THE POLYNOMIAL REPRESENTATION OF THE TYPE $A_{n-1}$ RATIONAL CHEREDNIK ALGEBRA IN CHARACTERISTIC $p \mid n$

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Abstract. We study the polynomial representation of the rational Cherednik algebra of type $A_{n-1}$ with generic parameter in characteristic $p$ for $p \mid n$. We give explicit formulas for generators for the maximal proper graded submodule, show that they cut out a complete intersection, and thus compute the Hilbert series of the irreducible quotient. Our methods are motivated by taking characteristic $p$ analogues of existing characteristic 0 results.

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1. Introduction

The present work presents a detailed study of the polynomial representation of the type $A_{n-1}$ rational Cherednik algebra over a field of characteristic $p$ dividing $n$. Rational Cherednik algebras were introduced by Etingof-Ginzburg in [EG02] as a rational degeneration of the double affine Hecke algebra dependent on two parameters $\hbar$ and $c$. In characteristic 0, their type $A$ representation theory has been the subject of extensive study. We refer the reader to [EM10] for a survey of these results.

In characteristic $p$ and especially in the modular case, much less is known about the representation theory of the rational Cherednik algebra. In this paper, we consider the modular case $p \mid n$. For $\hbar = 1$ and generic $c$, we provide a complete characterization of the irreducible quotient of the polynomial representation. We give explicit generators for the unique maximal proper graded submodule $J_c$, show that the irreducible quotient is a complete intersection, and compute its Hilbert series.

Our techniques are inspired by taking characteristic $p$ analogues of results about Cherednik algebras in characteristic 0. In particular, our explicit expression for generators of $J_c$ was obtained by converting expressions with complex residues to equivalent expressions dealing only with formal power series which may be interpreted in characteristic $p$. While we restrict our study to the polynomial representation in type $A$, we view it as a test case for this philosophy, which we believe may admit wider application.

We now state our results precisely and explain their relation to other recent work.

1.1. The rational Cherednik algebra in positive characteristic. We work over an algebraically closed field $k$ of characteristic $p > 0$ and fix $n$ so that $p \mid n$. Let $S_n$ denote the symmetric group on $n$ elements, $V = k^n$ its permutation representation, and $s_{ij} \in S_n$ the transposition permuting $i$ and $j$. Fix a basis $y_1, \ldots, y_n$ for $V$ and a dual basis $x_1, \ldots, x_n$ for $V^*$. Let $\mathfrak{h}$ and $\mathfrak{h}^*$ be the dual $(n-1)$-dimensional $S_n$-representations which are subrepresentation and quotient of $V$ and $V^*$, respectively given by

$$\mathfrak{h} = \text{span}\{y_i - y_j \mid i \neq j\} \text{ and } \mathfrak{h}^* = V^*/(x_1 + \cdots + x_n).$$

The action of $S_n$ on $\mathfrak{h}$ and $\mathfrak{h}^*$ is given explicitly by natural permutation of basis vectors.

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Fix constants $\hbar$ and $c$ in $k$. Denoting the tensor algebra of $\mathfrak{h} \oplus \mathfrak{h}^*$ by $T(\mathfrak{h} \oplus \mathfrak{h}^*)$, the type $A_{n-1}$ rational Cherednik algebra $\mathcal{H}_{1,c}(\mathfrak{h})$ is the quotient of $k[S_n] \rtimes T(\mathfrak{h} \oplus \mathfrak{h}^*)$ by the relations 

$$[x_i, x_j] = 0, \quad [y_i - y_j, y_l - y_m] = 0, \quad [y_i - y_j, x_i] = \hbar - cs_{ij} - c \sum_{t \neq i} s_{it},$$

for all $1 \leq i, j, l, m \leq n$ such that $i, j, l$ are distinct and $l \neq m$. There is a $\mathbb{Z}$-grading on $\mathcal{H}_{h,c}(\mathfrak{h})$ given by setting $\deg x = 1$ for $x \in \mathfrak{h}^*$, $\deg y = -1$ for $y \in \mathfrak{h}$, and $\deg g = 0$ for $g \in k[S_n]$. In addition, $\mathcal{H}_{h,c}(\mathfrak{h})$ admits a PBW decomposition

$$\mathcal{H}_{h,c}(\mathfrak{h}) = \text{Sym}(\mathfrak{h}) \otimes_k k[S_n] \otimes_k \text{Sym}(\mathfrak{h}^*).$$

For any $\alpha \neq 0$, $\mathcal{H}_{h,c}(\mathfrak{h})$ and $\mathcal{H}_{\alpha h,\alpha c}(\mathfrak{h})$ are isomorphic as algebras, so only the cases $\hbar = 0$ or $\hbar = 1$ need be considered. In this paper, we restrict our attention to $\hbar = 1$.

1.2. **Polynomial representation of the rational Cherednik algebra.** The rational Cherednik algebra $\mathcal{H}_{1,c}(\mathfrak{h})$ admits a $\mathbb{Z}_{\geq 0}$-graded representation on the polynomial ring $A = \text{Sym}(\mathfrak{h}^*)$, known as the polynomial representation. The actions of $\text{Sym}(\mathfrak{h}^*)$ and $k[S_n]$ on $A$ are by left multiplication and the $S_n$ action on $\mathfrak{h}^*$, respectively. The action of $\text{Sym}(\mathfrak{h})$ is implemented by letting $y \in \mathfrak{h}$ act by the Dunkl operator

$$D_y = \partial_y - \sum_{m \leq l} c(y, x_m - x_l) \frac{1 - s_{mi}}{x_m - x_l},$$

where we note that $\frac{1 - s_{mi}}{x_m - x_l} f$ is a polynomial for $f \in A$. Explicitly, we have

$$D_{y_i - y_j} = \partial_{y_i - y_j} - c \sum_{m \neq i} \frac{1 - s_{mi}}{x_i - x_m} + c \sum_{m \neq j} \frac{1 - s_{mj}}{x_j - x_m},$$

where $\partial_{y_i - y_j}$ is the differential operator satisfying $\partial_{y_i - y_j}(x) = \langle y_i - y_j, x \rangle$ for all $x \in \mathfrak{h}^*$.

1.3. **Maximal proper graded submodule and irreducible quotient of polynomial representation.** As described in [BC13, Section 2.5], there is a contravariant form

$$\beta_c : \text{Sym}(\mathfrak{h}^*) \otimes \text{Sym}(\mathfrak{h}) \to k$$

defined by setting $\beta_c(1, 1) = 1$ and imposing for all $x \in \mathfrak{h}^*, y \in \mathfrak{h}$, $f \in \text{Sym}(\mathfrak{h}^*), g \in \text{Sym}(\mathfrak{h})$ that

$$\beta_c(x f, g) = \beta_c(f, D_x(g)) \quad \text{and} \quad \beta_c(f, yg) = \beta_c(D_y(f), g).$$

where for $x \in \mathfrak{h}^*$ we denote by $D_x$ the Dunkl operator implementing the action of $\mathcal{H}_{1,c}(\mathfrak{h}^*)$ on its polynomial representation $\text{Sym}(\mathfrak{h})$.

The polynomial representation $\text{Sym}(\mathfrak{h}^*)$ has unique maximal graded proper submodule $J_c = \ker(\beta_c)$. By the definition of $\beta_c$, $J_c$ contains the ideal generated by all homogeneous vectors $f \in A$ of positive degree that are killed by all Dunkl operators $D_y$. Such $f$ are known as singular vectors. The quotient $L = A/J_c$ is an irreducible representation of $\mathcal{H}_{1,c}(\mathfrak{h})$. It inherits a $\mathbb{Z}_{\geq 0}$-grading from $A$, and for $L_j$ the degree $j$ subspace of $L$, we may define its Hilbert series as

$$h_L(t) = \sum_{j \geq 0} \dim L_j t^j.$$

1.4. **Statement of the main result.** For a formal power series $r(z)$, we denote by $[z^i]r(z)$ the coefficient of $z^i$ in $r(z)$. Throughout the paper, we will consider formal power series in $z$ considered as expansions of rational functions around $z = 0$. For $i = 1, \ldots, n - 1$, define the formal power series

$$F_i(z) = \frac{1}{1 - x_i z} \sum_{m = 0}^{p-1} \binom{c}{m} \left( \prod_{j=1}^{n} (1 - x_j z) - 1 \right)^m$$

for $\binom{c}{m} = \frac{c(c-1)\cdots(c-m+1)}{m!}$. Denote by $f_i$ the coefficients $f_i = [z^p] F_i(z)$.

**Theorem 4.1.** For generic $c$, $f_1, \ldots, f_{n-1}$ are linearly independent and generate the maximal proper graded submodule $J_c$ of the polynomial representation for $\mathcal{H}_{1,c}(\mathfrak{h})$. The irreducible quotient $L = A/J_c$ is a complete intersection with Hilbert series

$$h_L(t) = \left( \frac{1 - t^p}{1 - t} \right)^{n-1}.$$
Remark. In Theorem 4.1, by generic $c$ we mean $c$ avoiding finitely many values.

1.5. Connections to previous work. Our study is motivated by previous work on the representation theory of the type $A$ rational Cherednik algebra in both characteristic 0 and $p$. The type $A$ non-modular case $p \gg n$ was studied in [BFG06], and some properties of the maximal proper graded submodule of the polynomial representation were given in both modular and non-modular cases in [BC13]. In the modular case $p \mid n$, for $p = 2$ the polynomial representation associated to the $n$-dimensional permutation representation was studied in [Lia12].

Theorem 1.1 ([Lia12, Theorem 5.1]). The irreducible quotient of the polynomial representation associated to the $n$-dimensional permutation representation is a complete intersection with Hilbert series

$$h(t) = (1 + t)^n(1 + t^2).$$

The corresponding maximal proper graded submodule is generated by $n - 1$ elements of degree 2 and one element of degree 4.

It was further conjectured by Lian in [Lia12, Conjecture 5.2] that for all $p$ the corresponding irreducible is a complete intersection with $J_c$ having $n - 1$ generators in degree $p$ and a single generator in degree $p^2$. Our results are consistent with the restriction of Lian’s conjecture to the case when $\mathfrak{g}$ is the $(n - 1)$-dimensional quotient. It would be interesting to extend our work to prove Lian’s conjecture in full. For general $p \mid n$, a submodule of the maximal proper graded submodule was computed in [DS14, Proposition 6.1].

In characteristic 0, our results parallel the explicit decomposition of the polynomial representation of the type $A$ rational Cherednik algebra given in [BEG03, CE03]. There, the polynomial representation is irreducible unless $c = \frac{r}{n}$ for some integer $r$, and an explicit set of generators of the maximal proper graded submodule is known.

Proposition 1.2 ([CE03, Proposition 3.1]). If $\text{char}(k) = 0$ and $c = \frac{r}{n}$, the maximal proper graded submodule $J_c \subset A$ of the polynomial representation $A$ of $\mathcal{H}_{1,c}(\mathfrak{g})$ is generated by

$$\text{Res}_{\infty} \left[ \frac{dz}{z - x_j} \prod_{i=1}^{n} (z - x_i)^c \right] \text{ for } j = 1, \ldots, n - 1.$$ 

We interpret the characteristic $p$ analogue of Proposition 1.2 to mean that if $r = p$ and $p \mid n$, then since $p/n$ is equivalent to $0/0$ and thus an indeterminacy in characteristic $p$, taking $c = p/n$ in characteristic 0 should correspond to taking $c$ generic in characteristic $p$. While this substitution is of course invalid, Proposition 1.2 may be interpreted as a statement about certain formal power series. By using a power series version of this construction of generators which makes sense in characteristic $p$, we are able to mimic the arguments of [BEG03, CE03] to show that they cut out a complete intersection and generate the entire ideal. We believe that the philosophy of taking characteristic $p$ analogues of characteristic 0 results for the rational Cherednik algebra should apply more generally and hope to explore this further in future work.

1.6. Outline of the paper. The remainder of this paper is organized as follows. In Section 2, we check that the generators $f_i$ are linearly independent singular vectors. In Section 3, we show that they cut out a complete intersection. In Section 4, we put these facts together to conclude Theorem 4.1.

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2. An explicit construction of singular vectors

2.1. Definition of the singular vectors. In $A$, define the polynomials

$$g(z) = \prod_{j=1}^{n} (1 - x_j z) \quad \text{and} \quad F(z) = \sum_{m=0}^{p-1} \binom{c}{m} (g(z) - 1)^m.$$ 

In these terms, we have $F_i(z) = \frac{F(z)}{1 - x_i z}$ and $f_i = [z^p] F_i(z)$. We will show that $f_i$ are singular vectors.
2.2. Computation of some partial derivatives. We begin by computing some partial derivatives of $F$ which will be useful for computing the action of the Dunkl operators.

Lemma 2.1. We have $[z^0]g(z) = 1$ and $[z^1]g(z) = 0$, meaning $z^2 | g(z) - 1$.

Proof. For elementary symmetric polynomials $e_2, \ldots, e_n$, we have the expansion

$$g(z) = \prod_{j=1}^{n} (1 - x_j z) = 1 - z \sum_{i} x_i + z^2 e_2(x_1, \ldots, x_n) + \cdots + (-1)^n z^n e_n(x_1, \ldots, x_n).$$

Recalling that $\sum_i x_i = 0$ in $A$, we see that $[z^1]g(z) = 0$ and $[z^0]g(z) = 1$, so $z^2 | g(z) - 1$ as desired. \hfill \Box

Lemma 2.2. For some formal power series $V(z)$ with $[z^l]V(z) = 0$ for $l = 0, \ldots, p - 1$, we have

$$F'(z) = V(z) - \sum_{j=1}^{n} \frac{cz_j}{1 - x_j z} F(z).$$

Proof. We see easily that $\frac{\partial g}{\partial z} = -g(z) \sum_j \frac{x_j}{1 - x_j z}$. We now consider $\frac{\partial F}{\partial z}$. We compute

$$\frac{\partial F}{\partial z} = \sum_{m=1}^{p-1} m \binom{c}{m} (g(z) - 1)^{m-1} \frac{\partial g}{\partial z}$$

$$= -\sum_j \frac{x_j}{1 - x_j z} \sum_{m=0}^{p-2} c \binom{c-1}{m} (g(z) - 1)^m (g(z) - 1 + 1)$$

$$= -\sum_j \frac{x_j}{1 - x_j z} \left( \sum_{m=0}^{p-2} c \binom{c-1}{m} (g(z) - 1)^m + \sum_{m=1}^{p-1} c \binom{c-1}{m-1} (g(z) - 1)^m \right)$$

$$= -\sum_j \frac{cz_j}{1 - x_j z} F(z) + \sum_j \frac{x_j}{1 - x_j z} c \binom{c-1}{p-1} (g(z) - 1)^{p-1}.$$

Defining the formal power series

$$V(z) = \sum_j \frac{x_j}{1 - x_j z} c \binom{c-1}{p-1} (g(z) - 1)^{p-1},$$

we see that $F'(z) = V(z) - \sum_{j=1}^{n} \frac{cz_j}{1 - x_j z} F(z)$. It remains only to show that $[z^l]V(z) = 0$ for $l = 0, \ldots, p - 1$, which follows by noting that $(g(z) - 1)^{p-1} | V(z)$, applying Lemma 2.1, and noting $p \geq 2$. \hfill \Box

Lemma 2.3. For some formal power series $G(z)$ with $[z^l]G(z) = 0$ for $l = 0, \ldots, p$, we have

$$\partial_{y_2 - y_1} (F(z)) = G(z) - \left( \frac{z c}{1 - x_2 z} - \frac{z c}{1 - x_1 z} \right) F(z).$$

Proof. We may compute $\partial_{y_2 - y_1} (g(z)) = g(z) \left( -\frac{z}{1 - x_2 z} + \frac{z}{1 - x_1 z} \right)$. Using this, we see that

$$\partial_{y_2 - y_1} (F(z)) = \left( \sum_{m=1}^{p-1} m \binom{c}{m} (g(z) - 1)^{m-1} \right) \partial_{y_2 - y_1} (g(z))$$

$$= \left( -\frac{z}{1 - x_2 z} + \frac{z}{1 - x_1 z} \right) \left( \sum_{m=1}^{p-1} m \binom{c}{m} (g(z) - 1)^{m-1} \right) g(z)$$

$$= \left( -\frac{z}{1 - x_2 z} + \frac{z}{1 - x_1 z} \right) \left( \sum_{m=0}^{p-2} c \binom{c-1}{m} (g(z) - 1)^m + \sum_{m=0}^{p-2} c \binom{c-1}{m} (g(z) - 1)^{m+1} \right)$$

$$= \left( -\frac{z c}{1 - x_2 z} + \frac{z c}{1 - x_1 z} \right) \left( F(z) - \left( \frac{c-1}{p-1} \right) (g(z) - 1)^{p-1} \right).$$
Defining $G(z) = \left(\frac{z c}{1-z x_2} - \frac{z c}{1-x_1 z}\right) \left(\frac{c-1}{p-1}\right) (g(z) - 1)^{p-1}$, we have shown that

$$\partial_{y_2 - y_1}(F(z)) = G(z) - \left(\frac{z c}{1-z x_2} - \frac{z c}{1-x_1 z}\right) F(z)$$

It remains only to show that $[z^l]G(z) = 0$ for $l = 0, \ldots, p$, which follows by noting that $z(g(z) - 1)^{p-1} | G(z)$, applying Lemma 2.1, and noting $p \geq 2$. \qed

2.3. Proving $f_i$ are singular vectors. We now show that the $f_i$ are singular vectors summing to 0.

**Lemma 2.4.** For $c \neq 0$, we have $\sum_{i=1}^n f_i = 0$.

**Proof.** By Lemma 2.2, we have that

$$z F'(z) = z V(z) - \sum_{j=1}^n \frac{c x_j z}{1-x_j z} F(z) = z V(z) - c \sum_{j=1}^n \frac{1}{1-x_j z} F(z),$$

where we subtract $n c F(z)$ in the second equality. Notice that $[z^p]$ coefficient of the left side is 0 because $[z^{p-1}]F'(z) = 0$ and that $[z^p]z V(z) = 0$. Therefore, we conclude that $c \sum_i f_i = 0$, giving the desired. \qed

**Proposition 2.5.** The elements $f_i$ for $i = 1, \ldots, n - 1$ are singular vectors in $A$.

**Proof.** We must show that $f_i$ is annihilated by $D_{y_j - y_i}$ for all $j \neq i$. First, by symmetry it suffices to consider $f_1$. Because the Dunkl operators $D_{y_j - y_i}$ for all $i \neq j$ are spanned by $D_{y_j - y_i}$ for $1 < i \leq n$, it suffices to show $D_{y_1 - y_i}f_1 = 0$. Finally, because $f_1$ is symmetric in the $x_i$ for $i > 1$, it suffices to show that $D_{y_1 - y_i}f_1 = 0$.

Recall by Lemma 2.3 that $\partial_{y_2 - y_1}(F_1(z)) = G(z) - \left(\frac{z c}{1-z x_2} - \frac{z c}{1-x_1 z}\right) F(z)$ for a power series $G(z)$ with $[z^l]G(z) = 0$ for $l = 0, \ldots, p$. In terms of $G(z)$, we can calculate $\partial_{y_2 - y_1}(F_1(z))$ as

$$\partial_{y_2 - y_1}(F_1(z)) = -\frac{z}{(1-x_1 z)^2} F(z) + \frac{1}{1-x_1 z} \partial_{y_2 - y_1}(F(z))$$

$$= -\frac{z}{1-x_1 z} F_1(z) + \frac{1}{1-x_1 z} \left(\frac{z c}{1-x_1 z} - \frac{z c}{1-x_2 z}\right) F(z) + \frac{G(z)}{1-x_1 z}$$

$$= \left(\frac{z(c-1)}{1-x_1 z} - \frac{z c}{1-x_2 z}\right) F_1(z) + \frac{G(z)}{1-x_1 z}.$$
where \([z^l]V(z) = 0\) for \(l = 0, \ldots, p - 1\). From this, it follows that
\[
\frac{\partial F_1}{\partial z} = \frac{\partial}{\partial z} \left( \frac{F(z)}{1 - x_1 z} \right) = \frac{1}{1 - x_1 z} \frac{\partial F}{\partial z} + \frac{x_1}{(1 - x_1 z)^2} F(z)
\]
\[
= \frac{V(z)}{1 - x_1 z} - \frac{1}{1 - x_1 z} \sum_j c x_j \frac{\partial F}{\partial z} + \frac{x_1}{(1 - x_1 z)^2} F(z)
\]
\[
= \frac{V(z)}{1 - x_1 z} - \sum_j c x_j \frac{\partial F}{\partial z} F(z) + \frac{x_1}{1 - x_1 z} F_1(z).
\]
We now compute
\[
D_{y_2 - y_1}(F_1(z)) = \frac{G(z)}{1 - x_1 z} - \frac{z}{1 - x_1 z} F_1(z) + \sum_j \frac{z c}{1 - x_j z} F_1(z)
\]
\[
= \frac{G(z)}{1 - x_1 z} - z F_1(z) + z F_1(z) - \frac{z}{1 - x_1 z} F_1(z) + \sum_j \left( \frac{z c}{1 - x_j z} - z c \right) F_1(z)
\]
\[
= \frac{G(z)}{1 - x_1 z} - z F_1(z) - \frac{x_1 z^2}{1 - x_1 z} F_1(z) + \sum_j \frac{x_j c z^2}{1 - x_j z} F_1(z)
\]
\[
= \frac{G(z)}{1 - x_1 z} - z F_1(z) - \frac{z^2 \partial F_1}{\partial z} + \frac{z^2}{1 - x_1 z} V(z)
\]
\[
= \frac{G(z) + z^2 V(z)}{1 - x_1 z} - z F_1(z) - \frac{z^2 \partial F_1}{\partial z}.
\]
where in the second step we have subtracted \(n z c F_1(z)\). We note that \([z^p] \frac{G(z) + z^2 V(z)}{1 - x_1 z}\) is a linear combination of \([z^l](G(z) + z^2 V(z))\) for \(0 \leq l \leq p\), hence a linear combination of \([z^l]G(z)\) for \(0 \leq l \leq p\) and \([z^l]V(z)\) for \(0 \leq l \leq p - 2\). By Lemmas 2.2 and 2.3, these coefficients of \(G(z)\) and \(V(z)\) are all 0, hence \([z^p] \frac{G(z) + z^2 V(z)}{1 - x_1 z} = 0\).

We conclude that
\[
[z^p]D_{y_2 - y_1}(F_1(z)) = [z^p] \left( -z F_1(z) - z z^2 \frac{\partial F_1}{\partial z} \right).
\]
If \(b = [z^{p-1}](F_1(z))\), then \([z^p](-z F_1(z)) = -b\) and \([z^p](-z^2 \frac{\partial F_1}{\partial z}) = b\), which implies that
\[
D_{y_2 - y_1} F_1 = [z^p]D_{y_2 - y_1}(F_1(z)) = [z^p] \left( -z F_1(z) - z^2 \frac{\partial F_1}{\partial z} \right) = -b + b = 0.
\]

2.4. Proof of linear independence of \(f_i\).

**Proposition 2.6.** For generic \(c, f_1, \ldots, f_{n-1}\) are linearly independent degree \(p\) homogeneous polynomials.

**Proof.** We have the expansion
\[
F_i(z) = \frac{1}{1 - x_i z} \sum_{m=0}^{p-1} \binom{c}{m} (g(z) - 1)^m = \sum_{l=0}^{\infty} x_i^l z^l \sum_{m=0}^{p-1} \binom{c}{m} (g(z) - 1)^m.
\]

Because for any \(l\) the coefficient of \(z^l\) in each factor is a homogeneous polynomial of degree \(l\), we see that \([z^p]F_i(z)\) is homogeneous of degree \(p\).

For linear independence, suppose that \(\sum_{i=1}^{n-1} \lambda_i f_i = 0\) for some \(\lambda_i \in k\). Substitute \(x_n = -1\), \(x_j = 1\) and \(x_i = 0\) for \(i \neq j\), \(i < n\) so that \(g(z) = (1 - z)(1 + z) = 1 - z^2\) and hence
\[
F_j(z) = \sum_{l=0}^{\infty} z^l \sum_{m=0}^{p-1} \binom{c}{m} (-z^2)^m
\]
and
\[
F_i(z) = \sum_{l=0}^{\infty} \theta^l \sum_{m=0}^{p-1} \binom{c}{m} (-z^2)^m = \sum_{m=0}^{p-1} \binom{c}{m} (-z^2)^m \text{ for } i < n - 1, i \neq j.
\]
If \( p = 2 \), we see that \([z^p]F_j(z) = 1 - c\) and \([z^p]F_i(z) = -c\), so varying \( j \) implies that
\[
\lambda_j = c \sum_{i=1}^{n-1} \lambda_i \quad \text{for all} \quad j.
\]
In particular, all \( \lambda_i \) have common value \( \lambda \in k \) solving \((1 - c(n-1))\lambda = 0\), which for \( c \neq -1 \) and hence for \( c \) generic implies that \( \lambda = 0 \), giving linear independence.

If \( p > 2 \), we have
\[
[z^p]F_j(z) = f_j = \sum_{m=0}^{(p-1)/2} (-1)^m \binom{c}{m} = \binom{c - 1}{(p-1)/2},
\]
and \([z^p]F_i(z) = f_i = 0\) for \( i < n \) and \( i \neq j \). For \( c \notin \{1, \ldots, (p-1)/2\} \) and hence for generic \( c \), we have \((p-1)/2 \neq 0\), meaning that
\[
\sum_{i=1}^{n-1} \lambda_i f_i = \lambda_j \binom{c - 1}{(p-1)/2} = 0,
\]
which implies \( \lambda_j = 0 \). Varying \( j \) implies that \( \lambda_j = 0 \) for all \( j \), again yielding linear independence. \( \square \)

### 3. Complete intersection properties

Consider the ideal \( I_c = \langle f_1, \ldots, f_{n-1} \rangle \subset A \) generated by the \( f_i \). In this section, we will show that \( A/I_c \) is a complete intersection. Recall that for an ideal \( I \subset k[X_1, \ldots, X_m] \), the quotient ring \( k[X_1, \ldots, X_m]/I \) is a complete intersection if \( I \) has a minimal set of generators \( g_1, \ldots, g_t \) with \( \dim k[X_1, \ldots, X_m]/I = m - t \).

**Proposition 3.1.** For generic \( c \), the quotient \( A/I_c \) is a complete intersection.

**Proof.** By Proposition 2.6, \( I_c \) has a set of \( n-1 \) linearly independent and therefore minimal generators \{\( f_i \}\} in degree \( p \). We will now show that for generic \( c \), if \( a_1, \ldots, a_n \in k \) satisfy \( f_i(a_1, \ldots, a_n) = 0 \) for \( i = 1, \ldots, n \), then \( a_1 = \cdots = a_n = 0 \). Because \( f_n \) lies in the span of \( f_1, \ldots, f_{n-1} \) for \( c \neq 0 \) by Lemma 2.4, this will imply that \( \operatorname{supp}(A/I_c) = \{0, \ldots, 0\} \) and hence \( A/I_c \) has dimension 0, completing the proof.

Suppose that the \( a_i \) take values \{\( s_1, \ldots, s_r \}\}, where \( s_i \) occurs with multiplicity \( m_i > 0 \) so that \( g(z) = \prod_i (1 - s_i z)^{m_i}, \sum_i m_i s_i = 0, \) and \( \sum_i m_i = n \). We claim that for generic \( c \) this implies \( s_i = 0 \) for all \( i \). Let \( B = k[s_1, \ldots, s_r] / (\sum_i m_i s_i = 0) \); it suffices to check that with \( I_c \) interpreted as an ideal in \( B \), the quotient \( B/I_c \) is finite dimensional over \( k \).

Define the polynomial \( h(z) = \prod_{i=1}^r (z - s_i) \), and let the series expansion of \( F(z) \) be given by
\[
F(z) = \sum_{i=0} a_i(s_1, \ldots, s_r, c) z^i,
\]
where \( a_i \) has degree \( i \) in \( s_1, \ldots, s_r \) and \( a_0(s_1, \ldots, s_r, c) = 1 \). Define a degree \( p \) polynomial in \( z \) by
\[
\tilde{F}(z; s_1, \ldots, s_r, c) := \sum_{i=0}^p a_i(s_1, \ldots, s_r, c) z^{p-i}.
\]

In this notation, we have that
\[
F_i = [z^p] \frac{1}{1-s_i z} F(z) = \tilde{F}(s_i; s_1, \ldots, s_r, c) = 0
\]
for \( i = 1, \ldots, r \). Notice that \( \tilde{F}(s_i; s_1, \ldots, s_r, c) \) is a polynomial in \( s_1, \ldots, s_r \) of homogeneous degree \( p \). By the definition of \( F(z) \), we have
\[
\tilde{F}(z; s_1, \ldots, s_r, 0) = z^p,
\]
meaning that if \( c = 0 \), we have \( s_1 = \cdots = s_r = 0 \), hence \( B/I_0 \) is finite dimensional over \( k \).

Now, let \( B^d \) and \( I_c^d \) denote the degree \( d \) pieces of \( B \) and the homogeneous ideal \( I_c \), respectively. Choose a monomial basis \( \{t_i\} \) for \( B^d \) independent of \( c \), and let \( I_c^d \) be spanned by a finite set of polynomials \( \{g_j\} \) with \( g_j = \sum_i h_{ij}(c)t_i \). Notice that \dim \( I_c^d \) is given by the size of the maximal non-vanishing minor of the matrix \( H = (h_{ij}) \). On the other hand, we just showed that there is some degree \( d > 0 \) so that \( I_0^d = B^d \), meaning that \( \dim I_c^d < \dim B^d \) exactly when \( c \) lies in the zero set of all \( \dim B^d \) minors of \( H \). This implies that for all but finitely many \( c \) we have \( \dim I_c^d = \dim B^d \) and hence \( B/I_c \) is finite-dimensional over \( k \).
For each of the finitely many choices of \( r, m_1, \ldots, m_r > 0 \) with \( \sum_i m_i = n \), we conclude that for all but finitely many \( c \), if \( \{\alpha_1, \ldots, \alpha_n\} \) contains \( s_i \) with multiplicity \( m_i \), then \( \alpha_1 = \cdots = \alpha_n = 0 \). Taking the union over excluded possibilities for \( c \), we conclude that \( f_1(\alpha_1, \ldots, \alpha_n) = 0 \) implies \( \alpha_1 = \cdots = \alpha_n = 0 \) unconditionally for generic \( c \), so \( A/I_c \) is a complete intersection as desired. \( \square \)

**Remark.** Proposition 3.1 is a formal power series analogue of [CE03, Theorem 3.2]. However, our proof differs from the “residues by parts” argument which appears there, as the crucial [CE03, Lemma 3.2] fails in the modular case. It is interesting to note that our proof does not appear to translate to the characteristic 0 case, as no analogue of the polynomial \( \tilde{F}(z; s_1, \ldots, s_r, c) \) exists in that setting.

4. PROOF OF THE MAIN RESULT

We now put everything together to obtain our main result.

**Theorem 4.1.** For generic \( c, f_1, \ldots, f_{n-1} \) are linearly independent and generate the maximal proper graded submodule \( J_c \) of the polynomial representation for \( H_{1,c}(b) \). The irreducible quotient \( L = A/J_c \) is a complete intersection with Hilbert series

\[
h_L(t) = \left( \frac{1 - t^{p-1}}{1 - t} \right)^{n-1}.
\]

**Proof.** By [BC13, Proposition 3.4], the Hilbert series of \( L \) is

\[
h_L(t) = \left( \frac{1 - t^{p-1}}{1 - t} \right)^{n-1} h(t^p)
\]

for a polynomial \( h(t) \) with nonnegative integer coefficients. On the other hand, by Propositions 2.6 and 3.1, \( A/I_c \) is a complete intersection with \( n - 1 \) linearly independent degree \( p \) generators \( f_i \). Its Hilbert series is

\[
h_{A/I_c}(t) = \left( \frac{1 - t^{p-1}}{1 - t} \right)^{n-1}.
\]

By Proposition 2.5, the generators \( f_i \) of \( I_c \) are singular vectors, so \( I_c \subseteq J_c \), implying that \( h_{A/I_c}(t) \geq h_{A/J_c}(t) \) coefficient-wise. We conclude that \( h(t) = 1 \), hence \( h_{A/I_c}(t) = h_{A/J_c}(t) \) and \( I_c = J_c \), completing the proof. \( \square \)

**Remark.** In the proof of Proposition 3.1, we require that \( c \) avoids 0 for Lemma 2.4, \( c \) avoids \( \{-1, 1, \ldots, (p-1)/2\} \) for Proposition 2.6, and that \( c \) avoids a non-explicit finite set given by vanishing of a determinantal ideal. These are the only uses of the assumption that \( c \) is generic, so Proposition 3.1 and Theorem 4.1 hold for \( c \) avoiding these values.

**References**


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