Statistics 251
Section 2, Autumn 2020
Practice Final Exam
Time: 2 Hours

Name: ______________________________

CNetID: ______________________________

Instructions: This exam contains 6 problems. Please make sure you attempt all problems.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the blank pages stapled to the end of the exam. Please **do not write on the back side of pages**.

You will have 2 hours to complete this exam. You must take this exam between **10:30am** and **12:30pm** Chicago time on December 9, 2020. You must scan and upload the completed exam to Gradescope by **1:00pm Chicago time on December 9, 2020**.

The use of outside material including books, notes, calculators, and electronic devices is not allowed. Due to the coronavirus situation, this exam will be take-home. Please sign below to affirm that you have followed these rules.

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Signature: ______________________________

Formulas

Distribution	Probability mass function or density
Binomial $X \sim B(n,p)$	$\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, \cdots, n$
Geometric $X \sim \text{Geom}(p)$	$\mathbb{P}(X=k) = (1-p)^{k-1} p$, $k = 1, 2, 3, \cdots$
Poisson $X \sim \text{Poisson}(\lambda)$	$\mathbb{P}(X=k) = e^{-\lambda} \frac{\lambda^k}{k!}$, $k = 0, 1, 2, \cdots$
Normal $X \sim \mathcal{N}(\mu, \sigma^2)$	$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $x \in (-\infty, \infty)$
Exponential $X \sim \text{Exp}(\lambda)$	$f(t) = \lambda e^{-\lambda t}$, $x \in [0, \infty)$
Gamma $X \sim \text{Gamma}(\alpha, \lambda)$	$f(t) = \Gamma(\alpha)^{-1} \lambda^\alpha t^{\alpha-1} e^{-\lambda t}$, $x \in [0, \infty)$

The error function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{x^2}{2}}\, dx$$

has values given by the following table.

a	-2.58	-1.96	-1.65	-1.28	0	1.28	1.65	1.96	2.58
$\Phi(a)$	0.005	0.025	0.05	0.1	0.5	0.9	0.95	0.975	0.995

Problem 1 (10 points) Suppose A, B are two events such that $\mathbb{P}(A) = 0.3, \mathbb{P}(B) = 0.4$, and $\mathbb{P}(A \cup B) = 0.5$.

(a) (2.5 points) Find $\mathbb{P}(A \mid B)$

Answer:

(b) (2.5 points) Are A and B independent?

Answer:

(c) (2.5 points) Find $\mathbb{P}\left(A^c \cap B\right)$.

Answer:

(d) (2.5 points) Let $X = I_A, Y = I_B$. Find the correlation $\rho(X, Y)$.

Answer:

Problem 2 (10 points) If U is uniform on $(0, 2\pi)$ and Z is independent of X and exponential with rate 1, show that the random variables X and Y defined by

$$X = \sqrt{2Z}\cos U$$
$$Y = \sqrt{2Z}\sin U$$

are independent standard normal random variables.

Answer:

Problem 3 (10 points) Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables.

(a) (5 points) Calculate $\mathbb{E}[X_1 | X_1 + \cdots + X_n = x]$

Answer:

(b) (5 points) If X_1, X_2 are exponential with parameter λ, find the conditional variance $\mathrm{Var}(X_1 \mid X_1 + X_2 = x)$.

Answer:

Problem 4 (15 points) Let U_1 and U_2 be two independent uniform random variables on $[0,1]$. Define

$$X = \min(U_1, U_2)$$
$$Y = \max(U_1, U_2).$$

Find

(a) (5 points) the probability density function f_X of X

Answer:

(b) (5 points) the joint density function $f_{X,Y}$ of (X,Y)

Answer:

(c) (5 points) $\mathbb{P}(X \leq 1/2 \mid Y \geq 1/2)$

Answer:

Problem 5 (20 points) Let T_1 and T_2 be two independent exponential variables with rates λ_1 and λ_2, respectively. Define $T_{\min} = \min(T_1, T_2)$, and let $X_{\min}$ be a random variable which equals 1 if $T_1 < T_2$ and 2 if $T_2 < T_1$.

(a) (5 points) The distribution of $T_{\min}$ is exponential with some parameter λ. Find λ.

Answer:

(b) (5 points) Find $\mathbb{P}(X_{\min} = 1)$.

Answer:

(c) (10 points) Show that $T_{\min}$ and $X_{\min}$ are independent.

Answer:

Problem 6 (20 points) A box contains three coins, of which two are fair and one is unfair, meaning it lands heads with proability 1.

(a) (5 points) If you choose a coin at random and toss it, what is the probability it lands heads?

Answer:

(b) (15 points) If you choose a coin at random and get heads when tossing it, what is the probability it is the unfair coin?

Answer:

Problem 7 (15 points) Consider the sample average $\overline{X}_n = (X_1 + X_2 + \cdots + X_n)/n$ of n i.i.d. random variables $X_1, \ldots, X_n$ which are uniformly distributed on $[0, 1]$.

(a) (5 points) Use Markov's inequality to upper bound $\mathbb{P}(\overline{X}_n \geq 0.99)$.

Answer:

(b) (10 points) Use the Central Limit Theorem to find n so that $\mathbb{P}(\overline{X}_n < 0.51)$ is approximately 90%.

Answer: