Statistics 251
Section 2, Autumn 2020
Practice Midterm Solutions
October 26, 2020
Time Limit: 50 Minutes

Name: ____________________________

CNetID: ____________________________

Instructions: This exam contains 4 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the blank pages stapled to the end of the exam. Please **do not write on the back side of pages**.

You will have 50 minutes to complete this exam. You may choose any 50 minute period between 10:30am and 12 midnight Chicago time on October 26, 2020 to take the exam. You must scan and upload the completed exam to Gradescope by **12 midnight Chicago time on October 26, 2020**. Please write the 50 minute period you took the exam below.

Start Time: ____________________________

End Time: ____________________________

The use of outside material including books, notes, calculators, and electronic devices is not allowed. Due to the coronavirus situation, this exam will be take-home. Please sign below to affirm that you have followed these rules.

Signature: ____________________________

Formulas

Probability mass functions:

- Binomial with parameters n and p: $\mathbb{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$
- Poisson with parameter λ: $\mathbb{P}(X = k) = \frac{\lambda^k}{k!} e^{-\lambda}$
- Geometric with parameter p: $\mathbb{P}(X = k) = p(1-p)^{k-1}$
- Negative binomial with parameters r and p: $\mathbb{P}(X = k) = \binom{k-1}{r-1} p^r (1-p)^{k-r}$

Problem 1 (10 points) How many quintuples $(a_1, a_2, a_3, a_4, a_5)$ of non-negative integers satisfy $a_1 + a_2 + a_3 + a_4 + a_5 = 100$?

Answer: $\boxed{\binom{104}{4}}$ If we put 105 pebbles in a line and draw 4 dividers in the 104 spaces between them, this divides the pebbles into 5 ordered groups of size $b_1, \ldots, b_5 \geq 1$ with $b_1 + \cdots + b_5 = 105$. This can be done in $\binom{104}{4}$ ways, and setting $a_i = b_i - 1$ shows it is equivalent to the problem.

Problem 2 (20 points) Thirty people are invited to a party. Each person accepts the invitation, independently of all others, with probability $1/3$. Let X be the number of accepted invitations. Compute the following:

(a) (5 points) $\mathbb{E}[X]$

Answer: $\boxed{10}$. By linearity of expectation this is $30 \cdot 1/3 = 10$.

(b) (5 points) $\mathrm{Var}(X)$

Answer: $\boxed{\frac{20}{3}}$ Let X_i be the indicator random variable that person i accepts so that $X = \sum_{i=1}^{30} X_i$. Then $\mathbb{E}[X^2] = \sum_{i=1}^{30} \mathbb{E}[X_i^2] + 2\sum_{i<j} \mathbb{E}[X_i X_j] = \frac{30}{3} + \frac{30 \cdot 29}{9} = \frac{320}{3}$ so that $\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{20}{3}$.

(c) (5 points) $\mathbb{E}[X^2]$

Answer: $\boxed{\frac{320}{3}}$ This was computed above.

(d) (5 points) $\mathbb{E}[X^2 - 4X + 5]$

Answer: $\boxed{\frac{215}{3}}$ By linearity of expectation.

Problem 3 (20 points) Bob has noticed that during every given minute, there is a 1/720 chance that the Facebook page for his dry cleaning business will get a like, independently of what happens during any other minute. Let L be the total number of likes that Bob receives during a 24 hour period.

(a) (5 points) Compute $\mathbb{E}[L]$ and $\mathrm{Var}(X)$.

Answer: $\boxed{\mathbb{E}[L] = \frac{24 \cdot 60}{720} = 2 \text{ and } \mathrm{Var}(X) = 24 \cdot 60 \cdot \frac{1}{720} \cdot \frac{719}{720} = \frac{719}{360}.}$ We use linearity of expectation and the formula $np(1-p)$ for the variance of a binomial with parameters n, p.

(b) (5 points) Compute the probability that $L = 0$.

Answer: $\boxed{\left(\frac{719}{720}\right)^{24 \cdot 60}}$ The $24 \cdot 60$ events that Bob receives no likes during each minute are all independent and have probability $\frac{719}{720}$.

(c) (10 points) Use a Poisson approximation to approximate the probability that $L \geq 2$.

Answer: $\boxed{1 - 3e^{-2}}$ We approximate L by a Poisson with $\lambda = 2$.

Problem 4 (10 points) Consider an infinite sequence of independent tosses of a coin that comes up heads with probability p. Compute (in terms of p) the probability that the fifth head occurs on the tenth toss.

Answer: $\boxed{\binom{9}{4}p^5(1-p)^5}$ This is negative binomial with parameters $r = 10$ and p.

Problem 5 (20 points) A standard deck of 52 cards contains 4 aces. Suppose we choose a random ordering (all 52! permutations being equally likely). Compute the following:

(a) (5 points) The probability that all of the top 4 cards in the deck are aces.

Answer: $\boxed{\frac{24}{52 \cdot 51 \cdot 50 \cdot 49}}$ The number of outcomes with all 4 top cards aces is $4! \cdot 48!$ (4! ways to place the 4 aces and 48! ways to place the other cards), and the total number of outcomes is 52!.

(b) (5 points) The probability that none of the top 4 cards in the deck is an ace.

Answer: $\boxed{\frac{48 \cdot 47 \cdot 46 \cdot 45}{52 \cdot 51 \cdot 50 \cdot 49}}$ We may first place the aces in the bottom 48 positions in $48 \cdot 47 \cdot 46 \cdot 45$ ways and then the rest of the cards in 48! ways.

(c) (10 points) The expected number of aces among the top 4 cards in the deck.

Answer: $\boxed{\frac{4}{13}}$ Let X_i be the indicator that card i is an ace so that $X = \sum_{i=1}^{4} X_i$. Note that $\mathbb{E}[X_i] = \frac{1}{13}$, so by linearity we find that $\mathbb{E}[X] = \frac{4}{13}$.

Problem 6 (20 points) There are ten children: five attend school A, three attend school B, and two attend school C. Suppose that a pair of two children is chosen uniformly at random from the set of all possible pairs of children. Let X be the number of students in the random pair that attend school A and let Y be the number in the pair that attend school B.

(a) (10 points) Compute $\mathbb{E}[XY]$.

Answer: $\boxed{1/3}$ The product is non-zero and equal to 1 if and only if $X = Y = 1$, so the expectation is the probability that one child attends each of schools A and B. To compute this, draw 10 slots, with 5 for A, 3 for B, and 2 for C. The number of ways to assign a pair of children to slots with one in A and one in B is $2 \cdot 5 \cdot 3$, while the number of ways to assign a pair of children to slots with no constraint is $10 \cdot 9$. Dividing gives the answer.

(b) (10 points) Given that the two children in this pair attend the same school, what is the conditional probability that they both attend school A?

Answer: $\boxed{\frac{5}{7}}$ Let X be the event that they attend the same school and A, B, C the events that the school is A, B, C. Then we find $X = A \cup B \cup C$ and A, B, C are pairwise disjoint. Then we find that

$$\mathbb{P}(A) = \frac{5 \cdot 4}{10 \cdot 9} \qquad \mathbb{P}(B) = \frac{3 \cdot 2}{10 \cdot 9} \qquad \mathbb{P}(C) = \frac{2 \cdot 1}{10 \cdot 9}$$

so that

$$\mathbb{P}(A \mid X) = \frac{5 \cdot 4}{5 \cdot 4 + 3 \cdot 2 + 2 \cdot 1} = \frac{5}{7}.$$