54th International Mathematical Olympiad

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Problems (Day 1)

1. Prove that for any pair of positive integers k and n, there exist k positive integers $m_1, m_2, \ldots, m_k$ (not necessarily different) such that

$$1 + \frac{2^k - 1}{n} = \left(1 + \frac{1}{m_1}\right)\left(1 + \frac{1}{m_2}\right)\cdots\left(1 + \frac{1}{m_k}\right).$$

2. A configuration of 4027 points in the plane is called *Colombian* if it consists of 2013 red points and 2014 blue points, and no three of the points of the configuration are collinear. By drawing some lines, the plane is divided into several regions. An arrangement of lines is *good* for a Colombian configuration if the following two conditions are satisfied:

   - no line passes through any point of the configuration;
   - no region contains points of both colours.

   Find the least value of k such that for any Colombian configuration of 4027 points, there is a good arrangement of k lines.

3. Let the excircle of the triangle ABC opposite the vertex A be tangent to side BC at A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcentre of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

   The excircle of triangle ABC opposite the vertex A is the circle that is tangent to the line segment BC, to the ray AB beyond B, and to the ray AC beyond C. The excircles opposite B and C are similarly defined.

Problems (Day 2)

4. Let ABC be an acute-angled triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of triangle CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y, and H are collinear.

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5. Let $\mathbb{Q}_{>0}$ be the set of positive rational numbers. Let $f : \mathbb{Q}_{>0} \to \mathbb{R}$ be a function satisfying the following three conditions:
   (i) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x)f(y) \geq f(xy)$;
   (ii) for all $x, y \in \mathbb{Q}_{>0}$, we have $f(x+y) \geq f(x) + f(y)$;
   (iii) there exists a rational number $a > 1$ such that $f(a) = a$.

   Prove that $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.
6. Let $n \geq 3$ be an integer, and consider a circle with $n+1$ equally spaced points marked on it. Consider all labellings of these points with the numbers $0, 1, \ldots, n$ such that each label is used exactly once; two such labellings are considered to be the same if one can be obtained from the other by a rotation of the circle. A labelling is called *beautiful* if, for any four labels $a < b < c < d$ with $a + d = b + c$, the chord joining the points labeled a and d does not intersect the chord joining the points labeled b and c.

   Let M be the number of beautiful labellings, and let N be the number of ordered pairs (x, y) of positive integers such that $x + y \leq n$ and $\gcd(x, y) = 1$. Prove that

$$M = N + 1.$$

Solutions

1. We induct on k. The base case $k = 1$ holds with $m_1 = n$. If the statement holds for some k, we consider two cases. If $n = 2m - 1$ is odd, we have

$$1 + \frac{2^{k+1}-1}{n} = \frac{2m}{2m-1}\,\frac{2^{k+1}+2m-2}{2m} = \left(1 + \frac{1}{2m-1}\right)\left(1 + \frac{2^k-1}{m}\right),$$

   where $1 + \frac{2^k-1}{m}$ is the product of k terms of the desired form by the induction hypothesis, yielding the desired decomposition. If $n = 2m$ is even, we have

$$\begin{aligned} 1 + \frac{2^{k+1}-1}{n} &= \frac{2^{k+1}+2m-1}{2^{k+1}+2m-2}\;\frac{2^{k+1}+2m-2}{2m} \\ &= \left(1 + \frac{1}{2^{k+1}+2m-2}\right)\left(1 + \frac{2^k-1}{m}\right), \end{aligned}$$

   where $1 + \frac{2^k-1}{m}$ is the product of k terms of the desired form by the induction hypothesis, yielding the desired decomposition and completing the induction.

   This problem was proposed by the Olympiad problem committee from Japan.
2. The answer is 2013. We first show that a good arrangement with at most 2013 lines always exists. We begin with the following key lemma.

   LEMMA 1. *Any pair of points P and Q in a Colombian configuration C can be separated from the other points by two lines.*

   Proof. No three points in C are collinear, so each other point has one of finitely many positive distances to PQ. Choose $r > 0$ less than all such distances; the two lines parallel to and at a distance r from PQ have the desired property. ■

   Let $\mathcal{C}$ be the convex hull of our Colombian configuration C. If a red point R is a vertex of $\mathcal{C}$, draw a line ℓ_1 separating R from all other points. Next, place the other 2012 red points into 1006 pairs and apply Lemma 1 to draw 2012 lines separating them from the rest of C. Together with ℓ_1, these 2012 lines form a good arrangement. Otherwise, $\mathcal{C}$ has a side B_1B_2 consisting of blue points. Draw a line ℓ_1 separating B_1 and B_2 from the rest of C, place the 2012 remaining blue points into

1006 pairs, and apply Lemma 1. The resulting 2012 lines together with ℓ_1 form the desired good arrangement.

For the other direction, let $\mathcal{P} = A_1 \cdots A_{4026}$ be a regular 4026-gon with A_i red for i odd and blue for i even. Form a Colombian configuration C with the vertices of $\mathcal{P}$ and another blue point B. We claim that any good arrangement for C has at least 2013 lines. Note that each of the 4026 pairs (A_i, A_{i+1}) of neighboring points contains points in different regions, so A_iA_{i+1} intersects at least one of the lines. Each line intersects $\mathcal{P}$ at most twice, so an arrangement with k lines can produce at most $2k$ intersections, implying that $2k \geq 4026$ and hence $k \geq 2013$, as desired.

This problem was proposed by Ivan Guo from Australia. The current formulation of this problem was suggested during IMO jury meetings by Leonardo I. M. Sandoval from Mexico.

3. Let ω be the circumcircle of ABC, and let O_1 be the circumcenter of $A_1B_1C_1$. Because A_1, B_1, and C_1 are on the boundary of ABC and O_1 is outside of ABC, $A_1B_1C_1$ is obtuse. Without loss of generality, assume that $\angle B_1A_1C_1$ is obtuse so that O_1 and A lie on the same side of line B_1C_1.

LEMMA 2. *The second intersection A_0 of ω and the circumcircle of triangle AB_1C_1 is the midpoint of arc $\widehat{BAC}$.*

Proof. By the definition of A_0, we have $\angle A_0BC_1 = \angle A_0BA = \angle A_0CA = \angle A_0CB_1$ and $\angle A_0C_1A = \angle A_0B_1A$; hence, AC_1B and AB_1C are similar. But $BC_1 = CB_1$, so these two triangles are congruent; hence, $A_0B = A_0C$. Because $AA_0B_1C_1$ is cyclic, we have $\angle C_1A_0B_1 = \angle C_1AB_1 = \angle BAC$, so A_0 lies on $\widehat{BAC}$ with $BA_0 = CA_0$, implying that A_0 is the midpoint of $\widehat{BAC}$. ■

By Lemma 2, a spiral similarity centered at A_0 sends B_1C_1 to CB, so A_0 is the intersection of ω and the perpendicular bisector of B_1C_1, which is on the same side of BC as A. Recalling that A_0 is the circumcenter of $A_1B_1C_1$, and using this result for the analogous points B_0 and C_0, we obtain that $A_0C_1B_0A_1$ and $A_0A_1C_0B_1$ are kites with symmetry axes A_0B_0 and A_0C_0. Recalling that $C_1B_1AA_0$ is cyclic, we have $\angle CAB = \angle C_1A_0B_1 = 2\angle B_0A_0C_0 = \widehat{B_0C_0}$. By Lemma 2, B_0 and C_0 are the midpoints of $\widehat{ABC}$ and $\widehat{BCA}$, hence

$$\angle CAB = \widehat{B_0C_0} = 360^\circ - \widehat{ACC_0} - \widehat{B_0A} = 360^\circ - \frac{\widehat{BCA} + \widehat{ABC}}{2}$$
$$= 360^\circ - \frac{360^\circ - 2\angle BCA + 360^\circ - 2\angle ABC}{2} = \angle BCA + \angle ABC,$$

implying that $\angle CAB = 90^\circ$, so ABC has right angle at vertex A.

This problem was proposed by Alexander Polyanskiy from Russia.

4. Let L be the foot of the altitude of ABC from A, and let O_1 and O_2 be the centers of ω_1 and ω_2, respectively. Because $\angle WNB < \angle CNB = 90^\circ$, O_1 and N lie on the same side of BC. Likewise, O_2 and M lie on the same side of BC. Hence, segment O_1O_2 does not intersect line BC. In particular, W does not lie on line O_1O_2, and ω_1 and ω_2 intersect again at a point Z other than W.

Because XW is a diameter of ω_1, we have $XZ \perp WZ$. Likewise, we have $YZ \perp WZ$, so X, Y, and Z lie on a line perpendicular to line ZW. It suffices to show that $HZ \perp ZW$. First, quadrilaterals $BNHL$ and $CMHL$ are cyclic, so by power of a point, $AM \cdot AC = AH \cdot AL = AN \cdot AB$; hence, A lies on the radical axis ZW of ω_1 and ω_2. Second, by our previous argument, A is the radical center of circles ω_1, ω_2, and the circumcircles of quadrilaterals $BNHL$ and $CMHL$. In particular, we have

$AH \cdot AL = AZ \cdot AW$, so $ZHLW$ is cyclic. This shows that $\angle AZH = \angle ALW = 90°$, hence $HZ \perp AW$, completing the proof.

This problem was proposed by Warut Suksompong and Potcharapol Suteparuk from Thailand.

5. We claim that the only solution is $f(x) = x$ for all $x \in \mathbb{Q}_{>0}$.

By applying (i) and iterating (ii), we see that $f(n)f(x) \geq f(nx) \geq nf(x)$ for positive integer n. Setting $x = a$, we see that $f(n) \geq n$, because $f(a) = a > 0$.

We claim that f is non-negative and non-decreasing. Indeed, if $f(y) < 0$, setting $x = y$ and dividing by $f(y)$ in our first inequality shows that $f(n) \leq n$, hence $f(n) = n$. This chain of inequalities is thus an equality, so $f(n)f(x) = f(nx)$ for all x. Writing $y = \frac{p}{q} \in \mathbb{Q}_{>0}$, we have $f(q)f(\frac{p}{q}) = f(p)$, so $f(y) = y$, a contradiction. Hence, f is non-negative, thus also non-decreasing by (ii).

We claim now that $f(x) \geq x$ for all $x \geq 1$. First, note that $f(x) \geq f(\lfloor x \rfloor) \geq \lfloor x \rfloor > x - 1$. From (i) we know that $f(x)^n \geq f(x^n)$, so $f(x)^n \geq f(x^n) > x^n - 1$. But if $f(x) = x - \epsilon$ for some $\epsilon > 0$ and $x > 1$, then for all n we have $1 > x^n - f(x)^n \geq (x - f(x))(x^{n-1}) = \epsilon x^{n-1}$. Since $x > 1$, we can choose n such that $x^{n-1} > \frac{1}{\epsilon}$, a contradiction. Therefore, $f(x) \geq x$ for all $x > 1$, and we already know that $f(1) \geq 1$, yielding the claim.

We now show that $f(x) = x$ for $x \geq 1$. Note that $a^k = f(a)^k \geq f(a^k)$ for positive integers k by (i). We also have $f(a^k) \geq a^k$, so $f(a^k) = a^k$ for positive integers k. For $x \geq 1$ and k with $a^k > 2x$, we have $a^k = f(a^k) \geq f(x) + f(a^k - x) \geq x + (a^k - x) = a^k$. Equality thus holds, so $f(x) = x$ for $x \geq 1$.

Finally, for any integer n, we have $f(n) = n$ and $f(n)f(x) \geq f(nx) \geq nf(x)$, so equality holds, implying that $f(nx) = nf(x)$. In particular, for any $x = \frac{p}{q}$ in $\mathbb{Q}_{>0}$, we conclude that $qf(x) = f(p) = p$, hence $f(x) = x$, as desired.

This problem was proposed by Nikolai Nikolov from Bulgaria.

6. We will prove that there are $N + 1$ beautiful labelings for all $n \geq 2$.

Let $0 < x < 1$ be a real number. Define the beautiful labeling $C_n(x)$ as follows. For $0 \leq k \leq n$, let $W_k = e^{\frac{2\pi i k}{n}x}$, and let Z_k be the point that results from rearranging the W_k evenly in the same relative position; label Z_k by k. Note that $W_a W_b$ and $W_c W_d$ intersect iff $Z_a Z_b$ and $Z_c Z_d$ intersect.

Call such a labeling *cyclic*, and call it *degenerate* if two of the W_k coincide. Note that $C_n(x)$ is degenerate iff x is a reduced fraction with denominator at most n. Call such numbers *good*; there are N good fractions in $(0, 1)$.

LEMMA 3. *A labeling is beautiful if and only if it is non-degenerate cyclic.*

Proof. Let $C_n(x)$ be a non-degenerate cyclic labeling. For any $0 \leq a < b < c < d \leq n$ with $a + d = b + c$, arcs $\widehat{W_a W_b}$ and $\widehat{W_c W_d}$ have the same measure, so $W_a W_d \parallel W_b W_c$, implying that $C_n(x)$ is beautiful.

For the converse, induct on n with trivial base case $n = 2$. If all beautiful arrangements of $[0, n-1]$ are cyclic, for a beautiful arrangement A of $[0, n]$, form $A' = C_{n-1}(x)$ by removing n. Let x lie between consecutive good fractions p_1/q_1 and p_2/q_2 with $q_1, q_2 \leq n - 1$, and consider two cases.

Case 1: There is no fraction with denominator n between $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$. If $A \neq C_n(x)$, they can differ only in the location of n. Suppose W_n occurs directly between W_i and W_j in $C_n(x)$ in clockwise order. Note that $i + (n-1) = (i-1) + n$ and $j + (n-1) = (j-1) + n$, so the two corresponding pairs of chords do not intersect and $W_i, W_n, W_j, W_{i-1}, W_{n-1}, W_{j-1}$ occur in that order. In A, since $W_i W_{n-1}$ does not intersect $W_n W_{i-1}$, W_n lies on arc $\widehat{W_i W_{n-1}}$. Likewise, W_n must lie on arc $\widehat{W_{n-1} W_j}$. Thus W_n lies between W_i and W_j, so $A = C_n(x)$.

Case 2: There is a fraction $\frac{a}{n}$ with denominator n between $\frac{p_1}{q_1}$ and $\frac{p_2}{q_2}$. Since $\frac{p_2}{q_2} -$

$\frac{p_1}{q_1} \leq \frac{1}{n-1}$, there is a unique such fraction. Choose $x_1 \in (\frac{p_1}{q_1}, \frac{a}{n})$ and $x_2 \in (\frac{a}{n}, \frac{p_2}{q_2})$. We wish to show that either $A = C_n(x_1)$ or $A = C_n(x_2)$. In A', W_{q_1}, W_0, W_{q_2} occur in that clockwise order. It suffices to show that W_n lies on arc $\widehat{W_{q_1} W_{q_2}}$ in A. This follows by an analysis of chords $W_{q_1} W_{n-1}$, $W_n W_{q_2-1}$, $W_{q_1-1} W_n$, and $W_{n-1} W_{q_2}$ using the final argument of Case 1. ■

By Lemma 3, it remains for us to count non-degenerate cyclic labelings. We claim that $C_n(x) = C_n(y)$ iff there is no good fraction between x and y. As x varies, the ordering of points in $C_n(x)$ changes only when $C_n(x)$ is degenerate so that two points coincide. It follows that $C_n(x) = C_n(y)$ when there is no good fraction between x and y. If there is a good fraction p/q with $x < p/q < y$, in $C_n(y)$ there are at least p integers $1 \leq i \leq q$ such that W_0 is clockwise of W_{i-1} and W_i is clockwise of W_0, while in $C_n(x)$ there are fewer than p such integers. Hence, $C_n(x)$ and $C_n(y)$ differ, giving the claim.

We conclude that the number of non-degenerate cyclic labelings is one greater than the number of good fractions in (0, 1), hence equal to $N + 1$.

This problem was proposed by Alexander Golovanov and Mikhail Ivaniv from Russia.

Results

The IMO was held in Santa Marta, Colombia, on July 23–24, 2013. There were 527 competitors from 97 countries and regions. On each day contestants were given four and a half hours for three problems.

The top score of 41/42 was shared by Yutao Liu (China) and Eunsoo Jee (South Korea). The USA team won 4 gold and 2 silver medals, placing third behind China and Korea. The students' individual results were as follows.

- Ray Li, who finished 12th grade at Phillips Exeter Academy in Exeter, NH, won a silver medal.
- Mark Sellke, who finished 11th grade at William Henry Harrison High School in West Lafayette, IN, won a gold medal.
- Bobby Shen, who finished 12th grade at Dulles High School in Sugar Land, TX, won a gold medal.
- Thomas Swayze, who finished 12th grade at Canyon Crest Academy in San Diego, CA, won a silver medal.
- James Tao, who finished 11th grade at Illinois Mathematics and Science Academy in Aurora, IL, won a gold medal.
- Victor Wang, who finished 12th grade at Ladue Horton Watkins High School in St. Louis, MO, won a gold medal.