51st International Mathematical Olympiad

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Problems (Day 1)

1. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equality

$$f(\lfloor x \rfloor y) = f(x) \lfloor f(y) \rfloor$$

holds for all $x, y \in \mathbb{R}$. (Here $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.)

2. Let I be the incentre of triangle ABC and let Γ be its circumcircle. Let the line AI intersect Γ again at D. Let E be a point on the arc $\widehat{BDC}$ and F a point on the side BC such that

$$\angle BAF = \angle CAE < \tfrac{1}{2}\angle BAC.$$

Finally, let G be the midpoint of the segment IF. Prove that the lines DG and EI intersect on Γ.

3. Let $\mathbb{N}$ be the set of positive integers. Determine all functions $g : \mathbb{N} \to \mathbb{N}$ such that

$$\big(g(m) + n\big)\big(m + g(n)\big)$$

is a perfect square for all $m, n \in \mathbb{N}$.

Problems (Day 2)

4. Let P be a point inside triangle ABC. The lines AP, BP, and CP intersect the circumcircle Γ of triangle ABC again at the points K, L, and M, respectively. The tangent to Γ at C intersects the line AB at S. Suppose that $SC = SP$. Prove that $MK = ML$.

5. In each of six boxes B_1, B_2, B_3, B_4, B_5, B_6 there is initially one coin. There are two types of operation allowed:

Type 1: Choose a nonempty box B_j with $1 \le j \le 5$. Remove one coin from B_j and add two coins to B_{j+1}.

Type 2: Choose a nonempty box B_k with $1 \le k \le 4$. Remove one coin from B_k and exchange the contents of (possibly empty) boxes B_{k+1} and B_{k+2}.

Determine whether there is a finite sequence of such operations that results in boxes B_1, B_2, B_3, B_4, B_5 being empty and box B_6 containing exactly $2010^{2010^{2010}}$ coins. (Note that $a^{b^c} = a^{(b^c)}$.)

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6. Let $a_1, a_2, a_3, \ldots$ be a sequence of positive real numbers. Suppose that for some positive integer s, we have
$$a_n = \max\{a_k + a_{n-k} \mid 1 \le k \le n-1\}$$
for all $n > s$. Prove that there exist positive integers ℓ and N, with $\ell \le s$ and such that $a_n = a_\ell + a_{n-\ell}$ for all $n \ge N$.

Solutions

1. The answer is $f(x) = c$ for all x, where $c = 0$ or $1 \le c < 2$. To prove that these are the only possible solutions, consider two cases. First suppose that $\lfloor f(y) \rfloor = 0$ whenever $0 \le y < 1$. Then $f(\lfloor x \rfloor y) = f(x)\lfloor f(y) \rfloor = 0$ whenever $0 \le y < 1$. Since every real number can be represented as a product of the form $\lfloor x \rfloor y$ with $x \in \mathbb{R}$ and $0 \le y < 1$, in this case f is identically zero.

   Otherwise, suppose $\lfloor f(y_0) \rfloor \ne 0$ for some $0 \le y_0 < 1$. For any x_n satisfying $n \le x_n < n+1$, set $y = y_0$ and $x = x_n$ in the given equality to obtain $f(ny_0) = f(x_n)\lfloor f(y_0) \rfloor$. Letting $c_n = \frac{f(ny_0)}{\lfloor f(y_0) \rfloor}$, it follows that $f(x_n) = c_n$ for all $x_n \in [n, n+1)$. In particular, we have $\lfloor c_0 \rfloor = \lfloor f(y_0) \rfloor \ne 0$, hence $c_0 \ne 0$. Now set $x = y = 0$ in the given equality to obtain $c_0 = f(0) = f(0)\lfloor f(0) \rfloor = c_0 \lfloor c_0 \rfloor$, hence $\lfloor c_0 \rfloor = 1$. Finally, setting $y = 0$ and $x = n$ in the given equality, we find $c_n = f(n) = \frac{f(0)}{\lfloor f(0) \rfloor} = \frac{c_0}{\lfloor c_0 \rfloor} = c_0$. Therefore, in this case we have $f(x) = c_0$ for all x, and $\lfloor c_0 \rfloor = 1$.

   This problem was proposed by Pierre Bornsztein of France.

2. Let P be the second intersection of ray EI and Γ, and let segments PD and FI meet at M. We wish to show that $M = G$, or, equivalently, $FM = MI$. Let Q be the intersection of segments PD and AF. Applying Menelaus's theorem to triangle AFI and line QMD gives $\frac{FQ \cdot AD \cdot IM}{QA \cdot DI \cdot MF} = 1$. Hence it suffices to show that $\frac{FQ \cdot AD}{QA \cdot DI} = 1$ or equivalently that $AD/AQ = (DI + DA)/FA$.

   Triangles QAD and IAE are similar, so $AD/AQ = EA/AI$. Also, triangles ABF and AEC are similar, so we have $AF/AB = AC/AE$. Together these imply that $\frac{AD}{AQ} = \frac{AB \cdot AC}{AF \cdot AI}$. Now, let H be the intersection of BC and AD; notice that triangles DHC and DCA are similar, hence $DC^2 = DH \cdot DA$. Now because $\angle DCI = \angle CID$, we have $DC = DI$, hence $DA^2 - DI^2 = DA^2 - DC^2 = DA^2 - DH \cdot DA = DA \cdot HA$. On the other hand, notice that triangles ABH and ADC are similar, so $DA \cdot HA = AB \cdot AC$. Putting these together, we see that $\frac{AD}{AQ} = \frac{AB \cdot AC}{AF \cdot AI} = \frac{DA \cdot HA}{AF \cdot AI} = \frac{DI + DA}{FA}$, as needed.

   This problem was proposed by Wai Ming Tai of Hong Kong and Chongli Wang of China.

3. All functions of the form $g(n) = n + c$ for a constant nonnegative integer c satisfy the problem conditions. We claim that these are the only such functions.

   We first show that g must be injective. Suppose instead that $g(a) = g(b)$ for some $a \ne b$. Choose n so that $n + g(a) = p$ is prime and greater than $|a - b|$. From the hypothesis both $p(g(n) + a)$ and $p(g(n) + b)$ must be perfect squares, meaning that $g(n) + a$ and $g(n) + b$ are both divisible by p. But this is impossible, as $p > |a - b|$. Therefore, g is injective as claimed.

   We now show that $|g(k+1) - g(k)| = 1$ for all k. Suppose instead that some prime p divides $g(k+1) - g(k)$. Now, choose an integer n as follows. If $p^2 \mid g(k+1) - g(k)$, then take n so that $n + g(k+1)$ is divisible by p but not p^2. Otherwise, take n so that $n + g(k+1)$ is divisible by p^3 but not p^4. Note that the maximum power of p dividing $n + g(k+1)$ and $n + g(k)$ is odd. Now, the hypothesis implies that $(n + g(k+1))(g(n) + k + 1)$ and $(n + g(k))(g(n) + k)$ are both squares, meaning that $g(n) + k + 1$ and $g(n) + k$ are both divisible by p, a contradiction.

   For each k, we now have either $g(k+1) = g(k) + 1$ or $g(k+1) = g(k) - 1$. But g is injective, so if the latter occurs for some k, then it occurs for all $k' > k$, an impossibility because g takes positive values. Therefore, we have $g(k+1) = g(k) + 1$ for all k, hence $g(k) = k + g(1) - 1$.

This problem was proposed by Gabriel Carroll of the USA.

4. Without loss of generality, we may assume that S is on ray BA. Set $x_1 = \angle PAB$, $y_1 = \angle PBC$, $z_1 = \angle PCA$, $x_2 = \angle PAC$, $y_2 = \angle PBA$, and $z_2 = \angle PCB$. Because SC is tangent to Γ, we have $SC^2 = SA \cdot SB$ by the Power of a Point Theorem, and $\angle SCP = \angle SCM = \angle ACM + \angle ACS = z_1 + \angle ABC = z_1 + y_1 + y_2$. Because $SP = SC$, we have $SP^2 = SC^2 = SA \cdot SB$, so triangles SAP and SPB are similar. It follows that $\angle SPA = \angle SBP = y_2$ and that $\angle ASP = \angle BAP - \angle SPA = x_1 - y_2$. Now, $SP = SC$ implies $\angle SPC = \angle SCP = z_1 + y_1 + y_2$, so $\angle PSC = 180° - 2(z_1 + y_1 + y_2) = x_1 + x_2 + z_2 - (z_1 + y_1 + y_2)$. Notice that $\angle ASC = \angle BAC - \angle ACS = (x_1 + x_2) - (y_1 + y_2)$, so we have $\angle ASP = \angle ASC - \angle PSC = z_1 - z_2$. Combining our two computations of $\angle ASP$ yields $x_1 - y_2 = z_1 - z_2$ or $x_1 + z_2 = y_2 + z_1$. That is, we have $(\widehat{KB} + \widehat{BM})/2 = (\widehat{LA} + \widehat{AM})/2$, hence $\widehat{KM}/2 = \widehat{LM}/2$ and $MK = ML$.

   This problem was proposed by Marcin E. Kuczma of Poland.

5. The answer is yes. Although the problem specifies that the number of boxes is $n = 6$, the operations extend in the obvious way to general values of n. Our proof will consider several different values of n on the way to the final result. For this, it is convenient to let $(b_1, \dots, b_n)$ denote the n-box configuration where b_1 balls are in box B_1, b_2 balls are in box B_2, etc. Write $(b_1, \dots, b_n) \to (b'_1, \dots, b'_n)$ if we can obtain the configuration $(b'_1, \dots, b'_n)$ from $(b_1, \dots, b_n)$ following the rules in the n-box setting. We begin with two lemmas.

LEMMA 1. *Let a be a positive integer. Then* $(a, 0, 0) \to (0, 2^a, 0)$.

Proof. We will show that $(a, 0, 0) \to (a - k, 2^k, 0)$ for every $1 \le k \le a$, by inducting on k. For $k = 1$, applying a Type 1 operation to the first number gives $(a, 0, 0) \to (a - 1, 2, 0) = (a - 1, 2^1, 0)$. Now assume the statement holds for some $k < a$. Starting from $(a - k, 2^k, 0)$, repeatedly applying 2^k many Type 1 operations at the middle box yields $(a - k, 2^k, 0) \to \cdots \to (a - k, 0, 2^{k+1})$. A final Type 2 operation applied at the first box produces $(a - k, 0, 2^{k+1}) \to (a - k - 1, 2^{k+1}, 0)$, completing the induction. ■

LEMMA 2. *Define* $P_n = \underbrace{2^{2^{\cdot^{\cdot^{2}}}}}_{n}$. *Then* $(a, 0, 0, 0) \to (0, P_a, 0, 0)$ *for every positive integer a.*

Proof. We will show that $(a, 0, 0, 0) \to (a - k, P_k, 0, 0)$ for $1 \le k \le a$, by inducting on k. For $k = 1$, a Type 1 operation applied at the first box gives $(a, 0, 0, 0) \to (a - 1, P_1, 0, 0)$. Now assume that $(a, 0, 0, 0) \to (a - k, P_k, 0, 0)$ for some $a < k$. Applying Lemma 1 to the last three boxes, we obtain $(a - k, P_k, 0, 0) \to (a - k, 0, P_{k+1}, 0)$. A final Type 2 operation applied at the first box gives $(a - k, 0, P_{k+1}, 0) \to (a - k - 1, P_{k+1}, 0, 0)$, completing our induction. ■

We now describe the construction for the original 6-box setting. Write $A = 2010^{2010^{2010}}$. First, apply a Type 1 operation to B_5, giving $(1, 1, 1, 1, 1, 1) \to (1, 1, 1, 1, 0, 3)$. Second, apply Type 2 operations to B_4, B_3, B_2, and B_1 in this order, obtaining $(1, 1, 1, 1, 0, 3) \to (1, 1, 1, 0, 3, 0) \to (1, 1, 0, 3, 0, 0) \to (1, 0, 3, 0, 0, 0) \to (0, 3, 0, 0, 0, 0)$. Third, apply Lemma 2 twice, giving the sequence $(0, 3, 0, 0, 0, 0) \to (0, 0, P_3, 0, 0, 0) \to (0, 0, 0, P_{16}, 0, 0)$. It is easy to check that $P_{16} > A$, so there are more than $A = 2010^{2010^{2010}}$ coins in B_4 at this point. Fourth, decrease the number of coins in B_4 by applying Type 2 operations repeatedly to B_4 until its size decreases to $\frac{A}{4}$. This gives $(0, 0, 0, P_{16}, 0, 0) \to \cdots \to \left(0, 0, 0, \frac{A}{4}, 0, 0\right)$. Finally, apply Type 1 operations repeatedly to first empty B_4 and then B_5, obtaining $\left(0, 0, 0, \frac{A}{4}, 0, 0\right) \to \cdots \to \left(0, 0, 0, 0, \frac{A}{2}, 0\right) \to \cdots \to (0, 0, 0, 0, 0, A)$, as desired.

This problem was proposed by Hans Zantema of Netherlands.

Note. Following a practice established last year, Fields Medalist (and IMO gold medalist) Terence Tao hosted an online project for others to collaborate in solving this problem, which he identified as the most challenging problem on the exam (`http://polymathprojects.org/2010/07/08/minipolymath2-project-imo-2010-q5/`).

6. We generalize to the setting where the a_n may assume negative values. For any $r \in \mathbb{R}$, note that the transformation $a_n \mapsto a_n + rn$ does not change the problem conditions or the result to be proved. Picking $\ell \le s$ such that a_ℓ/ℓ is maximal, we can thus assume without loss of generality that $a_\ell = 0$. This means all of $a_1, \ldots, a_s$ are non-positive, hence all a_n are non-positive. Let $b_n = -a_n \ge 0$. For $n > s$, we have $b_n = \min\{b_k + b_{n-k} \mid 1 \le k \le n-1\}$ and in particular $b_n \le b_{n-\ell} + b_\ell = b_{n-\ell}$.

   From this, we draw two conclusions. First, all b_n must be bounded above by $M = \max\{b_1, \ldots, b_s\}$. Second, if we let S be the set of all linear combinations of the form $c_1b_1 + c_2b_2 + \cdots c_sb_s$, where the c_i are nonnegative integers, and let $T = \{x \le M : x \in S\}$, then since $b_n = \min\{b_k + b_{n-k} \mid 1 \le k \le n-1\}$, it is clear that every b_n must be in T. Crucially, T is a finite set.

   Now, for each integer i satisfying $\ell i + 1 > s$, let β_i denote the ℓ-tuple $(b_{\ell i+1}, b_{\ell i+2}, \ldots, b_{\ell i+\ell})$. By the previous paragraph, the number of such ℓ-tuples is at most $|T|^\ell$, a finite number. Further, because $b_n \le b_{n-\ell}$ for $n > s$, the individual indices of these β_i are non-increasing functions of i. Thus, there can only be finitely many i for which $\beta_i \ne \beta_{i+1}$. Let i_0 be greater than the largest such value; then, all ℓ-tuples β_i with $i \ge i_0$ are identical. Choosing $N = \ell(i_0 + 1)$ finishes the problem, since any $n \ge N$ gives $b_n = b_{n-\ell} = b_\ell + b_{n-\ell}$.

   This problem was proposed by Morteza Saghafian of Iran. This solution is by Evan O'Dorney.

Results. The IMO was held in Astana, Kazakhstan, on July 7–8, 2010. There were 517 competitors from 96 countries and regions. On each day contestants were given four and a half hours for three problems.

On this challenging exam, a perfect score was achieved by only one student, Zipei Nie (China). The USA team ranked third, behind China and Russia. Although the American team has consistently finished in the top ten at the IMO, this year's performance was particularly impressive because none of the team members were in their final year of high school. The students' individual results were as follows.

- Calvin Deng, who finished 9th grade at William G. Enloe High School in Raleigh, NC, won a silver medal.
- Ben Gunby, who finished 10th grade at Georgetown Day School in Washington, DC, won a gold medal.
- Xiaoyu He, who finished 10th grade at Acton-Boxborough Regional High School in Acton, MA, won a gold medal.
- In-Sung Na, who finished 11th grade at Northern Valley Regional High School in Old Tappan, NJ, won a silver medal.
- Evan O'Dorney from Danville, CA, who finished 11th grade (homeschooled through Venture School), won a gold medal. Furthermore, he placed 2nd overall with a score of 39/42. For his spectacular performance, he received a private congratulatory telephone call from the President of the United States, Barack Obama.
- Allen Yuan, who finished 11th grade at Detroit Country Day School in Beverly Hills, MI, won a silver medal.