# THE POLYNOMIAL REPRESENTATION OF THE TYPE $A_{n-1}$ RATIONAL CHEREDNIK ALGEBRA IN CHARACTERISTIC $p \mid n$

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Abstract. We study the polynomial representation of the rational Cherednik algebra of type $A_{n-1}$ with generic parameter in characteristic $p$ for $p \mid n$. We give explicit formulas for generators for the maximal proper graded submodule, show that they cut out a complete intersection, and thus compute the Hilbert series of the irreducible quotient. Our methods are motivated by taking characteristic $p$ analogues of existing characteristic 0 results.

Contents



## 1. Introduction

The present work presents a detailed study of the polynomial representation of the type $A_{n-1}$ rational Cherednik algebra over a field of characteristic $p$ dividing $n$. Rational Cherednik algebras were introduced by Etingof-Ginzburg in [EG02] as a rational degeneration of the double affine Hecke algebra dependent on two parameters $\hbar$ and $c$. In characteristic 0, their type $A$ representation theory has been the subject of extensive study. We refer the reader to [EM10] for a survey of these results.

In characteristic $p$ and especially in the modular case, much less is known about the representation theory of the rational Cherednik algebra. In this paper, we consider the modular case $p \mid n$. For $\hbar = 1$ and generic $c$, we provide a complete characterization of the irreducible quotient of the polynomial representation. We give explicit generators for the unique maximal proper graded submodule $J_c$, show that the irreducible quotient is a complete intersection, and compute its Hilbert series.

Our techniques are inspired by taking characteristic $p$ analogues of results about Cherednik algebras in characteristic 0. In particular, our explicit expression for generators of $J_c$ was obtained by converting expressions with complex residues to equivalent expressions dealing only with formal power series which may be interpreted in characteristic $p$. While we restrict our study to the polynomial representation in type $A$, we view it as a test case for this philosophy, which we believe may admit wider application.

We now state our results precisely and explain their relation to other recent work.

1.1. **The rational Cherednik algebra in positive characteristic.** We work over an algebraically closed field $k$ of characteristic $p > 0$ and fix $n$ so that $p \mid n$. Let $S_n$ denote the symmetric group on $n$ elements, $V = k^n$ its permutation representation, and $s_{ij} \in S_n$ the transposition permuting $i$ and $j$. Fix a basis $y_1, \ldots, y_n$ for $V$ and a dual basis $x_1, \ldots, x_n$ for $V^*$. Let $\mathfrak{h}$ and $\mathfrak{h}^*$ be the dual $(n-1)$-dimensional $S_n$-representations which are subrepresentation and quotient of $V$ and $V^*$, respectively given by

$$\mathfrak{h} = \text{span}\{y_i - y_j \mid i \neq j\} \text{ and } \mathfrak{h}^* = V^*/(x_1 + \cdots + x_n).$$

The action of $S_n$ on $\mathfrak{h}$ and $\mathfrak{h}^*$ is given explicitly by natural permutation of basis vectors.

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Fix constants $\hbar$ and $c$ in $k$. Denoting the tensor algebra of $\mathfrak{h} \oplus \mathfrak{h}^*$ by $T(\mathfrak{h} \oplus \mathfrak{h}^*)$, the *type* $A_{n-1}$ *rational Cherednik algebra* $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ is the quotient of $k[S_n] \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations

$$[x_i, x_j] = 0, \quad [y_i - y_j, y_l - y_m] = 0, \quad [y_i - y_j, x_i] = \hbar - cs_{ij} - c\sum_{t \neq i} s_{it}, \quad [y_i - y_j, x_l] = cs_{il} - cs_{jl}$$

for all $1 \leq i, j, l, m \leq n$ such that $i, j, l$ are distinct and $l \neq m$. There is a $\mathbb{Z}$-grading on $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ given by setting $\deg x = 1$ for $x \in \mathfrak{h}^*$, $\deg y = -1$ for $y \in \mathfrak{h}$, and $\deg g = 0$ for $g \in k[S_n]$. In addition, $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ admits a PBW decomposition

$$\mathcal{H}_{\hbar,c}(\mathfrak{h}) = \mathrm{Sym}(\mathfrak{h}) \otimes_k k[S_n] \otimes_k \mathrm{Sym}(\mathfrak{h}^*).$$

For any $\alpha \neq 0$, $\mathcal{H}_{\hbar,c}(\mathfrak{h})$ and $\mathcal{H}_{\alpha\hbar,\alpha c}(\mathfrak{h})$ are isomorphic as algebras, so only the cases $\hbar = 0$ or $\hbar = 1$ need be considered. In this paper, we restrict our attention to $\hbar = 1$.

1.2. **Polynomial representation of the rational Cherednik algebra.** The rational Cherednik algebra $\mathcal{H}_{1,c}(\mathfrak{h})$ admits a $\mathbb{Z}_{\geq 0}$-graded representation on the polynomial ring $A = \mathrm{Sym}(\mathfrak{h}^*)$, known as the *polynomial representation*. The actions of $\mathrm{Sym}(\mathfrak{h}^*)$ and $k[S_n]$ on $A$ are by left multiplication and the $S_n$ action on $\mathfrak{h}^*$, respectively. The action of $\mathrm{Sym}(\mathfrak{h})$ is implemented by letting $y \in \mathfrak{h}$ act by the *Dunkl operator*

$$D_y = \partial_y - \sum_{m<l} c\langle y, x_m - x_l\rangle \frac{1 - s_{ml}}{x_m - x_l},$$

where we note that $\frac{1-s_{ml}}{x_m - x_l} f$ is a polynomial for $f \in A$. Explicitly, we have

$$D_{y_i - y_j} = \partial_{y_i - y_j} - c\sum_{m \neq i} \frac{1 - s_{mi}}{x_i - x_m} + c\sum_{m \neq j} \frac{1 - s_{mj}}{x_j - x_m},$$

where $\partial_{y_i - y_j}$ is the differential operator satisfying $\partial_{y_i - y_j}(x) = \langle y_i - y_j, x\rangle$ for all $x \in \mathfrak{h}^*$.

1.3. **Maximal proper graded submodule and irreducible quotient of polynomial representation.** As described in [BC13, Section 2.5], there is a contravariant form

$$\beta_c : \mathrm{Sym}(\mathfrak{h}^*) \otimes \mathrm{Sym}(\mathfrak{h}) \to k$$

defined by setting $\beta_c(1, 1) = 1$ and imposing for all $x \in \mathfrak{h}^*, y \in \mathfrak{h}, f \in \mathrm{Sym}(\mathfrak{h}^*), g \in \mathrm{Sym}(\mathfrak{h})$ that

$$\beta_c(xf, g) = \beta_c(f, D_x(g)) \qquad \text{and} \qquad \beta_c(f, yg) = \beta_c(D_y(f), g).$$

where for $x \in \mathfrak{h}^*$ we denote by $D_x$ the Dunkl operator implementing the action of $\mathcal{H}_{1,c}(\mathfrak{h}^*)$ on its polynomial representation $\mathrm{Sym}(\mathfrak{h})$.

The polynomial representation $\mathrm{Sym}(\mathfrak{h}^*)$ has unique maximal graded proper submodule $J_c = \ker(\beta_c)$. By the definition of $\beta_c$, $J_c$ contains the ideal generated by all homogeneous vectors $f \in A$ of positive degree that are killed by all Dunkl operators $D_y$. Such $f$ are known as *singular vectors*. The quotient $L = A/J_c$ is an irreducible representation of $\mathcal{H}_{1,c}(\mathfrak{h})$. It inherits a $\mathbb{Z}_{\geq 0}$-grading from $A$, and for $L_j$ the degree $j$ subspace of $L$, we may define its Hilbert series as

$$h_L(t) = \sum_{j \geq 0} \dim L_j t^j.$$

1.4. **Statement of the main result.** For a formal power series $r(z)$, we denote by $[z^l]r(z)$ the coefficient of $z^l$ in $r(z)$. Throughout the paper, we will consider formal power series in $z$ considered as expansions of rational functions around $z = 0$. For $i = 1, \ldots, n-1$, define the formal power series

$$F_i(z) = \frac{1}{1 - x_i z} \sum_{m=0}^{p-1} \binom{c}{m} \left( \prod_{j=1}^{n} (1 - x_j z) - 1 \right)^m$$

for $\binom{c}{m} = \frac{c(c-1)\cdots(c-m+1)}{m!}$. Denote by $f_i$ the coefficients $f_i = [z^p]F_i(z)$.

**Theorem 4.1.** For generic $c$, $f_1, \ldots, f_{n-1}$ are linearly independent and generate the maximal proper graded submodule $J_c$ of the polynomial representation for $\mathcal{H}_{1,c}(\mathfrak{h})$. The irreducible quotient $L = A/J_c$ is a complete intersection with Hilbert series

$$h_L(t) = \left( \frac{1 - t^p}{1 - t} \right)^{n-1}.$$

**Remark.** In Theorem 4.1, by generic $c$ we mean $c$ avoiding finitely many values.

1.5. **Connections to previous work.** Our study is motivated by previous work on the representation theory of the type $A$ rational Cherednik algebra in both characteristic 0 and $p$. The type $A$ non-modular case $p \gg n$ was studied in [BFG06], and some properties of the maximal proper graded submodule of the polynomial representation were given in both modular and non-modular cases in [BC13]. In the modular case $p \mid n$, for $p = 2$ the polynomial representation associated to the $n$-dimensional permutation representation was studied in [Lia12].

**Theorem 1.1** ([Lia12, Theorem 5.1])**.** The irreducible quotient of the polynomial representation associated to the $n$-dimensional permutation representation is a complete intersection with Hilbert series

$$h(t) = (1+t)^n(1+t^2).$$

The corresponding maximal proper graded submodule is generated by $n-1$ elements of degree 2 and one element of degree 4.

It was further conjectured by Lian in [Lia12, Conjecture 5.2] that for all $p$ the corresponding irreducible is a complete intersection with $J_c$ having $n-1$ generators in degree $p$ and a single generator in degree $p^2$. Our results are consistent with the restriction of Lian's conjecture to the case when $\mathfrak{h}$ is the $(n-1)$-dimensional quotient. It would be interesting to extend our work to prove Lian's conjecture in full. For general $p \mid n$, a submodule of the maximal proper graded submodule was computed in [DS14, Proposition 6.1].

In characteristic 0, our results parallel the explicit decomposition of the polynomial representation of the type $A$ rational Cherednik algebra given in [BEG03, CE03]. There, the polynomial representation is irreducible unless $c = \frac{r}{n}$ for some integer $r$, and an explicit set of generators of the maximal proper graded submodule is known.

**Proposition 1.2** ([CE03, Proposition 3.1])**.** If $\mathrm{char}(k) = 0$ and $c = \frac{r}{n}$, the maximal proper graded submodule $J_c \subset A$ of the polynomial representation $A$ of $\mathcal{H}_{1,c}(\mathfrak{h})$ is generated by

$$\mathrm{Res}_\infty \left[ \frac{dz}{z - x_j} \prod_{i=1}^{n} (z - x_i)^c \right] \text{ for } j = 1, \ldots, n-1.$$

We interpret the characteristic $p$ analogue of Proposition 1.2 to mean that if $r = p$ and $p \mid n$, then since $p/n$ is equivalent to $0/0$ and thus an indeterminacy in characteristic $p$, taking $c = p/n$ in characteristic 0 should correspond to taking $c$ generic in characteristic $p$. While this substitution is of course invalid, Proposition 1.2 may be interpreted as a statement about certain formal power series. By using a power series version of this construction of generators which makes sense in characteristic $p$, we are able to mimic the arguments of [BEG03, CE03] to show that they cut out a complete intersection and generate the entire ideal. We believe that the philosophy of taking characteristic $p$ analogues of characteristic 0 results for the rational Cherednik algebra should apply more generally and hope to explore this further in future work.

1.6. **Outline of the paper.** The remainder of this paper is organized as follows. In Section 2, we check that the generators $f_i$ are linearly independent singular vectors. In Section 3, we show that they cut out a complete intersection. In Section 4, we put these facts together to conclude Theorem 4.1.

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## 2. An explicit construction of singular vectors

2.1. **Definition of the singular vectors.** In $A$, define the polynomials

$$g(z) = \prod_{j=1}^{n} (1 - x_j z) \qquad \text{and} \qquad F(z) = \sum_{m=0}^{p-1} \binom{c}{m} (g(z) - 1)^m.$$

In these terms, we have $F_i(z) = \frac{F(z)}{1 - x_i z}$ and $f_i = [z^p] \frac{F(z)}{1 - x_i z}$. We will show that $f_i$ are singular vectors.

2.2. **Computation of some partial derivatives.** We begin by computing some partial derivatives of $F$ which will be useful for computing the action of the Dunkl operators.

**Lemma 2.1.** We have $[z^0]g(z) = 1$ and $[z^1]g(z) = 0$, meaning $z^2 \mid g(z) - 1$.

*Proof.* For elementary symmetric polynomials $e_2, \ldots, e_n$, we have the expansion

$$g(z) = \prod_{j=1}^{n}(1 - x_j z) = 1 - z\sum_i x_i + z^2 e_2(x_1, \ldots, x_n) + \cdots + (-1)^n z^n e_n(x_1, \ldots, x_n).$$

Recalling that $\sum_i x_i = 0$ in $A$, we see that $[z^1]g(z) = 0$ and $[z^0]g(z) = 1$, so $z^2 \mid g(z) - 1$ as desired. □

**Lemma 2.2.** For some formal power series $V(z)$ with $[z^l]V(z) = 0$ for $l = 0, \ldots, p-1$, we have

$$F'(z) = V(z) - \sum_{j=1}^{n} \frac{cx_j}{1 - x_j z} F(z).$$

*Proof.* We see easily that $\frac{\partial g}{\partial z} = -g(z)\sum_j \frac{x_j}{1-x_j z}$. We now consider $\frac{\partial F}{\partial z}$. We compute

$$\begin{aligned}
\frac{\partial F}{\partial z} &= \sum_{m=1}^{p-1} m\binom{c}{m}(g(z)-1)^{m-1}\frac{\partial g}{\partial z} \\
&= -\sum_j \frac{x_j}{1-x_j z}\sum_{m=0}^{p-2} c\binom{c-1}{m}(g(z)-1)^m(g(z)-1+1) \\
&= -\sum_j \frac{x_j}{1-x_j z}\left(\sum_{m=0}^{p-2} c\binom{c-1}{m}(g(z)-1)^m + \sum_{m=1}^{p-1} c\binom{c-1}{m-1}(g(z)-1)^m\right) \\
&= -\sum_j \frac{x_j}{1-x_j z}\left(\sum_{m=0}^{p-1} c\binom{c}{m}(g(z)-1)^m - c\binom{c-1}{p-1}(g(z)-1)^{p-1}\right) \\
&= -\sum_j \frac{cx_j}{1-x_j z}F(z) + \sum_j \frac{x_j}{1-x_j z}c\binom{c-1}{p-1}(g(z)-1)^{p-1}.
\end{aligned}$$

Defining the formal power series

$$V(z) = \sum_j \frac{x_j}{1-x_j z}c\binom{c-1}{p-1}(g(z)-1)^{p-1},$$

we see that $F'(z) = V(z) - \sum_{j=1}^{n} \frac{cx_j}{1-x_j z}F(z)$. It remains only to show that $[z^l]V(z) = 0$ for $l = 0, \ldots, p-1$, which follows by noting that $(g(z)-1)^{p-1} \mid V(z)$, applying Lemma 2.1, and noting $p \geq 2$. □

**Lemma 2.3.** For some formal power series $G(z)$ with $[z^l]G(z) = 0$ for $l = 0, \ldots, p$, we have

$$\partial_{y_2 - y_1}(F(z)) = G(z) - \left(\frac{zc}{1-x_2 z} - \frac{zc}{1-x_1 z}\right)F(z).$$

*Proof.* We may compute $\partial_{y_2-y_1}(g(z)) = g(z)\left(-\frac{z}{1-x_2 z} + \frac{z}{1-x_1 z}\right)$. Using this, we see that

$$\begin{aligned}
\partial_{y_2-y_1}(F(z)) &= \left(\sum_{m=1}^{p-1} m\binom{c}{m}(g(z)-1)^{m-1}\right)\partial_{y_2-y_1}(g(z)) \\
&= \left(-\frac{z}{1-x_2 z} + \frac{z}{1-x_1 z}\right)\left(\sum_{m=1}^{p-1} m\binom{c}{m}(g(z)-1)^{m-1}\right)g(z) \\
&= \left(-\frac{z}{1-x_2 z} + \frac{z}{1-x_1 z}\right)\left(\sum_{m=0}^{p-2} c\binom{c-1}{m}(g(z)-1)^m + \sum_{m=0}^{p-2} c\binom{c-1}{m}(g(z)-1)^{m+1}\right) \\
&= \left(-\frac{zc}{1-x_2 z} + \frac{zc}{1-x_1 z}\right)\left(F(z) - \binom{c-1}{p-1}(g(z)-1)^{p-1}\right).
\end{aligned}$$

Defining $G(z) = \left(\frac{zc}{1-x_2z} - \frac{zc}{1-x_1z}\right)\binom{c-1}{p-1}(g(z)-1)^{p-1}$, we have shown that

$$\partial_{y_2-y_1}(F(z)) = G(z) - \left(\frac{zc}{1-x_2z} - \frac{zc}{1-x_1z}\right)F(z)$$

It remains only to show that $[z^l]G(z) = 0$ for $l = 0, \dots, p$, which follows by noting that $z(g(z)-1)^{p-1} \mid G(z)$, applying Lemma 2.1, and noting $p \geq 2$. □

## 2.3. **Proving $f_i$ are singular vectors.**

We now show that the $f_i$ are singular vectors summing to 0.

**Lemma 2.4.** For $c \neq 0$, we have $\sum_{i=1}^n f_i = 0$.

*Proof.* By Lemma 2.2, we have that

$$zF'(z) = zV(z) - \sum_{j=1}^n \frac{cx_jz}{1-x_jz}F(z) = zV(z) - c\sum_{j=1}^n \frac{1}{1-x_jz}F(z),$$

where we subtract $ncF(z)$ in the second equality. Notice that $[z^p]$ coefficient of the left side is 0 because $[z^{p-1}]F'(z) = 0$ and that $[z^p]zV(z) = 0$. Therefore, we conclude that $c\sum_i f_i = 0$, giving the desired. □

**Proposition 2.5.** The elements $f_i$ for $i = 1, \dots, n-1$ are singular vectors in $A$.

*Proof.* We must show that $f_i$ is annihilated by $D_{y_j-y_l}$ for all $j \neq l$. First, by symmetry it suffices to consider $f_1$. Because the Dunkl operators $D_{y_i-y_j}$ for all $i \neq j$ are spanned by $D_{y_i-y_1}$ for $1 < i \leq n$, it suffices to show $D_{y_i-y_1}f_1 = 0$. Finally, because $f_1$ is symmetric in the $x_i$ for $i > 1$, it suffices to show that $D_{y_2-y_1}f_1 = 0$.

Recall by Lemma 2.3 that $\partial_{y_2-y_1}(F(z)) = G(z) - \left(\frac{zc}{1-x_2z} - \frac{zc}{1-x_1z}\right)F(z)$ for a power series $G(z)$ with $[z^l]G(z) = 0$ for $l = 0, \dots, p$. In terms of $G(z)$, we can calculate $\partial_{y_2-y_1}(F_1(z))$ as

$$\begin{aligned}
\partial_{y_2-y_1}(F_1(z)) &= -\frac{z}{(1-x_1z)^2}F(z) + \frac{1}{1-x_1z}\partial_{y_2-y_1}(F(z)) \\
&= -\frac{z}{1-x_1z}F_1(z) + \frac{1}{1-x_1z}\left(\frac{zc}{1-x_1z} - \frac{zc}{1-x_2z}\right)F(z) + \frac{G(z)}{1-x_1z} \\
&= \left(\frac{z(c-1)}{1-x_1z} - \frac{zc}{1-x_2z}\right)F_1(z) + \frac{G(z)}{1-x_1z}.
\end{aligned}$$

In addition, we have that

$$\frac{1-s_{1i}}{x_1-x_i}(F_1(z)) = \frac{1}{x_1-x_i}\left(\frac{1}{1-x_1z} - \frac{1}{1-x_iz}\right)F(z) = \frac{z}{(1-x_iz)(1-x_1z)}F(z) = \frac{z}{1-x_iz}F_1(z).$$

Because $F_1(z)$ is invariant under $s_{ij}$ for $i, j > 1$, we now compute

$$\begin{aligned}
D_{y_2-y_1}(F_1(z)) &= \left(\partial_{y_2-y_1} - c\frac{1-s_{12}}{x_2-x_1} + c\sum_{j>1}\frac{1-s_{1j}}{x_1-x_j}\right)F_1(z) \\
&= \partial_{y_2-y_1}(F_1(z)) - c\frac{1-s_{12}}{x_2-x_1}F_1(z) + c\sum_{j>1}\frac{1-s_{1j}}{x_1-x_j}F_1(z) \\
&= \frac{G(z)}{1-x_1z} + \left(\frac{z(c-1)}{1-x_1z} - \frac{zc}{1-x_2z}\right)F_1(z) + \frac{zc}{1-x_2z}F_1(z) + \sum_{j>1}\frac{zc}{1-x_jz}F_1(z) \\
&= \frac{G(z)}{1-x_1z} - \frac{z}{1-x_1z}F_1(z) + \sum_j \frac{zc}{1-x_jz}F_1(z).
\end{aligned}$$

To show that the $z^p$ coefficient in $D_{y_2-y_1}(F_1(z))$ vanishes, we must consider $\frac{\partial F_1}{\partial z}$. By Lemma 2.2, we have

$$\frac{\partial F}{\partial z} = V(z) - \sum_j \frac{cx_j}{1-x_jz}F(z),$$

where $[z^l]V(z) = 0$ for $l = 0, \dots, p-1$. From this, it follows that

$$\begin{aligned}\frac{\partial F_1}{\partial z} = \frac{\partial}{\partial z}\left(\frac{F(z)}{1-x_1z}\right) &= \frac{1}{1-x_1z}\frac{\partial F}{\partial z} + \frac{x_1}{(1-x_1z)^2}F(z) \\ &= \frac{V(z)}{1-x_1z} - \frac{1}{1-x_1z}\sum_j \frac{cx_j}{1-x_jz}F(z) + \frac{x_1}{(1-x_1z)^2}F(z) \\ &= \frac{V(z)}{1-x_1z} - \sum_j \frac{cx_j}{1-x_jz}F_1(z) + \frac{x_1}{1-x_1z}F_1(z).\end{aligned}$$

We now compute

$$\begin{aligned}D_{y_2-y_1}(F_1(z)) &= \frac{G(z)}{1-x_1z} - \frac{z}{1-x_1z}F_1(z) + \sum_j \frac{zc}{1-x_jz}F_1(z) \\ &= \frac{G(z)}{1-x_1z} - zF_1(z) + zF_1(z) - \frac{z}{1-x_1z}F_1(z) + \sum_j \left(\frac{zc}{1-x_jz} - zc\right)F_1(z) \\ &= \frac{G(z)}{1-x_1z} - zF_1(z) - \frac{x_1z^2}{1-x_1z}F_1(z) + \sum_j \frac{x_jcz^2}{1-x_jz}F_1(z) \\ &= \frac{G(z)}{1-x_1z} - zF_1(z) - z^2\frac{\partial F_1}{\partial z} + z^2\frac{V(z)}{1-x_1z} \\ &= \frac{G(z)+z^2V(z)}{1-x_1z} - zF_1(z) - z^2\frac{\partial F_1}{\partial z},\end{aligned}$$

where in the second step we have subtracted $nzcF_1(z)$. We note that $[z^p]\,\frac{G(z)+z^2V(z)}{1-x_1z}$ is a linear combination of $[z^l](G(z)+z^2V(z))$ for $0 \le l \le p$, hence a linear combination of $[z^l]G(z)$ for $0 \le l \le p$ and $[z^l]V(z)$ for $0 \le l \le p-2$. By Lemmas 2.2 and 2.3, these coefficients of $G(z)$ and $V(z)$ are all 0, hence $[z^p]\,\frac{G(z)+z^2V(z)}{1-x_1z} = 0$. We conclude that

$$[z^p]D_{y_2-y_1}(F_1(z)) = [z^p]\left(-zF_1(z) - z^2\frac{\partial F_1}{\partial z}\right).$$

If $b = [z^{p-1}](F_1(z))$, then $[z^p](-zF_1(z)) = -b$ and $[z^p]\left(-z^2\frac{\partial F_1}{\partial z}\right) = b$, which implies that

$$D_{y_2-y_1}f_1 = [z^p]D_{y_2-y_1}(F_1(z)) = [z^p]\left(-zF_1(z) - z^2\frac{\partial F_1}{\partial z}\right) = -b + b = 0. \qquad \square$$

2.4. **Proof of linear independence of $f_i$.**

**Proposition 2.6.** For generic $c$, $f_1, \dots, f_{n-1}$ are linearly independent degree $p$ homogeneous polynomials.

*Proof.* We have the expansion

$$F_i(z) = \frac{1}{1-x_iz}\sum_{m=0}^{p-1}\binom{c}{m}(g(z)-1)^m = \sum_{l=0}^{\infty} x_i^l z^l \sum_{m=0}^{p-1}\binom{c}{m}(g(z)-1)^m.$$

Because for any $l$ the coefficient of $z^l$ in each factor is a homogeneous polynomial of degree $l$, we see that $[z^p]F_i(z)$ is homogeneous of degree $p$.

For linear independence, suppose that $\sum_{i=1}^{n-1}\lambda_i f_i = 0$ for some $\lambda_i \in k$. Substitute $x_n = -1$, $x_j = 1$ and $x_i = 0$ for $i \neq j, i < n$ so that $g(z) = (1-z)(1+z) = 1 - z^2$ and hence

$$F_j(z) = \sum_{l=0}^{\infty} z^l \sum_{m=0}^{p-1}\binom{c}{m}(-z^2)^m$$

and

$$F_i(z) = \sum_{l=0}^{\infty} 0^l z^l \sum_{m=0}^{p-1}\binom{c}{m}(-z^2)^m = \sum_{m=0}^{p-1}\binom{c}{m}(-z^2)^m \text{ for } i < n-1,\ i \neq j.$$

If $p = 2$, we see that $[z^2]F_j(z) = 1 - c$ and $[z^2]F_i(z) = -c$, so varying $j$ implies that

$$\lambda_j = c \sum_{i=1}^{n-1} \lambda_i \text{ for all } j.$$

In particular, all $\lambda_i$ have common value $\lambda \in k$ solving $(1 - c(n-1))\lambda = 0$, which for $c \neq -1$ and hence for $c$ generic implies that $\lambda = 0$, giving linear independence.

If $p > 2$, we have

$$[z^p]F_j(z) = f_j = \sum_{m=0}^{(p-1)/2} (-1)^m \binom{c}{m} = \binom{c-1}{(p-1)/2}$$

and $[z^p]F_i(z) = f_i = 0$ for $i < n$ and $i \neq j$. For $c \notin \{1, 2, \ldots, (p-1)/2\}$ and hence for generic $c$, we have $\binom{c-1}{(p-1)/2} \neq 0$, meaning that

$$\sum_{i=1}^{n-1} \lambda_i f_i = \lambda_j \binom{c-1}{(p-1)/2} = 0,$$

which implies $\lambda_j = 0$. Varying $j$ implies that $\lambda_j = 0$ for all $j$, again yielding linear independence. □

## 3. Complete intersection properties

Consider the ideal $I_c = \langle f_1, \ldots, f_{n-1} \rangle \subset A$ generated by the $f_i$. In this section, we will show that $A/I_c$ is a complete intersection. Recall that for an ideal $I \subset k[X_1, \ldots, X_m]$, the quotient ring $k[X_1, \ldots, X_m]/I$ is a complete intersection if $I$ has a minimal set of generators $g_1, \ldots, g_l$ with $\dim k[X_1, \ldots, X_m]/I = m - l$.

**Proposition 3.1.** For generic $c$, the quotient $A/I_c$ is a complete intersection.

*Proof.* By Proposition 2.6, $I_c$ has a set of $n-1$ linearly independent and therefore minimal generators $\{f_i\}$ in degree $p$. We will now show that for generic $c$, if $\alpha_1, \ldots, \alpha_n \in k$ satisfy $f_i(\alpha_1, \ldots, \alpha_n) = 0$ for $i = 1, \ldots, n$, then $\alpha_1 = \cdots = \alpha_n = 0$. Because $f_n$ lies in the span of $f_1, \ldots, f_{n-1}$ for $c \neq 0$ by Lemma 2.4, this will imply that $\operatorname{supp}(A/I_c) = \{(0, \ldots, 0)\}$ and hence $A/I_c$ has dimension 0, completing the proof.

Suppose that the $\alpha_i$ take values $\{s_1, \ldots, s_r\}$, where $s_i$ occurs with multiplicity $m_i > 0$ so that $g(z) = \prod_i (1 - s_i z)^{m_i}$, $\sum_i m_i s_i = 0$, and $\sum_i m_i = n$. We claim that for generic $c$ this implies $s_i = 0$ for all $i$. Let $B = k[s_1, \ldots, s_r]/(\sum_i m_i s_i = 0)$; it suffices to check that with $I_c$ interpreted as an ideal in $B$, the quotient $B/I_c$ is finite dimensional over $k$.

Define the polynomial $h(z) = \prod_{i=1}^r (z - s_i)$, and let the series expansion of $F(z)$ be given by

$$F(z) = \sum_{i \geq 0} a_i(s_1, \ldots, s_r, c) z^i,$$

where $a_i$ has degree $i$ in $s_1, \ldots, s_r$ and $a_0(s_1, \ldots, s_r, c) = 1$. Define a degree $p$ polynomial in $z$ by

$$\widetilde{F}(z; s_1, \ldots, s_r, c) := \sum_{i=0}^{p} a_i(s_1, \ldots, s_r, c) z^{p-i}.$$

In this notation, we have that

$$f_i = [z^p] \frac{1}{1 - s_i z} F(z) = \widetilde{F}(s_i; s_1, \ldots, s_r, c) = 0$$

for $i = 1, \ldots, r$. Notice that $\widetilde{F}(s_i; s_1, \ldots, s_r, c)$ is a polynomial in $s_1, \ldots, s_r$ of homogeneous degree $p$. By the definition of $F(z)$, we have

$$\widetilde{F}(z; s_1, \ldots, s_r, 0) = z^p,$$

meaning that if $c = 0$, we have $s_1 = \cdots = s_r = 0$, hence $B/I_0$ is finite dimensional over $k$.

Now, let $B^d$ and $I_c^d$ denote the degree $d$ pieces of $B$ and the homogeneous ideal $I_c$, respectively. Choose a monomial basis $\{t_i\}$ for $B^d$ independent of $c$, and let $I_c^d$ be spanned by a finite set of polynomials $\{g_j\}$ with $g_j = \sum_i h_{ji}(c) t_i$. Notice that $\dim I_c^d$ is given by the size of the maximal non-vanishing minor of the matrix $H = (h_{ji})$. On the other hand, we just showed that there is some degree $d > 0$ so that $I_0^d = B^d$, meaning that $\dim I_c^d < \dim B^d$ exactly when $c$ lies in the zero set of all $\dim B^d$ minors of $H$. This implies that for all but finitely many $c$ we have $\dim I_c^d = \dim B^d$ and hence $B/I_c$ is finite-dimensional over $k$.

For each of the finitely many choices of $r, m_1, \dots, m_r > 0$ with $\sum_i m_i = n$, we conclude that for all but finitely many $c$, if $\{\alpha_1, \dots, \alpha_n\}$ contains $s_i$ with multiplicity $m_i$, then $\alpha_1 = \cdots = \alpha_n = 0$. Taking the union over excluded possibilities for $c$, we conclude that $f_i(\alpha_1, \dots, \alpha_n) = 0$ implies $\alpha_1 = \cdots = \alpha_n = 0$ unconditionally for generic $c$, so $A/I_c$ is a complete intersection as desired. □

**Remark.** Proposition 3.1 is a formal power series analogue of [CE03, Theorem 3.2]. However, our proof differs from the "residues by parts" argument which appears there, as the crucial [CE03, Lemma 3.2] fails in the modular case. It is interesting to note that our proof does not appear to translate to the characteristic 0 case, as no analogue of the polynomial $\widetilde{F}(z; s_1, \dots, s_r, c)$ exists in that setting.

## 4. Proof of the main result

We now put everything together to obtain our main result.

**Theorem 4.1.** For generic $c$, $f_1, \dots, f_{n-1}$ are linearly independent and generate the maximal proper graded submodule $J_c$ of the polynomial representation for $\mathcal{H}_{1,c}(\mathfrak{h})$. The irreducible quotient $L = A/J_c$ is a complete intersection with Hilbert series

$$h_L(t) = \left(\frac{1 - t^p}{1 - t}\right)^{n-1}.$$

*Proof.* By [BC13, Proposition 3.4], the Hilbert series of $L$ is

$$h_L(t) = \left(\frac{1 - t^p}{1 - t}\right)^{n-1} h(t^p)$$

for a polynomial $h(t)$ with nonnegative integer coefficients. On the other hand, by Propositions 2.6 and 3.1, $A/I_c$ is a complete intersection with $n-1$ linearly independent degree $p$ generators $f_i$. Its Hilbert series is

$$h_{A/I_c}(t) = \left(\frac{1 - t^p}{1 - t}\right)^{n-1}.$$

By Proposition 2.5, the generators $f_i$ of $I_c$ are singular vectors, so $I_c \subseteq J_c$, implying that $h_{A/I_c}(t) \geq h_{A/J_c}(t)$ coefficient-wise. We conclude that $h(t) = 1$, hence $h_{A/I_c}(t) = h_{A/J_c}(t)$ and $I_c = J_c$, completing the proof. □

**Remark.** In the proof of Proposition 3.1, we require that $c$ avoids 0 for Lemma 2.4, $c$ avoids $\{-1, 1, \dots, (p-1)/2\}$ for Proposition 2.6, and that $c$ avoids a non-explicit finite set given by vanishing of a determinantal ideal. These are the only uses of the assumption that $c$ is generic, so Proposition 3.1 and Theorem 4.1 hold for $c$ avoiding these values.

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