52nd International Mathematical Olympiad

ZUMING FENG
Phillips Exeter Academy
Exeter, NH 03833
zfeng@exeter.edu

YI SUN
Phillips Exeter Academy
Exeter, NH 03833
yisun@mit.edu

Problems (Day 1)

1. Given any set $A = \{a_1, a_2, a_3, a_4\}$ of four distinct positive integers, we denote the sum $a_1 + a_2 + a_3 + a_4$ by s_A. Let n_A denote the number of pairs (i, j) with $1 \leq i < j \leq 4$ for which $a_i + a_j$ divides s_A. Find all sets A of four distinct positive integers which achieve the largest possible value of n_A.
2. Let $\mathcal{S}$ be a finite set of at least two points in the plane. Assume that no three points of $\mathcal{S}$ are collinear. A *windmill* is a process that starts with a line ℓ going through a single point $P \in \mathcal{S}$. The line rotates clockwise about the *pivot* P until the first time that the line meets some other point belonging to $\mathcal{S}$. This point, Q, takes over as the new pivot, and the line now rotates clockwise about Q, until it next meets a point of $\mathcal{S}$. This process continues indefinitely, with the pivot always being a point from $\mathcal{S}$.

   Show that we can choose a point P in $\mathcal{S}$ and a line ℓ going through P such that the resulting windmill uses each point of $\mathcal{S}$ as a pivot infinitely many times.
3. Let f be a real-valued function defined on the set of real numbers that satisfies

   $$f(x + y) \leq yf(x) + f(f(x))$$

   for all real numbers x and y. Prove that $f(x) = 0$ for all $x \leq 0$.

Problems (Day 2)

4. Let $n > 0$ be an integer. We are given a balance and n weights of weight 2^0, $2^1, \ldots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed.

   Determine the number of ways in which this can be done.
5. Let f be a function from the set of integers to the set of positive integers. Suppose that, for any two integers m and n, the difference $f(m) - f(n)$ is divisible by $f(m - n)$. Prove that, for all integers m and n with $f(m) \leq f(n)$, the number $f(n)$ is divisible by $f(m)$.
6. Let ABC be an acute triangle with circumcircle Γ. Let ℓ be a tangent line to Γ, and let ℓ_a, ℓ_b and ℓ_c be the lines obtained by reflecting ℓ in the lines BC, CA and AB, respectively. Show that the circumcircle of the triangle determined by the lines ℓ_a, ℓ_b and ℓ_c is tangent to the circle Γ.

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Solutions

1. For any positive integer k, the sets $\{k, 5k, 7k, 11k\}$ and $\{k, 11k, 19k, 29k\}$ achieve the maximum value of $n_A = 4$.

   In general, let $A = \{a_1, a_2, a_3, a_4\}$ be labeled so that $0 < a_1 < a_2 < a_3 < a_4$. Then $a_2 + a_4$ and $a_3 + a_4$ are both strictly between $s_A/2$ and s_A, and so cannot divide s_A. Thus $n_A = 4$ is maximal. If the other pair-sums all divide s_A we must have

$$\begin{cases} a_2 + a_3 = (1/2)\ s_A & \text{(because } a_1 + a_4 \text{ also divides } s_A) \\ a_1 + a_3 = (1/n)\ s_A \\ a_1 + a_2 = (1/m)\ s_A \end{cases}$$

   where $2 < n < m$ (if $2 = n$ then $a_2 = a_1$ and if $n = m$ then $a_3 = a_2$). Subtracting the first equation from the sum of the last two gives $(1/n + 1/m - 1/2)\, s_a = 2a_1$, so that $1/n + 1/m > 1/2$. This equation can hold only if $n = 3$ and either $m = 4$ or $m = 5$. Now if $m = 4$ then $a_1 = (1/24)s_A$ and $A = \{k, 5k, 7k, 11k\}$; if $m = 5$, we have $a_1 = (1/60)s_A$ and $A = \{k, 11k, 19k, 29k\}$.

   This problem was proposed by Fernando Campos García of Mexico.

2. Call a point in $\mathcal{S}$ a *vertex*, and direct all lines so that they have right and left sides. Call a direction *ordinary* if no line with that direction passes through two vertices, and call a line ordinary if it has an ordinary direction. Let $n = |\mathcal{S}|$. Call a line a *balancing line* if it passes through exactly one vertex and has exactly $\lfloor (n-1)/2 \rfloor$ other vertices to its right.

   We first show that there exists an ordinary balancing line through any vertex P. Start with any ordinary directed line through P with, say, k points to its right. Rotating it 180° about P gives a line with $n - 1 - k$ points to its right. The number of points to the right of the ordinary lines in this process changes in increments of 1, so some ordinary line which occurs has exactly $\lfloor (n-1)/2 \rfloor$ points to its right. This is the desired ordinary balancing line.

   Now, choose any ordinary balancing line ℓ; we claim that the windmill starting from ℓ uses each vertex as a pivot infinitely often. In any windmill, the number of points to the right of the ordinary lines remains fixed, as at each pivot change the old and new pivots switch sides. Thus, because ℓ was balancing, each ordinary line in the windmill is balancing. Now, by definition, lines of each ordinary direction (which are hence balancing) occur infinitely often in this windmill. But there can be at most one balancing line in any direction, so the balancing lines we constructed through each vertex are the unique ones in their respective directions and must appear infinitely often, as needed.

   This problem was proposed by Geoff Smith of the United Kingdom.

3. Let $f(0) = a$ and $f(f(0)) = b$. Setting $x = 0$ in the given identity we obtain $f(y) \le ay + b$. Applying this to the last term of the given yields

$$f(x + y) \le yf(x) + af(x) + b. \tag{1}$$

   Substituting $x = z + a$, $y = -a$ gives $f(z) \le b$ for all $z \in \mathbb{R}$. Now applying this to the last term of the identity gives

$$f(x + y) \le yf(x) + b. \tag{2}$$

   Replacing x and y in (2) with $x + y$ and $-y$ gives $f(x) \le -yf(x + y) + b$. For $y < 0$, we can multiply (2) by $-y$ and add it to the last inequality, giving $f(x) \le -y^2 f(x) + b - yb$, or $f(x) \le b\left(\frac{1-y}{1+y2}\right)$ when $y < 0$. As $y \to -\infty$ the last expression gets arbitrarily close to zero, so we have $f(x) \le 0$ for all $x \in \mathbb{R}$.

Setting $x = 2a - 1$ and $y = 1 - a$ in (1) gives $f(2a-1) \geq 0$. Since $f(x) \leq 0$ always, this means $f(2a-1) = 0$. The given identity with $y = 0$ forces $f(x) \leq f(f(x))$ always, so $0 = f(2a-1) \leq f(f(2a-1)) = a$. This means that $a = f(0) = 0$ and thus $b = 0$ as well. Setting $y = -x$ in (2) and using these facts gives $0 \leq -xf(x)$ for all x. For $x < 0$, this implies that $f(x) \geq 0$; but we know $f(x) \leq 0$ always and $f(0) = 0$, so in fact $f(x) = 0$ whenever $x \leq 0$.

This problem was proposed by Igor Voronovich of Belarus. This solution is based on one by Oleg Golberg. (There are non-constant functions that satisfy the identity.)

4. The answer is $(2n-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-1)$.

   Call a sequence of moves *valid* if the right pan is never heavier than the left pan when making these moves. It suffices to give a $(2n+1)$-to-1 mapping between valid sequences for weights $2^0, \ldots, 2^n$ and weights $2^0, \ldots, 2^{n-1}$.

   For a valid sequence of moves of weights $2^0, 2^1, \ldots, 2^n$, if we remove the move of putting weight 2^0 in this sequence and divide the remaining weights by 2, we obtain a valid sequence of moves of weights $2^0, \ldots, 2^{n-1}$. On the other hand, for a valid sequence S of weights $2^0, \ldots, 2^{n-1}$, doubling each weight gives a valid sequence S' of weights $2^1, \ldots, 2^n$. Note that the difference in weight between the left and right pans is always at least 2 after the first move in S'. Therefore, modifying S' by adding weight 2^0 to the left pan on the first move or to either pan on any move after the first yields $2n+1$ valid sequences of weights $2^0, \ldots, 2^n$. These two constructions give the desired mapping.

   This problem was proposed by Morteza Saghafian of Iran.

5. Setting $n = 0$ in the given gives $f(m) \mid f(m) - f(0)$, hence $f(m) \mid f(0)$ for all m, while taking $m = 0$ yields $f(-n) \mid f(0) - f(n)$ for all n. Together, these show that $f(-n) \mid f(n)$ for all n, implying $f(n) = f(-n)$. It therefore suffices to show that for all $m, n > 0$ either $f(m) \mid f(n)$ or $f(n) \mid f(m)$.

   Assume the contrary and pick $m > n > 0$ violating the desired with $m+n$ minimal. Since $m - n > 0$ and $(m-n) + n = m < m + n$, the minimality of $m+n$ implies that either $f(n) \mid f(m-n)$ or $f(m-n) \mid f(n)$. If $f(n) \mid f(m-n)$, then $f(n) \mid f(m) - f(m-n)$ implies $f(n) \mid f(m)$, a contradiction. Therefore, $f(n) \nmid f(m-n)$, hence $f(m-n) \mid f(n)$ and $f(m-n) < f(n)$. Note that $f(m) \mid f(n) - f(n-m) = f(n) - f(m-n)$. Since $f(n) - f(m-n) > 0$, this means $f(m) < f(n)$. Now, by the given, we have $f(n) \mid f(m) - f(m-n)$, where $|f(m) - f(m-n)| < f(n)$ because $f(m), f(m-n) < f(n)$. Hence, it must be that $f(m) = f(m-n)$, implying $f(m) \mid f(n)$, a contradiction.

   This problem was proposed by Mahyar Sefidgaran of Iran. This solution is by Oleg Golberg.

6. Let A_1, B_1, C_1 be the intersections of ℓ_b and ℓ_c, ℓ_c and ℓ_a, and ℓ_a and ℓ_b. Let ℓ be tangent to Γ at T. Define points A_2, B_2, C_2 (distinct from A, B, C) on Γ so that $\widehat{TA} = \widehat{A_2A}$, $\widehat{TB} = \widehat{B_2B}$, and $\widehat{TC} = \widehat{C_2C}$. Let lines AB, BC, CA meet ℓ at C_3, A_3, B_3, respectively. Without loss of generality, we suppose that ℓ is such that B lies inside triangle $B_1A_3C_3$; other configurations are analogous. Let B_1B_2 intersect Γ again at H. We claim there is a homothety $\mathbf{H}$ centered at H sending $A_2B_2C_2$ to $A_1B_1C_1$ and Γ to the circumcircle Γ_1 of triangle $A_1B_1C_1$. Since H lies on Γ, such an $\mathbf{H}$ would show that Γ and Γ_1 are tangent.

   We first show that corresponding sides of $A_1B_1C_1$ and $A_2B_2C_2$ are parallel; by symmetry, it suffices to show that $B_2C_2 \parallel B_1C_1$. Let S be the intersection of lines B_2C_2 and ℓ. Since $\widehat{B_2T} = 2\widehat{BT}$ and $\widehat{TC_2} = 2\widehat{TC}$, we have $\angle B_2ST = \angle B_2C_2T - \angle STC_2 = 2(\angle BC_2T - \angle CTC_2)$. Because BC_2CT is cyclic, we have $2(\angle BC_2T - \angle CTC_2) = 2(\angle BCT - \angle CTC_2) = 2\angle BA_3T$. Because B_1A_3 and ℓ are reflections of each other across line BC, we have $2\angle BA_3T = \angle B_1A_3T$. Combining these equalities gives $\angle B_2ST = \angle B_1A_3T$, hence $B_2C_2 \parallel B_1C_1$.

It remains to show that A_1A_2 and C_1C_2 pass through H; by symmetry, it suffices to do so for C_1C_2. We claim first that the intersection I of B_1B and C_1C lies on Γ. Indeed, by definition A_1B_1, AB, ℓ concur at C_3, B_1C_1, BC, ℓ concur at A_3, and C_1A_1, CA, ℓ concur at B_3. By reflection properties, line AB (through C_3) bisects $\angle A_3C_3B_1$, and line BC (through A_3) bisects $\angle B_1A_3C_3$, so B is the incenter of triangle $B_1C_3A_3$ in our configuration. We see similarly that C is the excenter of triangle $C_1A_3B_3$. Computing, we see $\angle ABI = 180^\circ - \angle B_3BC_3 = 180^\circ - \left(90^\circ + \frac{\angle B_1A_3C_3}{2}\right) = 90^\circ - \frac{\angle B_1A_3C_3}{2}$ and $\angle ACI = \angle CB_3A_1 - \angle CC_1B_3 = \frac{\angle A_3B_3A_1}{2} - \frac{\angle A_3C_1B_3}{2} = \frac{\angle B_3A_3C_1}{2} = \frac{180^\circ - \angle B_1A_3C_3}{2} = 90^\circ - \frac{\angle B_1A_3C_3}{2}$. Hence, $\angle ACI = \angle ABI$, and $ACBI$ is cyclic, so I lies on Γ.

By Pascal's theorem on the (self-intersecting) cyclic hexagon B_2HC_2BIC, the intersection B_1 of B_2H and BI, the intersection X of C_2B and CB_2, and the intersection of HC_2 and IC all lie on B_1X. Now, because $\widehat{B_2B} = \widehat{BT}$ and $\widehat{C_2C} = \widehat{CT}$, CB_2 and BC_2 are the reflections of CT and BT across BC. Thus, their intersection X is the reflection of T across BC and lies on the reflection B_1C_1 of TS across BC. This means that B_1X is the same line as B_1C_1. Therefore, HC_2 passes through the intersection of IC and B_1C_1, which is C_1 because I lies on CC_1. Thus, C_1C_2 passes through H, as needed.

This problem was proposed by the Olympiad problem committee of Japan.

Results.

The IMO was held in Amsterdam, The Netherlands, on July 18–19, 2011. There were 564 competitors from 101 countries and regions. On each day contestants were given four and a half hours for three problems.

On this challenging exam, a perfect score was achieved by only one student, Lisa Sauermann (Germany). With this result, she becomes the most successful IMO participant of all time, having won 4 gold medals and 1 silver medal in her 5 participations. Each member of the USA team won a gold medal, ranking the USA 2nd among all 101 participating countries, behind China. This impressive performance is only the second time the entire USA team has won gold medals. The students' individual results were as follows.

- Wenyu Cao, who finished 12th grade at Phillips Academy in Andover, MA, won a gold medal.
- Ben Gunby, who finished 11th grade at Georgetown Day School in Washington, DC, won a gold medal.
- Xiaoyu He, who finished 11th grade at Acton-Boxborough Regional High School in Acton, MA, won a gold medal.
- Mitchell Lee, who finished 11th grade at Thomas Jefferson High School for Science and Technology in Alexandria, VA, won a gold medal.
- Evan O'Dorney from Danville, CA, who finished 12th grade (homeschooled through Venture School), won a gold medal.
- David Yang, who finished 10th grade at Phillips Exeter Academy in Exeter, NH, won a gold medal.