The top fourteen students on the 2012 USAJMO were (in alphabetical order):

Ernest Chiu	10	West Windsor Plainsboro High School	Plainsboro, NJ
Paolo Gentili	10	Canyon Crest Academy	San Diego, CA
Courtney Guo	10	International School of Beijing	Beijing, China
Steven Hao	10	Lynbrook High School	San Jose, CA
Andrew He	9	Monta Vista High School	Cupertino, CA
Calvin Huang	10	Henry M Gunn High School	Palo Alto, CA
Shashwat Kishore	9	Unionville High School	Kennett Square, PA
Laura Pierson	6	Berkeley Math Circle, University of California	Berkeley, CA
Tahsin Saffat	9	Westview High School	Portland, OR
David Stoner	9	South Aiken High School	Aiken, SC
Ashwath Thirumalai	9	Harker High School	San Jose, CA
Jerry Wu	10	Mission San Jose High School	Fremont, CA
Isaac Xia	10	Concord-Carlisle Regional High School	Concord, MA
Jesse Zhang	9	University of Colorado	Boulder, CO

53rd International Mathematical Olympiad

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Problems (Day 1)

1. Given triangle ABC the point J is the centre of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC.

   Prove that M is the midpoint of ST.

   (The *excircle* of ABC opposite the vertex A is the circle tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C.)

2. Let $n \geq 3$ be an integer, and let $a_2, a_3, \ldots, a_n$ be positive real numbers such that $a_2 a_3 \cdots a_n = 1$. Prove that

$$(1 + a_2)^2 (1 + a_3)^3 \cdots (1 + a_n)^n > n^n.$$

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3. The *liar's guessing game* is a game played between two players A and B. The rules of the game depend on two positive integers k and n which are known to both players.

   At the start of the game A chooses integers x and N with $1 \le x \le N$. Player A keeps x secret, and truthfully tells N to player B. Player B now tries to obtain information about x by asking player A questions as follows: each question consists of B specifying an arbitrary set S of positive integers (possibly one specified in some previous question), and asking A whether x belongs to S. Player B may ask as many such questions as he wishes. After each question, player A must immediately answer it with *yes* or *no*, but is allowed to lie as many times as she wants; the only restriction is that, among any $k+1$ consecutive answers, at least one answer must be truthful.

   After B has asked as many questions as he wants, he must specify a set X of at most n positive integers. If x belongs to X, then B wins; otherwise, he loses. Prove that:

   1. If $n \ge 2^k$, then B can guarantee a win.
   2. For all sufficiently large k, there exists an integer $n \ge 1.99^k$ such that B cannot guarantee a win.

Problems (Day 2)

4. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, c that satisfy $a + b + c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

   (Here $\mathbb{Z}$ denotes the set of integers.)

5. Let ABC be a triangle with $\angle BCA = 90°$, and let D be the foot of the altitude from C. Let X be a point in the interior of segment CD. Let K be the point on the segment AX such that $BK = BC$. Similarly, let L be the point on the segment BX such that $AL = AC$. Let M be the point of intersection of AL and BK.

   Show that $MK = ML$.

6. Find all positive integers n for which there exist non-negative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = 1.$$

Solutions

1. Notice that $\angle KAJ = \frac{A}{2}$ and that

$$\angle KGJ = \angle MCJ - \angle GMC = \angle MCJ - \angle KMB = \frac{A+B}{2} - \frac{B}{2} = \frac{A}{2},$$

   so $\angle KAJ = \angle KGJ$, hence $AKJG$ is cyclic. In particular, $\angle AGC = \angle AKJ = 90°$, meaning that $AG \perp GJ$, so $AG \parallel ML$. Now, CML is an isosceles triangle with altitude CJ, so because $AT \parallel ML$, ACT is isosceles with altitude LG. In the same way, we can show that ABS is isosceles with altitude BF.

   Notice now that $\angle SAT = A + \angle SAB + \angle CAT = A + \frac{B}{2} + \frac{C}{2}$, where we have used the fact that SAB and CAT are isosceles. On the other hand, we see that $\angle MGT = 90 + \angle KGJ = 90 + \angle KAJ = 90 + \frac{A}{2}$, which implies that $\angle MGT = \angle SAT$, hence $SA \parallel MG$. Because G is the midpoint of AT, this implies that MG is the midline of triangle AST, so M is the midpoint of ST.

   This problem was proposed by Evangelos Psychas of Greece.

2. By the AM-GM inequality, for every k with $2 \leq k \leq n$, we have

$$(1+a_k)^k = \left(\frac{1}{k-1} + \frac{1}{k-1} + \cdots + \frac{1}{k-1} + a_k\right)^k \geq \frac{k^k a_k}{(k-1)^{k-1}},$$

where equality holds if and only if $a_k = \frac{1}{k-1}$. Multiplying these inequalities for each k from 2 to n yields

$$\begin{aligned}(1+a_2)^2(1+a_3)^3 \cdots (1+a_n)^n &\geq 2^2 a_2 \cdot \frac{3^3 a_3}{2^2} \cdots \frac{n^n a_n}{(n-1)^{n-1}} \\ &= n^n a_2 a_3 \cdots a_n = n^n.\end{aligned}$$

Equality holds only if $a_k = \frac{1}{k-1}$ for each k, implying that $a_2 a_3 \cdots a_n = \frac{1}{(n-1)!}$, an impossibility for $n \geq 3$. Thus, the inequality is strict, as needed.

This problem was proposed by Angelo di Pasquale of Australia.

3. (1) Let T be the set of possible values of x given the answers to B's questions. We give a strategy for B to reduce $|T|$ to at most 2^k, upon which he can specify the set T to guarantee a win.

Suppose $|T| > 2^k$, and let $t_0, t_1, \ldots, t_{2^k}$ be $2^k + 1$ distinct elements of T. Let B start by asking repeatedly about the set $\{t_{2^k}\}$. If A says *no* to the first $k+1$ of these questions, then t_{2^k} is excluded, reducing the size of T by 1.

If A says *yes* to any question, B stops asking about $\{t_{2^k}\}$ and asks about the sets $U_0, U_1, \ldots, U_{k-1}$, where

$$U_i = \{t_j \mid j \text{ has a 0 in the } i\text{th digit in binary}\}.$$

Construct the binary number $d = \overline{d_{k-1}d_{k-2}\cdots d_0}$ by $d_i = 0$ if A said *no* to U_i and $d_i = 1$ otherwise. If x were equal to t_d, then A would have lied in her answers to the previous $k+1$ questions. So $x \neq t_d$. In this case also, B has reduced $|T|$ by one. He can repeat until $|T| = 2^k$ and then specify the set T to win.

(2) Let λ be a real number with $1.99 < \lambda < 2$. Because $\lambda > 1.99$, for sufficiently large k, we have $1.99^k + 2 < (2-\lambda)\lambda^{k+1}$. Choose n to be an integer so that $1.99^k \leq n < 1.99^k + 1$, meaning that $n + 1 < (2-\lambda)\lambda^{k+1}$.

Player A will choose $N = n+1$ and an arbitrary x. Let $m_i(t)$ be the number of consecutive answers, ending at the tth answer, which are inconsistent with $x = i$ for $i = 1, 2, \ldots, n+1$, with $m_i(0) = 0$. Define

$$L(t) = \sum_{i=1}^{n+1} \lambda^{m_i(t)}.$$

Player A will use the strategy of giving the answer which minimizes $L(t)$, irrespective of her choice of x. We will show that this is a valid strategy and that B can guarantee that any number was not chosen by A.

We first show by induction that $L(t) < \lambda^{k+1}$. For the base case, we have $L(0) = n + 1 < (2-\lambda)\lambda^{k+1}$. Suppose now that $L(t) < \lambda^{k+1}$; if B asks about a set S, A chooses between two possible values for $L(t+1)$:

$$L_1 = |S| + \sum_{i \notin S} \lambda^{m_i(t)+1}, \qquad L_2 = (n+1-|S|) + \sum_{i \in S} \lambda^{m_i(t)+1}.$$

Recalling that $n+1 < (2-\lambda)\lambda^{k+1}$ and $\lambda L(t) < \lambda^{k+2}$, we have

$$\frac{L_1+L_2}{2} = \frac{n+1}{2} + \frac{1}{2}\sum_{i=1}^{n+1} \lambda^{m_i(t)+1} = \frac{n+1+\lambda L(t)}{2} < \lambda^{k+1},$$

so we see that $L(t+1) = \min\{L_1, L_2\} < \lambda^{k+1}$, completing the induction.

It is therefore impossible for any $m_i(t)$ to reach a value $k+1$ or higher, so $m_i(t) \leq k$ for all i and t. This means that A's strategy never violates the rules. Because A's answers are independent of x, there is no number that B can guarantee that A chose, so B cannot guarantee a win.

This problem was proposed by David Arthur and Jacob Tsimerman of Canada.

4. There are three classes of solutions, namely:

- for a fixed integer m, the function $f(n) = mn^2$,
- for a fixed non-zero integer m, the function

$$f(n) = \begin{cases} 0 & \text{if } n \text{ is even,} \\ m & \text{if } n \text{ is odd,} \end{cases} \quad \text{and}$$

- for a fixed non-zero integer m, the function

$$f(n) = \begin{cases} 0 & \text{if } n \equiv 0 \pmod 4, \\ 4m & \text{if } n \equiv 2 \pmod 4, \\ m & \text{if } n \text{ is odd.} \end{cases}$$

All of these functions satisfy the given identity. It remains to show they are the only solutions. Setting $a = b = c = 0$ in the identity yields $3f(0)^2 = 6f(0)^2$, hence $f(0) = 0$. Setting $a = 0$ and $c = -b$ in the identity yields $f(b)^2 + f(-b)^2 = 2f(b)f(-b)$, hence $(f(b) - f(-b))^2 = 0$, so $f(b) = f(-b)$ for every $b \in \mathbb{Z}$.

Setting $c = -a-b$ in the given and using the fact that $f(-a-b) = f(a+b)$, we obtain

$$f(a)^2 + f(b)^2 + f(a+b)^2 = 2f(a)f(b) + 2f(b)f(a+b) + 2f(a+b)f(a).$$

Rearranging and factoring yields

$$(f(a+b) - f(a) - f(b))^2 = 4f(a)f(b). \tag{1}$$

Now, if $f(a) = 0$ for any a, then from (1) (or from the original identity) we get $f(a+b) = f(b)$ for every $b \in \mathbb{Z}$, so f has period a. If $f(1) = 0$ this means that f is identically zero, and we have a solution in the first class. If $f(2) = 1$ the periodicity means that f is in the second class. Otherwise we may assume that $f(1) = m \neq 0$ and $f(2) \neq 0$.

In this case applying (1) with $a = b = 1$ implies that $f(2) = 4m$, and then applying (1) with $a = 2$ and $b = 1$ implies that $f(3) = m$ or $f(3) = 9m$. If $f(3) = m$, then applying (1) with $a = 3$ and $b = 1$ implies that $f(4) \in \{0, 4m\}$. On the other hand, applying (1) with $a = b = 2$ shows that $f(4) \in \{0, 16m\}$. Because m is non-zero, this shows that $f(4) = 0$. Now the periodicity of f places it the third class.

In the only remaining case, we have $f(0) = 0$, $f(1) = m$, $f(2) = 4m$, and $f(3) = 9m$. We claim by induction on n that $f(n) = mn^2$ for all positive integers n. The base cases $n = 0, 1, 2, 3$ hold by assumption. Suppose that $f(\ell) = m\ell^2$ for all $\ell \leq n$ and that $n \geq 3$. Now equation (1) with $a = n$ and $b = 1$ implies that $f(n+1) \in \{(n+1)^2m, (n-1)^2m\}$, while equation (1) with $a = n-1$ and $b = 2$ implies that $f(n+1) \in \{(n+1)^2m, (n-3)^2m\}$. Because $n \geq 3$, $(n-3)^2m \neq (n-1)^2m$, so we must have $f(n+1) = (n+1)^2m$, completing the induction and placing f in the first class of solutions.

This problem was proposed by Liam Baker of South Africa.

5. Extend segment AX through X to a point E satisfying $BE \perp AX$. Extend segment BX through X to a point F satisfying $AF \perp BX$. Let lines BE and AF meet at Y so that Y is the orthocenter of triangle ABX.

It is easy to see that $BDFY$ is cyclic, and so $AD \cdot AB = AF \cdot AY$ by power of a point. Consequently, using the fact that triangles ACD and ABC are similar, we have $AL^2 = AC^2 = AD \cdot AB = AF \cdot AY$. It follows that triangles AFL and ALY are similar, implying that $\angle ALY = \angle AFL = 90°$ and $YL^2 = YF \cdot YA$. In exactly the same way, we can show that $\angle BKY = 90°$ and $YK^2 = YE \cdot YB$.

Because X is the orthocenter of triangle ABY, $ABEF$ is cyclic, from which it follows that $YF \cdot YA = YE \cdot YB$ by power of a point. Combining this with our previous observations, we find that $YL^2 = YF \cdot YA = YE \cdot YB = YK^2$. Now, notice that YKM and YLM are right triangles which share side YM and have $YK = YL$, so they are congruent, implying that $MK = ML$.

This problem was proposed by Josef Tkadlec of the Czech Republic.

6. There exist such numbers $a_1, a_2, \ldots, a_n$ if and only if $n \equiv 1, 2 \pmod 4$.

We first show this is necessary. For $a_1, a_2, \ldots, a_n$ satisfying the conditions, let $M = \max\{a_1, a_2, \ldots, a_n\}$ and let $N = \sum_{i=1}^{n} i \cdot 3^{M-a_i}$ so that $\frac{1}{3^{a_1}} + \frac{2}{3^{a_2}} + \cdots + \frac{n}{3^{a_n}} = \frac{N}{3^M}$, which means that $N = 3^M$ is odd. Thus, we have

$$1 + 2 + \cdots + n \equiv \sum_{i=1}^{n} i \cdot 3^{M-a_i} = N \equiv 1 \pmod 2,$$

which can happen only if $n \equiv 1, 2 \pmod 4$.

We now show that $n \equiv 1, 2 \pmod 4$ is sufficient. Call a sequence $b_1, b_2, \ldots, b_n$ *feasible* if there exist nonnegative integers $a_1, a_2, \ldots, a_n$ such that

$$\frac{1}{2^{a_1}} + \frac{1}{2^{a_2}} + \cdots + \frac{1}{2^{a_n}} = \frac{b_1}{3^{a_1}} + \frac{b_2}{3^{a_2}} + \cdots + \frac{b_n}{3^{a_n}} = 1.$$

We wish to show the sequence $1, 2, \ldots, n$, which we denote by α_n, is feasible for $n \equiv 1, 2 \pmod 4$. We first give a method of generating feasible sequences.

LEMMA. *Let $b_1, b_2, \ldots, b_n$ be feasible. For non-negative integers u and v with $u + v = 3b_k$, the sequence $b_1, b_2, \ldots, b_{k-1}, u, v, b_{k+1}, \ldots, b_n$ is feasible.*

Proof. Let $a_1, \ldots, a_n$ be exponents making $b_1, \ldots, b_n$ feasible, meaning that

$$\left(\frac{1}{2^{a_1}} + \cdots + \frac{1}{2^{a_{k-1}}}\right) + \frac{1}{2^{a_k}} + \left(\frac{1}{2^{a_{k+1}}} + \cdots + \frac{1}{2^{a_n}}\right) = 1$$

$$\left(\frac{b_1}{3^{a_1}} + \cdots + \frac{b_{k-1}}{3^{a_{k-1}}}\right) + \frac{b_k}{3^{a_k}} + \left(\frac{b_{k+1}}{3^{a_{k+1}}} + \cdots + \frac{b_n}{3^{a_n}}\right) = 1.$$

Observe that

$$\frac{1}{2^{a_k+1}} + \frac{1}{2^{a_k+1}} = \frac{1}{2^{a_k}} \qquad \text{and} \qquad \frac{u}{3^{a_k+1}} + \frac{v}{3^{a_k+1}} = \frac{b_k}{3^{a_k}},$$

so the previous relations show that

$$\left(\frac{1}{2^{a_1}}+\cdots+\frac{1}{2^{a_{k-1}}}\right)+\frac{1}{2^{a_k+1}}+\frac{1}{2^{a_k+1}}+\left(\frac{1}{2^{a_{k+1}}}+\cdots+\frac{1}{2^{a_n}}\right)=1$$

$$\left(\frac{b_1}{3^{a_1}}+\cdots+\frac{b_{k-1}}{3^{a_{k-1}}}\right)+\frac{u}{3^{a_k+1}}+\frac{v}{3^{a_k+1}}+\left(\frac{b_{k+1}}{3^{a_{k+1}}}+\cdots+\frac{b_n}{3^{a_n}}\right)=1,$$

so the exponents $a_1, \ldots, a_{k-1}, a_k+1, a_k+1, a_{k+1}, \ldots, a_n$ show feasibility. ■

We induct on n to show that α_n is feasible when $n \equiv 1, 2 \pmod 4$. First, for the base cases $n = 1, 2, 5, 6$, we may take the sequences of exponents to be (0), $(1, 1)$, $(2, 2, 2, 3, 3)$, and $(2, 2, 3, 3, 3, 3)$, respectively. Now, suppose that for some $n \geq 9$, α_k is feasible for all $k < n$. If $n \equiv 2 \pmod 4$, then $n-1 \equiv 1 \pmod 4$, so α_{n-1} is feasible by the inductive hypothesis. Applying the lemma to α_{n-1} with $u = \frac{n}{2}$ and $v = n$, we see that α_n is feasible. If $n \equiv 1 \pmod 4$, then $n-7 \equiv 2 \pmod 4$ with $n-7 > 0$ and $\frac{n-5}{2} \geq n-7$. By the inductive hypothesis, α_{n-7} is feasible. Now, beginning with α_{n-7}, we apply the lemma seven times with $(b_k, u, v) = \left(\frac{n-5}{2}, \frac{n-3}{2}, n-6\right)$, $\left(\frac{n-3}{2}, \frac{n-3}{2}, n-3\right)$, $\left(\frac{n-3}{2}, \frac{n-1}{2}, n-4\right)$, $\left(\frac{n-1}{2}, \frac{n-1}{2}, n-1\right)$, $\left(\frac{n-1}{2}, \frac{n-3}{2}, n\right)$, $\left(\frac{n-3}{2}, \frac{n-5}{2}, n-2\right)$, and $\left(\frac{n-5}{2}, \frac{n-5}{2}, n-5\right)$ to obtain that α_n is feasible, completing the induction.

This problem was proposed by Dušan Djukić of Serbia.

Results

The IMO was held in Mar del Plata, Argentina, on July 10–11, 2011. There were 548 competitors from 100 countries and regions. On each day contestants were given four and a half hours for three problems.

On this challenging exam, a perfect score was achieved by only one student, Jeck Lim (Singapore). The USA team won 5 gold medals and 1 silver medal, placing third behind Korea and China. The students' individual results were as follows.

- Xiaoyu He, who finished 12th grade at Acton-Boxborough Regional High School in Acton, MA, won a silver medal.
- Ravi Jagadeesan, who finished 10th grade at Phillips Exeter Academy in Exeter, NH, won a gold medal.
- Mitchell Lee from Oakton, VA, who finished 12th grade (homeschooled), won a gold medal.
- Bobby Shen, who finished 11th grade at Dulles High School in Sugar Land, TX, won a gold medal and placed third overall with a score of 39/42.
- Thomas Swayze, who finished 11th grade at Canyon Crest Academy in San Diego, CA, won a gold medal.
- David Yang, who finished 11th grade at Phillips Exeter Academy in Exeter, NH, won a gold medal.