

# Axiomatic Attribution for Multilinear Functions

[Extended Abstract]\*

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## ABSTRACT

We study the *attribution* problem. That is, given a real-valued characteristic function  $f$  of  $n$  variables and initial and final values  $r$  and  $s$  for its independent variables, our objective is to divide the responsibility for the change  $f(s) - f(r)$  in the characteristic function among each of its independent variables. We call these assigned responsibilities *attributions*, and we would like the attributions to form a complete partition of the total change.<sup>1</sup> When  $r = 0$ , the attribution problem coincides with a standard cost sharing model from the social choice literature (cf. Moulin [2]), where the characteristic function is the cost function, the independent variables are the demands of the agents, and the attributions are cost shares for the agents.<sup>2</sup>

We follow the cost sharing literature in identifying good attribution methods axiomatically (for a classical example, see Friedman and Moulin [1]). We consider:

- **Additivity** – attributions are additive in the characteristic function,
- **Dummy** – if the characteristic function does not depend on a variable, then its attribution is zero, and
- **Affine Scale Invariance** – attributions are invariant under simultaneous affine transformation of the characteristic function and the variables.

First, we show that when the characteristic function is the sum of a multilinear function and an additively separable one, every attribution method satisfying these axioms is a random order method. Intuitively, a multilinear function is determined by its values on the vertices of a hypercube, so its attributions should depend on these values alone, leading to the space of random order methods. The proof proceeds by using this idea to count dimensions.

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<sup>1</sup>Here the characteristic function may be nonlinear and the changes in the variables large, ruling out a naive approach based on linear approximation of the characteristic function.

<sup>2</sup>A significant difference is that cost sharing assumes monotone cost functions, while attribution relaxes this requirement. Therefore, while negative cost shares do not make sense, negative attributions are possible in some contexts.

Second, in our main result, we show that there is a unique attribution method satisfying these axioms and **Anonymity** (which requires attributions to be invariant under relabeling of the variables) if and only if the characteristic function is the sum of a multilinear function and an additively separable one.<sup>3</sup> The main technical tool is the use of Stokes' Theorem to compare attribution methods. The resulting method coincides with the classical Aumann-Shapley and Shapley-Shubik methods, and thus we term it the *Aumann-Shapley-Shubik* method. When the characteristic function is multilinear, our result prescribes this method for use; to this end, we provide a computationally efficient implementation.

Together, our results single out the class of multilinear characteristic functions as a particularly nice one for attribution problems. We give several examples of natural attribution problems where such functions arise, including pay-per-click advertising, website traffic analysis, portfolio analysis, and performance analysis of sports teams.

## Categories and Subject Descriptors

H.1.m [Information Systems]: Models and Principles—*Miscellaneous*; J.4 [Computer Applications]: Social and Behavioral Sciences—*Economics*

## General Terms

Economics, Theory

## Keywords

Cost sharing, Axioms, Attribution, Causality

## REFERENCES

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- [3] REDEKOP, J. Increasing marginal cost and the monotonicity of Aumann-Shapley pricing. mimeo, University of Waterloo, 1996.

<sup>3</sup>Unlike this result, most axiomatic results in cost sharing quantify over all cost functions and may not apply to a smaller class (see Redekop [3] for an example for the axiomatization of the Aumann-Shapley method in [1]).