_{\mathbb{H}} - \overline{X}$. Then $\overline{X}$ and $Z_{\mathbb{H}}$ are independent. Moreover, $\overline{X}(e^{-s/2})$ evolves as a standard linear Brownian motion. Finally $Z_{\mathbb{H}}$ is a log-correlated field whose covariance is given by

$$\mathbb{E}[Z_{\mathbb{H}}(x)Z_{\mathbb{H}}(y)] = 2\log\frac{|x| \vee |y|}{|x-y|}. \tag{C.3}$$

We use $\overline{X}$ and $Z_{\mathbb{H}}$ in Section 5 and Appendix E. In particular, we use the following fact in Appendix E.

Lemma C.3. *For $z \in \mathbb{H} \cup \mathbb{R}$, we have $\mathbb{E}[\overline{X}(z)^2] = \mathbb{E}[X_{\mathbb{H}}(z)X_{\mathbb{H}}(0)]$.*

Proof. Since $\lim_{y\to 0}\mathbb{E}[Z_{\mathbb{H}}(z)Z_{\mathbb{H}}(y)] = 0$, we have $\mathbb{E}[\overline{X}^2(z)] = \lim_{y\to 0}\mathbb{E}[\overline{X}(z)\overline{X}(y)] = \mathbb{E}[X_{\mathbb{H}}(z)X_{\mathbb{H}}(0)]$. □

We now state a general result of existence of moments of GMC measure covering all situations encountered in the main text. Concretely, we will use the case when $F(x)$ below equals $\frac{\gamma}{2}F_\tau(x)$ or 0, where F_τ is as in (2.5).

Lemma C.4 (Moments of GMC). *Fix $\gamma \in (0, 2)$ and $\alpha < Q$. Let $F : [0, 1] \mapsto \mathbb{R}$ be a continuous Gaussian field independent of $Y_\infty(x)$, and $f : [0, 1] \mapsto (0, +\infty)$ be a continuous bounded function. Then*

- *for $p < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q - \alpha)$ we have $\mathbb{E}\left[\left(\int_0^1 e^{F(x)}\sin(\pi x)^{-\frac{\alpha\gamma}{2}}f(x)e^{\frac{\gamma}{2}Y_\infty(x)}dx\right)^p\right] < \infty$;*
- *for $\chi \in \{\frac{\gamma}{2}, \frac{2}{\gamma}\}$, $u \in \mathfrak{B} = \{z : 0 < \mathrm{Im}(z) < \frac{3}{4}\mathrm{Im}(\tau)\}$, and $p < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q - \alpha)$, we have*
$$\mathbb{E}\left[\left|\int_0^1 e^{F(x)}\sin(\pi x)^{-\frac{\alpha\gamma}{2}}\Theta_\tau(x+u)^{\frac{\chi\gamma}{2}}f(x)e^{\frac{\gamma}{2}Y_\infty(x)}dx\right|^p\right] < \infty;$$
- *for $p < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q - \alpha \vee \gamma)$ and $y \in [0, 1]$ we have*
$$\mathbb{E}\left[\left|\int_0^1 e^{F(x)+\frac{\gamma^2}{4}\mathbb{E}[Y_\infty(x)Y_\infty(y)]}\sin(\pi x)^{-\frac{\alpha\gamma}{2}}\Theta_\tau(x+u)^{\frac{\chi\gamma}{2}}f(x)e^{\frac{\gamma}{2}Y_\infty(x)}dx\right|^p\right] < \infty.$$

Proof. For the first claim, since a positive function is integrated against the GMC measure, we are in the classical case of the existence of moments of GMC with insertion of weight α. Following [DKRV16, Lemma 3.10], adapted to the case of one-dimensional GMC, the condition is thus $\alpha < Q$ and $p < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q-\alpha)$.

The second claim is more difficult since the integrand $\Theta_\tau(x+u)^{\frac{\chi\gamma}{2}}$ is a complex valued quantity. For the case of positive moments $p \geq 0$ one can simply use the bound

$$\mathbb{E}\left[\left|\int_0^1 e^{F(x)} \sin(\pi x)^{-\frac{\alpha\gamma}{2}} \Theta_\tau(x+u)^{\frac{\chi\gamma}{2}} f(x) e^{\frac{\gamma}{2} Y_\infty(x)} dx\right|^p\right] \leq M \mathbb{E}\left[\left(\int_0^1 e^{F(x)} \sin(\pi x)^{-\frac{\alpha\gamma}{2}} f(x) e^{\frac{\gamma}{2} Y_\infty(x)} dx\right)^p\right],$$

which is valid for some constant $M > 0$. The claim then reduces to the first case.

For negative moments corresponding to $p < 0$, we know by Lemma B.4 that for all $x \in (0,1)$, $\Theta_\tau(x+u)^{\frac{\chi\gamma}{2}}$ is strictly contained in a half-space, touching the boundary of the half-space only at $x = 0, 1$. Let $v_1 \in \mathbb{C}$ be a normal vector contained in the half-space, and let $v_2 \in \mathbb{C}$ be perpendicular to v_1. We have $\Theta_\tau(x+u)^{\frac{\chi\gamma}{2}} = h_1(x) v_1 + h_2(x) v_2$ with $h_1(x) > 0$ except possibly at $x = 0, 1$. This implies the upper bound

$$\mathbb{E}\left[\left|\int_0^1 e^{F(x)} \sin(\pi x)^{-\frac{\alpha\gamma}{2}} \left(h_1(x) v_1 + h_2(x) v_2\right) f(x) e^{\frac{\gamma}{2} Y_\infty(x)} dx\right|^p\right] \leq M' \mathbb{E}\left[\left(\int_0^1 e^{F(x)} \sin(\pi x)^{-\frac{\alpha\gamma}{2}} h_1(x) f(x) e^{\frac{\gamma}{2} Y_\infty(x)} dx\right)^p\right]$$

for some $M' > 0$. Therefore we can again apply the first case to show finiteness.

Lastly, the third claim can be treated the exact same way as for the second claim except the bound on p changes to $p < \frac{4}{\gamma^2} \wedge \frac{2}{\gamma}(Q - \alpha \vee \gamma)$ due to the $\frac{\gamma^2}{4} \mathbb{E}[Y_\infty(x) Y_\infty(y)]$ term, which should be understood as a γ insertion resulting in a modification on the bound on p as in [DKRV16, Lemma 3.10]. □

Finally, we state Girsanov's theorem in a form used frequently in the main text.

Theorem C.5. *Let $Y(x)$ be either of the Gaussian fields $Y_\infty(x)$ or $Y_\tau(x)$ on $[0,1]$ defined in Section 2.1. Let $\mathcal{X}$ be a Gaussian variable measurable with respect to Y, and let F be a bounded continuous function. Then we have*

$$\mathbb{E}\left[e^{\mathcal{X} - \frac{1}{2}\mathbb{E}[\mathcal{X}^2]} F((Y(x))_{x \in [0,1]})\right] = \mathbb{E}\left[F((Y(x) + \mathbb{E}[\mathcal{X} Y(x)])_{x \in [0,1]})\right]. \tag{C.4}$$

Theorem C.5 means that under the reweighing by the Radon-Nikodym derivative $e^{\mathcal{X} - \frac{1}{2}\mathbb{E}[\mathcal{X}^2]}$, the law of Y is that of $Y(x) + \mathbb{E}[\mathcal{X} Y(x)])_{x \in [0,1]}$ under the original probability. Therefore, (C.4) holds if F is a non-negative measurable function. We will frequently apply this result to the case where F is a moment of the GMC measure constructed from the field $Y(x)$. More precisely, let $f : [0,1] \mapsto (0, +\infty)$ be a continuous bounded function and $p < \frac{4}{\gamma^2}$. Then one has

$$\mathbb{E}\left[e^{\mathcal{X} - \frac{1}{2}\mathbb{E}[\mathcal{X}^2]} \left(\int_0^1 f(x) e^{\frac{\gamma}{2} Y(x)} dx\right)^p\right] = \mathbb{E}\left[\left(\int_0^1 f(x) e^{\frac{\gamma}{2}\mathbb{E}[\mathcal{X} Y(x)]} e^{\frac{\gamma}{2} Y(x)} dx\right)^p\right]. \tag{C.5}$$

Appendix D. Hypergeometric differential equations

For complex parameters A, B, C and a function g on $\mathbb{C}$, the (inhomogeneous) hypergeometric differential equation with inhomogeneous part $g(w)$ is the second order ODE

$$\Big(w(1-w)\partial_{ww} + (C - (1 + A + B)w)\partial_w - AB\Big) f(w) = g(w) \tag{D.1}$$

for an unknown function $f(w)$. This appendix presents background on the hypergeometric differential equation [DLMF, Chapter 15]. Throughout this appendix we assume that C is not an integer. Moreover, we assume $\arg w \in (-\pi, \pi)$ when considering fractional power of w so that the branch cut is at $(-\infty, 0)$.

D.1. **Homogeneous hypergeometric differential equations.** We now assume that $g(w)=0$ so that the equation (D.1) is homogeneous. Solving the second order ODE in power series gives the following result.

Lemma D.1. *Fix $X \in \{0, 1-C\}$. When $g(w)=0$, solutions to* (D.1) *which can be written in the form $w^X f(w)$ with f analytic in $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$ form a one dimensional linear space.*

The *Gauss hypergeometric function* ${}_2F_1(A,B,C;w)$ is defined to be the solution to (D.1) with $g=0$ satisfying Property (R) from Definition 4.3 and ${}_2F_1(A,B,C;0)=1$. Set

$$v_1(w) := {}_2F_1(A,B,C;w) \qquad \text{and} \qquad v_2(w) := {}_2F_1(1+A-C, 1+B-C, 2-C; w).$$

Then $w^{1-C}v_2(w)$ is also a solution to (D.1). Moreover, on any open subset of $\{w \in \mathbb{C} : w \neq (-\infty, 0] \cup [1,\infty)\}$, equation (D.1) has 2-dimensional solution space spanned by $v_1(w)$ and $w^{1-C}v_2(w)$.

The power series coefficient of ${}_2F_1(A,B,C;w)$ is characterized by $a_0 = 1$ and $\frac{a_{n+1}}{a_n} = \frac{(n+A)(n+B)}{(n+1)(n+C)}$ for $n \in \mathbb{N}$. If $\mathrm{Re}(C) > \mathrm{Re}(A+B)$, this power series converges absolutely on the closed unit disk $\overline{\mathbb{D}}$. We find:

Lemma D.2. *If* $\mathrm{Re}(C) > \mathrm{Re}(A+B)$*, both $v_1(w)$ and $v_2(w)$ satisfy Property (R) from Definition 4.3.*

A separate basis of solutions to (D.1) with similar good behavior at $w=1$ is given by

$${}_2F_1(A,B,1+A+B-C,1-w) \qquad \text{and} \qquad (1-w)^{C-A-B}{}_2F_1(C-A, C-B, 1+C-A-B, 1-w).$$

These two bases of solutions are related by *connection equations*, one of which is

$$\text{(D.2)} \quad {}_2F_1(A,B,1+A+B-C,1-w) = \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)}v_1(w) + \frac{\Gamma(2-C)\Gamma(C-A-B)}{\Gamma(1-A)\Gamma(1-B)}w^{1-C}v_2(w).$$

If $\mathrm{Re}(C) > \mathrm{Re}(A+B)$, the coefficients in the connection equation (D.2) satisfy *Euler's identity*

$$\text{(D.3)} \qquad v_1(1) = {}_2F_1(A,B,C,1) = \frac{\Gamma(C)\Gamma(C-A-B)}{\Gamma(C-A)\Gamma(C-B)} \quad \text{and} \quad v_2(1) = \frac{\Gamma(2-C)\Gamma(C-A-B)}{\Gamma(1-A)\Gamma(1-B)}.$$

Moreover, the quantity $\frac{{}_2F_1(A,B,C,w)}{\Gamma(C)}$ is holomorphic as a function of A,B,C. Since Γ is meromorphic on $\mathbb{C}$ with poles at $\{0,-1,-2,\cdots\}$ and has no zeros, we have the following.

Lemma D.3. *Let $V = \{(A,B,C) \in \mathbb{C}^3 : \mathrm{Re}(C) > \mathrm{Re}(A+B)$ and $C \notin \mathbb{Z}\}$. Both functions $(w,A,B,C) \mapsto v_1$ and $(w,A,B,C) \mapsto v_2$ are continuous on $\overline{\mathbb{D}} \times V$ and analytic on $\mathbb{D} \times V$. Moreover, if $(A,B,C) \in V$, then $v_1(1) \neq 0$ and $v_2(1) \neq 0$.*

D.2. **Inhomogeneous hypergeometric differential equations.** If $g(w)$ is not identically zero, then any solution to (D.1) can be written as

$$\text{(D.4)} \qquad f(w) = f_{\mathrm{homog}}(w) + f_{\mathrm{part}}(w),$$

where $f_{\mathrm{part}}(w)$ is a particular solution to (D.1) and $f_{\mathrm{homog}}(w)$ solves the homogeneous version of (D.1). We will give a particular solution to (D.1) in terms of power series. We use the following notion of integration. Fix $\beta \in \mathbb{C} \setminus \{1,2,\cdots\}$. If $f(w)$ admits the series form $f(w) = \sum_{n=0}^{\infty} a_n w^n$ for $w \in \mathbb{D}$ such that

$$\text{(D.5)} \qquad \sum_{n=1}^{\infty} \frac{|a_n|}{n} < \infty,$$

we use the notation

$$\text{(D.6)} \qquad \int_0^w t^{-\beta} f(t)dt := w^{1-\beta} \sum_{n=0}^{\infty} \frac{a_n}{n-\beta+1} w^n \qquad \text{for } w \in \overline{\mathbb{D}}.$$

When $t^{-\beta}f(t)$ is integrable on $[0,w]$, (D.6) is a numerical equality, but extends the definition beyond the domain using the power series form. The following lemma characterizes the power series in (D.6); its proof is an easy exercise left to the reader.

Lemma D.4. *Using the notation (D.6), we have*

$$\frac{\partial}{\partial w}\left(\int_0^w t^{-\beta} f(t)dt\right) = w^{-\beta} f(w) \qquad \text{for } w \in \mathbb{D}.$$

Moreover, $w^{\beta}\int_0^w t^{-\beta} f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n-\beta+1} w^{n+1}$ satisfies Property (R).

Lemma D.5. *Assume that*

$$C \text{ is not an integer and } \operatorname{Re}(C-A-B)\in(0,1). \tag{D.7}$$

Fix $X\in\{0,1-C\}$. Suppose $g(w)=w^X\widetilde{g}(w)$, and $\widetilde{g}(w)$ is a function satisfying Property (R). For each $a\in\mathbb{C}$, there exists a unique function f_a satisfying Property (R) such that $f_a(1)=a$ and $w^Xf_a(w)$ solves equation (D.1)*. Moreover, $(w,A,B,C)\mapsto f_a(w)$ is continuous on $\overline{\mathbb{D}}\times U$ and analytic on $\mathbb{D}\times U$, where $U=\{(A,B,C)\in\mathbb{C}^3:\operatorname{Re}(C-A-B)\in(0,1)$ and $C\notin\mathbb{Z}\}$.*

Proof. It is elementary to check that if $\sum_{n=0}^\infty a_nt^n$ satisfies Property (R) and $\sum_{n=0}^\infty b_nt^n$ satisfies condition (D.5), then the series $(\sum_0^\infty a_nt^n)(\sum_0^\infty b_nt^n)$ satisfies condition (D.5). By (D.7), the series $(1-t)^{A+B-C}$ satisfies condition (D.5). Therefore $\frac{v_2(t)g(t)}{(1-t)^{C-A-B}}$ (resp., $\frac{v_1(t)g(t)}{t^{1-C}(1-t)^{C-A-B}}$) can be written as t^X (resp., t^{C-1+X}) times a power series satisfying (D.5). Let

$$f_{\text{part}}(w):=-\frac{v_1(w)}{1-C}\int_0^w\frac{v_2(t)g(t)}{(1-t)^{C-A-B}}dt+\frac{v_2(w)}{1-C}w^{1-C}\int_0^w\frac{v_1(t)g(t)}{t^{1-C}(1-t)^{C-A-B}}dt,\ \text{for } w\in\overline{\mathbb{D}}, \tag{D.8}$$

where both expressions in (D.8) represent series defined using (D.6). Then by Lemma D.5,

$$\text{both } w^{-X}\int_0^w\frac{v_2(t)g(t)}{(1-t)^{C-A-B}}dt \text{ and } w^{1-C-X}\int_0^w\frac{v_1(t)g(t)}{t^{1-C}(1-t)^{C-A-B}}dt \text{ satisfy Property (R).} \tag{D.9}$$

Hence $w^{-X}f_{\text{part}}(w)$ satisfies Property (R). Moreover, $w^{-X}f_{\text{part}}(w)$ depends on A,B,C analytically.

A direct computation using Lemma D.4 shows that f_{part} is a particular solution to (D.1), where we note that the form of f_{part} is motivated by variation of parameters and the fact that the Wronskian for the homogeneous fundamental solutions $\{v_1(w),w^{1-C}v_2(w)\}$ is $(1-C)w^{-C}(1-w)^{C-A-B-1}$. Since $v_1(1)\neq0$ by (D.3), if $X=0$, then

$$f_a(w)=f_{\text{part}}(w)+(a-f_{\text{part}}(1))\frac{v_1(w)}{v_1(1)} \tag{D.10}$$

is the desired function, which is unique by Lemma D.1. If $X=1-C$, we conclude similarly with $w^{1-C}v_2(w)$ in place of $v_1(w)$. □

Lemma D.6. *Suppose $U\subset\mathbb{C}$ is an open set and $g(w,\alpha)$ is a function which is (w,α)-regular on $\overline{\mathbb{D}}\times U$ in the sense of Definition 4.10. Suppose we are in the setting of Lemma D.5. For $\alpha\in U$, let $f(w,\alpha)$ be defined as $f_a(w)$ in Lemma D.5 with $g=g(w,\alpha)$. Then $f(w,\alpha)$ is (w,α)-regular on $\overline{\mathbb{D}}\times U$.*

Proof. Recall that ${}_2F_1(A,B,C,w)$ is holomorphic for $C\notin\{0,-1,-2,\ldots\}$. By the same argument as in (D.9), we see that both $w^{-X}\int_0^w\frac{v_2(t)g(t)}{(1-t)^{C-A-B}}dt$ and $w^{1-C-X}\int_0^w\frac{v_1(t)g(t)}{t^{1-C}(1-t)^{C-A-B}}dt$ are (w,α)-regular on $\overline{\mathbb{D}}\times U$. Therefore $w^{-X}f_{\text{part}}$ is (w,α)-regular on $\overline{D}\times U$ with f_{part} from equation (D.8). If $X=0$, we obtain Lemma D.6 by (D.10). If $X=1-C$, we can use the counterpart of (D.10) with $w^{1-C}v_2$ in place of v_1. □

We now state Lemma D.8, a simple fact used in the proof of Lemma 4.5 concerning the behavior of the solutions near 0. We do not require Property (R) here, allowing us to remove the condition $\operatorname{Re}(C-A-B)\in(0,1)$. To prove Lemma D.8, we use the following variant of Lemma D.5, proved by the same argument as Lemma D.5.

Lemma D.7. *Suppose C is not an integer. Fix $X\in\{0,1-C\}$. Suppose $g(w)=w^X\widetilde{g}(w)$, and $\widetilde{g}(w)$ is an analytic function on $\mathbb{D}$. Let f_{part} be defined as in* (D.8)*. Then f_{part} is a particular solution to* (D.1)*. Moreover, $w^{-X}f_{part}(w)$ is an analytic function on $\mathbb{D}$.*

Lemma D.8. *Suppose A,B,C,X,g are as in Lemma D.7 with $\operatorname{Re}(1-C)>0$. Given $\theta_0\in[0,2\pi)$, let $D=\{z=re^{\mathbf{i}\theta}:r\in(0,1),\theta\neq\theta_0\}$. Suppose f solves equation* (D.1) *on an open set $U\subset D$. Then f can be extended to an analytic function on D such that as $w\in D$ tends to 0, $f(w)$ tends to a finite number.*

Proof. Since w^X restricted to U can be analytically extended to D, Lemma D.8 follows from Lemma D.7, (D.4), and the solution structure of the homogeneous hypergeometric equation. □

Appendix E. Proof of OPE lemmas

In this appendix, we provide the proofs of Lemmas 5.5 and 5.6 concerning OPE. Similar estimates have been performed in the works [KRV19a], [RZ18], [RZ20]. The paper [KRV19a] introduced the method to study the reflection principle and the reflection coefficient as required for Lemma 5.6. The generalization to complex valued observables, which is necessary for the u-deformed block, has been performed in [RZ20]. We will be quite brief in places where the detailed arguments in [RZ20] can be adapted straightforwardly.

Proof of Lemma 5.5. Let $g(u) := \int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}}\Theta_\tau(u+x)^{\frac{\gamma^2}{4}}e^{\pi\gamma Px}dx$ and $f(u) = \mathbb{E}[g(u)^{-\frac{\alpha}{\gamma}+\frac{1}{2}}]$. Recall the remark below Definition 3.3. Due to the prefactor $\sin(\pi u)^{l_\chi}$ in $\phi^{\alpha}_{\frac{\gamma}{2}}(u,q)$, we can write $\phi^{\alpha}_{\frac{\gamma}{2}}(u,q)$ as $\Sigma(u)f(u)$ where Σ is differentiable at 0. Here we drop the dependence of Σ and f in q,P,γ for simplicity.

For $t\in[0,1]$, let $g(t,u) := (1-t)g(0)+tg(u)$. Then

$$f(u)-f(0) = \int_0^1 \partial_t\mathbb{E}[g(t,u)^{-\frac{\alpha}{\gamma}+\frac{1}{2}}]dt = \left(-\frac{\alpha}{\gamma}+\frac{1}{2}\right)\int_0^1 \mathbb{E}[(g(u)-g(0))g(t,u)^{-\frac{\alpha}{\gamma}-\frac{1}{2}}]dt.$$

We claim that uniformly in $t\in[0,1]$,

(E.1)
$$\lim_{u\to 0} u^{-2l_0-1}\mathbb{E}[(g(u)-g(0))g(t,u)^{-\frac{\alpha}{\gamma}-\frac{1}{2}}] = (1-e^{\pi\gamma P-2\mathbf{i}\pi l_0})C\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}-\frac{\gamma^2}{4}}e^{\pi\gamma Px}dx\right)^{-\frac{\alpha}{\gamma}-\frac{1}{2}}\right],$$

where

$$C = e^{2\mathbf{i}\pi l_0-\frac{\mathbf{i}\pi\gamma^2}{2}}q^{-\frac{\alpha\gamma}{12}-\frac{\gamma^2}{24}}\eta(q)^{-\frac{\alpha\gamma}{2}-\frac{\gamma^2}{4}}\Theta'_\tau(0)^{2l_0}\frac{\Gamma(1-\frac{\alpha\gamma}{2})\Gamma(-1+\frac{\alpha\gamma}{2}-\frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})}.$$

Since $1+2l_0\in(0,1)$ and Σ is differentiable at 0, Equation (E.1) yields that

$$\begin{aligned}&\lim_{u\to 0}\sin(\pi u)^{-2l_0-1}\left(\phi^{\alpha}_{\frac{\gamma}{2}}(u,q)-\phi^{\alpha}_{\frac{\gamma}{2}}(0,q)\right)\\ =&\pi^{-2l_0-1}\Sigma(0)(1-e^{\pi\gamma P-2\mathbf{i}\pi l_0})\left(-\frac{\alpha}{\gamma}+\frac{1}{2}\right)C\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}-\frac{\gamma^2}{4}}e^{\pi\gamma Px}dx\right)^{-\frac{\alpha}{\gamma}-\frac{1}{2}}\right].\end{aligned}$$

Recall $\Theta'_\tau(0) = -2\pi\eta(q)^3$ from (B.3). Plugging in the value of $\Sigma(0)$ and the definitions of $W^+_{\frac{\gamma}{2}}$ and $A^q_{\gamma,P}$, we get Lemma 5.5.

It remains to prove (E.1). For all $t\in[0,1]$, by Girsanov's theorem (Theorem C.5) we have

$$\mathbb{E}[(g(u)-g(0))g(t,u)^{-\frac{\alpha}{\gamma}-\frac{1}{2}}] = \int_0^1 \Theta_\tau(y)^{-\frac{\alpha\gamma}{2}}\left(\Theta_\tau(u+y)^{\frac{\gamma^2}{4}}-\Theta_\tau(y)^{\frac{\gamma^2}{4}}\right)e^{\pi\gamma Py}\mathbb{E}(y,u,t)dy,$$

where

$$\mathbb{E}(y,u,t) = \mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}q^{\frac{\gamma^2}{12}}\eta(q)^{\frac{\gamma^2}{2}}\frac{\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}}}{|\Theta_\tau(x-y)|^{\frac{\gamma^2}{2}}}\left((1-t)\Theta_\tau(x)^{\frac{\gamma^2}{4}}+t\Theta_\tau(u+x)^{\frac{\gamma^2}{4}}\right)e^{\pi\gamma Px}dx\right)^{-\frac{\alpha}{\gamma}-\frac{1}{2}}\right].$$

For $\delta\in(\frac{1}{1-2l_0},1)$, as $u\to 0$, we have

$$\max_{y\in[u^{1-\delta},1-u^{1-\delta}]} u^{-2l_0-1}\Theta_\tau(y)^{-\frac{\alpha\gamma}{2}}\left(\Theta_\tau(u+y)^{\frac{\gamma^2}{4}}-\Theta_\tau(y)^{\frac{\gamma^2}{4}}\right)e^{\pi\gamma Py} = O(u^{\delta(1-2l_0)-1}) = o(1),$$

On the other hand, $\int_0^1\mathbb{E}(y,u,t)dy$ is uniformly bounded for u small enough. Therefore, uniformly in $t\in[0,1]$,

$$\lim_{u\to 0}u^{-2l_0-1}\int_{u^{1-\delta}}^{1-u^{1-\delta}}\Theta_\tau(y)^{-\frac{\alpha\gamma}{2}}\left(\Theta_\tau(u+y)^{\frac{\gamma^2}{4}}-\Theta_\tau(y)^{\frac{\gamma^2}{4}}\right)e^{\pi\gamma Py}\mathbb{E}(y,u,t)dy = 0. \tag{E.2}$$

Now we switch our attention to the integral on $[0, u^{1-\delta}]$. By a change of variable $y = uz$, we have

$$\lim_{u\to 0} u^{-2l_0-1}\int_0^{u^{1-\delta}} \Theta_\tau(y)^{-\frac{\alpha\gamma}{2}}\Big(\Theta_\tau(u+y)^{\frac{\gamma^2}{4}} - \Theta_\tau(y)^{\frac{\gamma^2}{4}}\Big)e^{\pi\gamma P y}\mathbb{E}(y,u,t)dy$$

$$= \lim_{u\to 0} u^{-2l_0}\int_0^{u^{-\delta}} \Theta_\tau(uz)^{-\frac{\alpha\gamma}{2}}\Big(\Theta_\tau(u+uz)^{\frac{\gamma^2}{4}} - \Theta_\tau(uz)^{\frac{\gamma^2}{4}}\Big)e^{\pi\gamma P uz}$$

$$\times\,\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}q^{\frac{\gamma^2}{12}}\eta(q)^{\frac{\gamma^2}{2}}\frac{\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}}}{|\Theta_\tau(x-uz)|^{\frac{\gamma^2}{2}}}\Big((1-t)\Theta_\tau(x)^{\frac{\gamma^2}{4}} + t\Theta_\tau(u+x)^{\frac{\gamma^2}{4}}\Big)e^{\pi\gamma P x}dx\right)^{-\frac{\alpha}{\gamma}-\frac12}\right]dz$$

(E.3)
$$= C'\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}-\frac{\gamma^2}{4}}e^{\pi\gamma P x}dx\right)^{-\frac{\alpha}{\gamma}-\frac12}\right],$$

where $C' := e^{2\mathbf{i}\pi l_0 - \frac{\mathbf{i}\pi\gamma^2}{2}}q^{-\frac{\alpha\gamma}{12}-\frac{\gamma^2}{24}}\eta(q)^{-\frac{\alpha\gamma}{2}-\frac{\gamma^2}{4}}\int_0^\infty \lim_{u\to 0} u^{-2l_0}\Theta_\tau(uz)^{-\frac{\alpha\gamma}{2}}\Big(\Theta_\tau(u+uz)^{\frac{\gamma^2}{4}} - \Theta_\tau(uz)^{\frac{\gamma^2}{4}}\Big)e^{\pi\gamma P uz}dz$. The convergence above is uniformly in $t \in [0,1]$, where we have applied the dominated convergence theorem. The prefactor $e^{2\mathbf{i}\pi l_0 - \frac{\mathbf{i}\pi\gamma^2}{2}}$ comes from applying (B.21) to replace $|\Theta_\tau(x)|^{-\frac{\gamma^2}{2}}$ by $e^{\mathbf{i}\pi\frac{\gamma^2}{2}}\Theta_\tau(x)^{-\frac{\gamma^2}{2}}$ and then pulling the phase factor outside the expectation.

For $|z| \le u^{-\delta}$, the limit $\lim_{u\to 0} u^{-2l_0}\Theta_\tau(uz)^{-\frac{\alpha\gamma}{2}}\Big(\Theta_\tau(u+uz)^{\frac{\gamma^2}{4}} - \Theta_\tau(uz)^{\frac{\gamma^2}{4}}\Big)e^{\pi\gamma P uz}$ is given by

$$\lim_{u\to 0} u^{-2l_0}(uz\Theta_\tau'(0))^{-\frac{\alpha\gamma}{2}}\Big((u(1+z)\Theta_\tau'(0))^{\frac{\gamma^2}{4}} - (uz\Theta_\tau'(0))^{\frac{\gamma^2}{4}}\Big) = \Theta_\tau'(0)^{2l_0}z^{-\frac{\alpha\gamma}{2}}\Big((1+z)^{\frac{\gamma^2}{4}} - z^{\frac{\gamma^2}{4}}\Big).$$

Substituting this equation into (E.3) yields

$$C' = e^{2\mathbf{i}\pi l_0 - \frac{\mathbf{i}\pi\gamma^2}{2}}\Theta_\tau'(0)^{2l_0}q^{-\frac{\alpha\gamma}{12}-\frac{\gamma^2}{24}}\eta(q)^{-\frac{\alpha\gamma}{2}-\frac{\gamma^2}{4}}\int_0^\infty z^{-\frac{\alpha\gamma}{2}}\Big((1+z)^{\frac{\gamma^2}{4}} - z^{\frac{\gamma^2}{4}}\Big)dz.$$

Since $\alpha \in (\frac{\gamma}{2}, \frac{2}{\gamma})$ the integral above over z is absolutely convergent and can be explicitly evaluated as

$$\int_0^\infty z^{-\frac{\alpha\gamma}{2}}\Big((1+z)^{\frac{\gamma^2}{4}} - z^{\frac{\gamma^2}{4}}\Big)dz = \frac{\Gamma(1-\frac{\alpha\gamma}{2})\Gamma(-1+\frac{\alpha\gamma}{2}-\frac{\gamma^2}{4})}{\Gamma(-\frac{\gamma^2}{4})}.$$

Therefore $C = C'$, hence (E.3) remains true if C' is replaced by C.

Applying the same argument to $\lim_{u\to 0} u^{-2l_0-1}\int_{1-u^{1-\delta}}^1 \Theta_\tau(y)^{-\frac{\alpha\gamma}{2}}\Big(\Theta_\tau(u+y)^{\frac{\gamma^2}{4}} - \Theta_\tau(y)^{\frac{\gamma^2}{4}}\Big)e^{\pi\gamma P y}\mathbb{E}(y,u,t)dy$ implies that this limit equals $-e^{\pi\gamma P - 2\mathbf{i}\pi l_0}C\mathbb{E}\left[\Big(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}-\frac{\gamma^2}{4}}e^{\pi\gamma P x}dx\Big)^{-\frac{\alpha}{\gamma}-\frac12}\right]$. Combining with (E.2), we get (E.1).

Proof of Lemma 5.6. Recall the field $X_{\mathbb{H}}$ on $\mathbb{H}$ and the map ϕ from Section C. Consider a sample of F_τ as in (2.5) in Section 2.1, independent of $X_{\mathbb{H}}$. Let $Y_\infty := X_{\mathbb{H}}(\phi(x))$ for $x \in [0,1]$ as in Lemma C.2, and let $Y_\tau := Y_\infty + F_\tau$. Throughout this section we work under this particular coupling of $X_{\mathbb{H}}, Y_\infty, Y_\tau$.

We write $u = \mathbf{i}t$ and work with small $t > 0$. For a Borel set $I \subseteq [0,1]$, we introduce the notation

(E.4)
$$K_I(\mathbf{i}t) := \int_I e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}}\Theta_\tau(\mathbf{i}t+x)^{\frac{\gamma\chi}{2}}e^{\pi\gamma P x}dx, \qquad \text{and} \qquad s = -\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}$$

so that the difference in Lemma 5.6 equals $\mathbb{E}[K_{[0,1]}(\mathbf{i}t)^s] - \mathbb{E}[K_{[0,1]}(0)^s]$.

Throughout this section we assume α to be close enough to Q such that

(E.5)
$$\chi(Q-\alpha) < h_\alpha := (1 + (\frac{2}{\gamma} - \frac{\gamma}{2})\alpha - \frac{4}{\gamma^2})(\frac{\gamma\alpha}{2} - 1)^{-1} \wedge (\frac{\gamma\alpha}{2} - 1)(1 - \frac{\gamma\alpha}{2} + \gamma^2)^{-1} \wedge 1$$

and then fix $h \in (\chi(Q-\alpha), h_\alpha)$. This condition on h corresponds to the conditions [RZ20, Equation (5.18)] and [RZ20, Equation (5.52)] respectively for the case $\chi = \frac{2}{\gamma}$ and $\chi = \frac{\gamma}{2}$ needed in the proof of Lemma E.1 stated below. Notice it is indeed possible to choose α close enough to Q such that (E.5) holds since as α

tends to Q, $\chi(Q-\alpha)$ converges to 0 and h_α to fix positive number depending only on γ. Notice also the condition $\chi(Q-\alpha)<1$ included in (E.5). Now let

$$g_\tau(t) := e^{\frac{\gamma}{2}F_\tau(0)}e^{\frac{\gamma}{2}\overline{X}(4\pi t^{1+h})} \text{ and } \sigma_t := \Theta'_\tau(0)^{\frac{\gamma\chi}{2}-\frac{\alpha\gamma}{2}}(2\pi)^{\frac{\gamma^2}{4}}t^{\frac{\gamma\chi}{2}+\frac{\gamma}{2}(1+h)(Q-\alpha)}e^{-\frac{\gamma^2}{8}\mathbb{E}[F_\tau(0)^2]}g_\tau(t). \tag{E.6}$$

Recall $\mathcal{B}^\lambda$ from (5.10) with $\lambda=\frac{Q-\alpha}{2}$. Let M is an exponential random variable with rate $(Q-\alpha)$, namely, $\mathbb{P}[M>x]=e^{-(Q-\alpha)x}$ for $x>0$. Consider an independent coupling of $(M,\mathcal{B}^\lambda,Z_{\mathbb{H}})$, which is also independent of $(X_{\mathbb{H}},F_\tau)$ above. Recall $\rho(\alpha,1,e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P})$ from (5.11), defined in terms of $(\mathcal{B}^\lambda,Z_{\mathbb{H}})$.

The OPE method in [KRV19a, RZ20] gives

Lemma E.1. *As $t\to 0$, the difference $\mathbb{E}[K_{[0,1]}(\mathbf{i}t)^s]-\mathbb{E}[K_{[0,1]}(0)^s]$ can be written as*

$$\mathbb{E}[(K_{(t,1-t)}(\mathbf{i}t)+\mathbf{i}^{\frac{\gamma\chi}{2}}\sigma_t e^{\frac{\gamma}{2}M}\rho(\alpha,1,e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P}))^s]-\mathbb{E}[K_{(t,1-t)}(\mathbf{i}t)^s]+o(t^{\chi(Q-\alpha)}).$$

Lemma E.2. $\lim_{t\to0} t^{\chi(\alpha-Q)}\mathbb{E}\left[\sigma_t^{\frac{2}{\gamma}(Q-\alpha)}K_{(t,1-t)}(\mathbf{i}t)^{s-\frac{2}{\gamma}(Q-\alpha)}\right]=C\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\gamma}{2}(2Q-\alpha-\chi)}e^{\pi\gamma P x}dx\right)^{\frac{\alpha+\chi-2Q}{\gamma}}\right]$ *where* $C:=(2\pi)^{(\alpha-Q)(\frac{2}{\gamma}-\alpha)}(q^{\frac16}\eta(q))^{(Q-\alpha)(\alpha+\chi-2Q)}\Theta'_\tau(0)^{(Q-\alpha)(\chi-\alpha)}e^{\mathbf{i}\pi(Q-\alpha)(\alpha+\chi-2Q)}(q^{-\frac{1}{12}}\eta(q))^{(Q-\alpha)(2\alpha-\frac{4}{\gamma})}$.

Given Lemmas E.1 and E.2, Lemma 5.6 can be proved as follows. Since the density of $e^{\frac{\gamma}{2}M}$ is $\frac{2}{\gamma}(Q-\alpha)v^{-\frac{2}{\gamma}(Q-\alpha)-1}1_{v>1}dv$, we have

$$\begin{aligned}
&\mathbb{E}[(K_{(t,1-t)}(\mathbf{i}t)+\mathbf{i}^{\frac{\gamma\chi}{2}}\sigma_t e^{\frac{\gamma}{2}M}\rho(\alpha,1,e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P}))^s]-\mathbb{E}[K_{(t,1-t)}(\mathbf{i}t)^s]\\
&=\frac{2}{\gamma}(Q-\alpha)\mathbb{E}\left[\int_1^\infty\frac{dv}{v^{\frac{2}{\gamma}(Q-\alpha)+1}}\left(\left(K_{(t,1-t)}(\mathbf{i}t)+\mathbf{i}^{\frac{\gamma\chi}{2}}\sigma_t\rho(\alpha,1,e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P})v\right)^s-K_{(t,1-t)}(\mathbf{i}t)^s\right)\right]\\
&=\mathbf{i}^{\chi(Q-\alpha)}\frac{2}{\gamma}(Q-\alpha)\mathbb{E}\left[\int_{u_t}^\infty\frac{du(1+u)^s}{u^{\frac{2}{\gamma}(Q-\alpha)+1}}\left(\rho(\alpha,1,e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P})\sigma_t\right)^{\frac{2}{\gamma}(Q-\alpha)}K_{(t,1-t)}(\mathbf{i}t)^{s-\frac{2}{\gamma}(Q-\alpha)}\right]\\
&=\mathbf{i}^{\chi(Q-\alpha)}\frac{2}{\gamma}(Q-\alpha)\overline{R}(\alpha,1,e^{\pi\gamma P-\frac{\mathbf{i}\pi\gamma\chi}{2}})\mathbb{E}\left[\int_{u_t}^\infty\frac{du(1+u)^s}{u^{\frac{2}{\gamma}(Q-\alpha)+1}}\sigma_t^{\frac{2}{\gamma}(Q-\alpha)}K_{(t,1-t)}(\mathbf{i}t)^{s-\frac{2}{\gamma}(Q-\alpha)}\right],
\end{aligned}$$

where we have applied the change of variable $u=\frac{\mathbf{i}^{\frac{\gamma\chi}{2}}\sigma_t\rho(\alpha,1,e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P})}{K_{(t,1-t)}(\mathbf{i}t)}v$ with $u_t:=\frac{\mathbf{i}^{\frac{\gamma\chi}{2}}\sigma_t\rho(\alpha,1,e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P})}{K_{(t,1-t)}(\mathbf{i}t)}$ being random.

Since $\lim_{t\to0}u_t=0$ almost surely, one obtains by simple arguments of uniform integrability

$$\begin{aligned}
&\mathbb{E}[(K_{(t,1-t)}(\mathbf{i}t)+\mathbf{i}^{\frac{\gamma\chi}{2}}\sigma_t e^{\frac{\gamma}{2}M}\rho(\alpha,1,e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P}))^s]-\mathbb{E}[K_{(t,1-t)}(\mathbf{i}t)^s]\\
&=\mathbf{i}^{\chi(Q-\alpha)}\frac{2}{\gamma}(Q-\alpha)\overline{R}(\alpha,1,e^{\pi\gamma P-\frac{\mathbf{i}\pi\gamma\chi}{2}})\left(\int_0^\infty\frac{du(1+u)^s}{u^{\frac{2}{\gamma}(Q-\alpha)+1}}\right)\mathbb{E}\left[\sigma_t^{\frac{2}{\gamma}(Q-\alpha)}K_{(t,1-t)}(\mathbf{i}t)^{s-\frac{2}{\gamma}(Q-\alpha)}\right]+o(t^{\chi(Q-\alpha)})\\
&=-\mathbf{i}^{\chi(Q-\alpha)}\frac{\Gamma(\frac{2\alpha}{\gamma}-\frac{4}{\gamma^2})\Gamma(\frac{2Q-\alpha-\chi}{\gamma})}{\Gamma(\frac{\alpha}{\gamma}-\frac{\chi}{\gamma})}\overline{R}(\alpha,1,e^{\pi\gamma P-\frac{\mathbf{i}\pi\gamma\chi}{2}})\mathbb{E}\left[\sigma_t^{\frac{2}{\gamma}(Q-\alpha)}K_{(t,1-t)}(\mathbf{i}t)^{s-\frac{2}{\gamma}(Q-\alpha)}\right]+o(t^{\chi(Q-\alpha)}).
\end{aligned}$$

Using (B.3), we can simplify the prefactor C as

$$\begin{aligned}
C&=(2\pi)^{(\alpha-Q)(\frac{2}{\gamma}-\alpha)}q^{\frac16(Q-\alpha)(\chi+\frac{2}{\gamma}-2Q)}\eta(q)^{(Q-\alpha)(3\alpha+\chi-2Q-\frac{4}{\gamma})}\Theta'_\tau(0)^{(Q-\alpha)(\chi-\alpha)}e^{\mathbf{i}\pi(Q-\alpha)(\alpha+\chi-2Q)}\\
&=(2\pi)^{(Q-\alpha)(\frac{\gamma}{3}-\frac{\chi}{3}+\frac{2}{3\gamma})}q^{\frac16(Q-\alpha)(\chi+\frac{2}{\gamma}-2Q)}\Theta'_\tau(0)^{(Q-\alpha)(\frac{2\chi}{3}-\frac{4}{3\gamma}-\frac{2}{3\chi})}e^{\mathbf{i}\pi(Q-\alpha)(\frac{4}{3\gamma}-\frac{2\chi}{3}-\frac{4}{3\chi})}.
\end{aligned}$$

Since $1+2l_\chi=\chi(Q-\alpha)$, combing Lemmas E.1 and E.2, we obtain Lemma 5.6.

Proof of Lemma E.1. Applying the coordinate change in Lemma C.2 and using the fact $|\phi^{-1}(0)'|=(2\pi)^{-1}$, we can find a function $f(t,x)$ such that for each Borel set $I\subset[0,1]$ we have

$$K_I(\mathbf{i}t):=\int_{\phi(I)}e^{\frac{\gamma}{2}X_{\mathbb{H}}(y)}|(2\pi)^{-1}y|^{-\frac{\alpha\gamma}{2}}(\mathbf{i}t+(2\pi)^{-1}y)^{\frac{\gamma\chi}{2}}e^{\frac{\gamma}{2}F_\tau(\phi^{-1}(y))}f(t,y)dy. \tag{E.7}$$

Although f has an explicit expression, we will only need the following two facts. Firstly, $f(t,y)$ is bounded on $[0,0.1\operatorname{Im}\tau]\times\mathbb{R}$. Moreover, by Lemma 2.5, the following two limits exist:

(E.8)

$$f(0,0^+):=\lim_{y\to0^+}f(0,y)=\mathbb{E}[e^{-\frac{\gamma^2}{8}F_\tau(0)}]|\Theta'_\tau(0)|^{-\frac{\alpha\gamma}{2}}\Theta'_\tau(0)^{\frac{\gamma\chi}{2}} \text{ and } f(0,0^-):=\lim_{y\to0^-}f(0,y)=e^{-\frac{\mathbf{i}\chi\gamma}{2}}e^{\pi\gamma P}f(0,0^+).$$

Let $\widetilde{Z}$ be independent of $(X_{\mathbb{H}}, F_\tau)$ and with the law of $Z_{\mathbb{H}}$. Let $\widetilde{X}_{\mathbb{H}} = \overline{X} + \widetilde{Z}$. Let $\tilde{K}_{[0,t^{1+h})\cup(1-t^{1+h},1]}(\mathbf{i}t)$ be defined as in (E.7) with $\widetilde{X}_{\mathbb{H}}$ in place of $X_{\mathbb{H}}$ and I set to be $[0,t^{1+h})\cup(1-t^{1+h},1]$. By the argument in [RZ20], for $h \in (\chi(Q-\alpha), h_\alpha)$, the difference $\mathbb{E}[K_{[0,1]}(\mathbf{i}t)^s] - \mathbb{E}[K_{[0,1]}(0)^s]$ can be written as

$$\mathbb{E}[(K_{(t,1-t)}(\mathbf{i}t) + \tilde{K}_{[0,t^{1+h})\cup(1-t^{1+h},1]}(\mathbf{i}t))^s] - \mathbb{E}\left[(K_{(t,1-t)}(\mathbf{i}t))^s\right] + o(t^{\chi(Q-\alpha)}). \tag{E.9}$$

More precisely, if $f(t,x)$ is replaced by a certain piecewise constant function and $e^{\frac{\gamma}{2}F_\tau(\phi^{-1}(y))}$ is replaced by 1 in (E.7), then this claim is a special case of [RZ20][2] with (β_1, q) there replaced by (α, s). However, since f is bounded and $e^{\frac{\gamma}{2}F_\tau(\phi^{-1}(y))}$ is independent of $X_{\mathbb{H}}$ with uniformly bounded positive moments of all order, the exact same argument works for our case as well.

We now claim that $\mathbb{E}[(K_{(t,1-t)}(\mathbf{i}t) + \tilde{K}_{[0,t^{1+h})\cup(1-t^{1+h},1]}(\mathbf{i}t))^s]$ in (E.9) can be replaced by

$$\mathbb{E}[(K_{(t,1-t)}(\mathbf{i}t) + \tilde{K}_{[0,t^{1+h})\cup(1-t^{1+h},1]}(\mathbf{i}t))^s] = \mathbb{E}[(K_{(t,1-t)}(\mathbf{i}t) + (\mathbf{i}t)^{\frac{\gamma\chi}{2}}\tilde{A}\tilde{S})^s] + o(t^{\chi(Q-\alpha)}), \tag{E.10}$$

where $\tilde{A} = e^{\frac{\gamma}{2}F_\tau(0)}(2\pi)^{\frac{\alpha\gamma}{2}}$ and $\tilde{S} := \int_{|y|\le\phi(t^{1+h})} e^{\frac{\gamma}{2}\tilde{X}_{\mathbb{H}}(y)}|y|^{-\frac{\alpha\gamma}{2}}\left(f(0,0^+)1_{x>0} + f(0,0^-)1_{x<0}\right)dy$. To see this, one can write the inequalities, for a constant $C>0$,

$$\begin{aligned}
&\left|\mathbb{E}[(K_{(t,1-t)}(\mathbf{i}t) + \tilde{K}_{[0,t^{1+h})\cup(1-t^{1+h},1]}(\mathbf{i}t))^s] - \mathbb{E}[(K_{(t,1-t)}(\mathbf{i}t) + (\mathbf{i}t)^{\frac{\gamma\chi}{2}}\tilde{A}\tilde{S})^s]\right| \\
&\le |s|\int_0^1 dv\,\mathbb{E}\left[\left|\tilde{K}_{[0,t^{1+h})\cup(1-t^{1+h},1]}(\mathbf{i}t) - (\mathbf{i}t)^{\frac{\gamma\chi}{2}}\tilde{A}\tilde{S}\right|\left|v\tilde{K}_{[0,t^{1+h})\cup(1-t^{1+h},1]}(\mathbf{i}t) + (1-v)(\mathbf{i}t)^{\frac{\gamma\chi}{2}}\tilde{A}\tilde{S}\right|^{s-1}\right] \\
&\le C|t| = o(t^{\chi(Q-\alpha)}).
\end{aligned}$$

Notice $C|t| = o(t^{\chi(Q-\alpha)})$ holds because of (E.5). Given (E.10), we again arrive at a setting very close to the one treated in [RZ20]. Following [RZ20][3], for any constant $A>0$, the difference $\mathbb{E}[(K_{(t,1-t)}(\mathbf{i}t) + (\mathbf{i}t)^{\frac{\gamma\chi}{2}}A\tilde{S})^s] - \mathbb{E}\left[(K_{(t,1-t)}(\mathbf{i}t))^s\right]$ equals

$$\mathbb{E}[(K_{(t,1-t)}(\mathbf{i}t) + \tilde{\sigma}_t A e^{\frac{\gamma}{2}M}\rho(\alpha,1,e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P}))^s] - \mathbb{E}[K_{(t,1-t)}(\mathbf{i}t)^s] + o(t^{\chi(Q-\alpha)})$$

where $\tilde{\sigma}_t = \mathbf{i}^{\frac{\gamma\chi}{2}}\Theta_\tau'(0)^{\frac{\gamma\chi}{2}-\frac{\alpha\gamma}{2}}(2\pi)^{\frac{\gamma^2}{4}-\frac{\alpha\gamma}{2}}t^{\frac{\gamma\chi}{2}+\frac{\gamma}{2}(1+h)(Q-\alpha)}e^{-\frac{\gamma^2}{8}\mathbb{E}[F_\tau(0)^2]}e^{\frac{\gamma}{2}\overline{X}(4\pi t^{1+h})}$. In our case we need to take $A = \tilde{A}$ with $\tilde{A} = e^{\frac{\gamma}{2}F_\tau(0)}(2\pi)^{\frac{\alpha\gamma}{2}}$ being a random constant. The argument of [RZ20] can be adapted to this slightly more general case simply by writing the same inequalities as used above,

$$\begin{aligned}
&\left|\mathbb{E}[(K_{(t,1-t)}(\mathbf{i}t) + (\mathbf{i}t)^{\frac{\gamma\chi}{2}}A\tilde{S})^s] - \mathbb{E}[(K_{(t,1-t)}(\mathbf{i}t) + \tilde{\sigma}_t A e^{\frac{\gamma}{2}M}\rho(\alpha,1,e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P}))^s]\right| \\
&\le |s|\int_0^1 dv\,\mathbb{E}\left[\left|\tilde{K}_{[0,t^{1+h})\cup(1-t^{1+h},1]}(\mathbf{i}t) - (\mathbf{i}t)^{\frac{\gamma\chi}{2}}\tilde{A}\tilde{S}\right|\left|K_{(t,1-t)}(\mathbf{i}t) + v(\mathbf{i}t)^{\frac{\gamma\chi}{2}}A\tilde{S} + (1-v)\tilde{\sigma}_t A e^{\frac{\gamma}{2}M}\rho(\alpha,1,e^{-\mathbf{i}\pi\frac{\gamma\chi}{2}+\pi\gamma P})\right|^{s-1}\right] \\
&\le o(t^{\chi(Q-\alpha)}).
\end{aligned}$$

Finally note that $\tilde{\sigma}_t e^{\frac{\gamma}{2}F_\tau(0)}(2\pi)^{\frac{\alpha\gamma}{2}} = \mathbf{i}^{\frac{\gamma\chi}{2}}\sigma_t$. We obtain Lemma E.1.

□

It remains to prove Lemma E.2. We first use the Girsanov Theorem C.5 to get the following.

Lemma E.3. *Let $\mathbb{P}$ be the probability measure corresponding to $X_{\mathbb{H}}$ and F_τ. Fix $a>0$ and let $\mathbb{Q}$ be the probability measure given by $d\mathbb{Q} = \mathbb{E}[g_\tau(t)^a]^{-1}g_\tau(t)^a d\mathbb{P}$. Then for small enough t, the $\mathbb{Q}$-law of $\{Y_\tau(x)\}_{x\in[t,1-t]}$ is the same as the $\mathbb{P}$-law of $\{Y_\tau(x) + a\mathbb{E}[Y_\tau(x)Y_\tau(0)]\}_{x\in[t,1-t]}$.*

Proof. Due to the independence of F_τ and $X_{\mathbb{H}}$, we separate the reweighing effect of F_τ and $X_{\mathbb{H}}$. By Girsanov's theorem (Theorem C.5), the $\mathbb{Q}$-law of $X_{\mathbb{H}}$ equals the $\mathbb{P}$-law of $X_{\mathbb{H}} + a\mathbb{E}[X_{\mathbb{H}}(\cdot)\overline{X}(4\pi t^{1+h})]$. By the mean value property of Green function on $\mathbb{H}$, we have $\mathbb{E}[X_{\mathbb{H}}(\cdot)\overline{X}(4\pi t^{1+h})] = \mathbb{E}[X_{\mathbb{H}}(\cdot)X_{\mathbb{H}}(0)]$. Therefore, for t small enough, restricted to $\phi([t,1-t])$, the $\mathbb{Q}$-law of $X_{\mathbb{H}}$ is given by the $\mathbb{P}$-law of $X_{\mathbb{H}} + a\mathbb{E}[X_{\mathbb{H}}(\cdot)X_{\mathbb{H}}(0)]$. Hence restricted to $[t,1-t]$, the $\mathbb{Q}$-law of Y_∞ is given by the $\mathbb{P}$-law of $Y_\infty + a\mathbb{E}[Y_\infty(\cdot)Y_\infty(0)]$. By Girsanov's theorem, the $\mathbb{Q}$-law of F_τ equals the $\mathbb{P}$-law of $F_\tau + a\mathbb{E}[F_\tau(\cdot)F_\tau(0)]$. Since $\mathbb{E}[Y_\tau(x)Y_\tau(0)] = \mathbb{E}[Y_\infty(\cdot)Y_\infty(0)] + \mathbb{E}[F_\tau(\cdot)F_\tau(0)]$, we conclude the proof. □

[2] In [RZ20], see equations (5.10) through (5.19) for the case $\chi = \frac{2}{\gamma}$ and equations (5.47) through (5.55) for $\chi = \frac{\gamma}{2}$.

[3] See this time in [RZ20] equation (5.40) for $\chi = \frac{2}{\gamma}$ and equation (5.51) for $\chi = \frac{\gamma}{2}$.

Proof of Lemma E.2. By the Girsanov Theorem C.5 and Lemma E.3, $\mathbb{E}\left[g_\tau(t)^{\frac{2}{\gamma}(Q-\alpha)}K_{(t,1-t)}(\mathbf{i}t)^{s-\frac{2}{\gamma}(Q-\alpha)}\right]$ equals

$$\mathbb{E}\left[g_\tau(t)^{\frac{2}{\gamma}(Q-\alpha)}\right]\mathbb{E}\left[\left(\int_t^{1-t} e^{\frac{\gamma}{2}Y_\tau(x)}e^{(Q-\alpha)\mathbb{E}[Y_\tau(x)Y_\tau(0)]}\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}}\Theta_\tau(\mathbf{i}t+x)^{\frac{\gamma\chi}{2}}e^{\pi\gamma Px}dx\right)^{\frac{\alpha+\chi-2Q}{\gamma}}\right].$$

for small enough t. As $t\to 0$, the second terms converge to $\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\gamma}{2}(2Q-\alpha-\chi)}e^{\pi\gamma Px}dx\right)^{\frac{\alpha+\chi-2Q}{\gamma}}\right]$.

On the other hand, $\mathbb{E}\left[g_\tau(t)^{\frac{2}{\gamma}(Q-\alpha)}\right]=\mathbb{E}[e^{(Q-\alpha)F_\tau(0)}]\mathbb{E}[e^{(Q-\alpha)\overline{X}(4\pi t^{1+h})}]$. By Lemma C.3,

$$\mathbb{E}[e^{(Q-\alpha)\overline{X}(4\pi t^{1+h})}]=e^{\frac{1}{2}(Q-\alpha)^2\mathbb{E}[\overline{X}(4\pi t^{1+h})^2]}=e^{\frac{1}{2}(Q-\alpha)^2\mathbb{E}[X_{\mathbb{H}}(4\pi t^{1+h})X_{\mathbb{H}}(0)]}.$$

By (C.2), $\lim_{t\to 0}t^{(1+h)(Q-\alpha)^2}\mathbb{E}[e^{(Q-\alpha)\overline{X}(4\pi t^{1+h})}]=(2\pi)^{-(Q-\alpha)^2}$. Lastly $\mathbb{E}[e^{(Q-\alpha)F_\tau(0)}]=|q^{-1/12}\eta(q)|^{-2(Q-\alpha)^2}$. Combining all of these terms gives the desired claim. □

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