Statistics 251, Autumn 2020 — Homework 7

Due date: 11:30am on Monday, November 16, 2020 on Gradescope.

Instructions: Please present your solutions in a legible, coherent manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Points may be deducted for unclear or messy solutions.

Collaboration and Academic Integrity: You are encouraged to collaborate on homework. However, you must write your solutions alone and **understand what you write**. When submitting your homework, list in the space below any sources you used (in print, online, or human) other than the textbook or the teaching staff.

1. [10pts] Recall the density function of a normal distribution with parameters (μ, σ^2) is given by
$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{(x-\mu)^2}{2\sigma^2}).$$
Prove that if X_1, X_2 are independent normal random variables with parameters (μ_1, σ_1^2), (μ_2, σ_2^2), then $X_1 + X_2$ is a normal random variable with parameters $(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.
2. [10pts] Recall the density function of a gamma distribution with parameters (α, λ) is given by
$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$
Prove that if X and Y are independent gamma random variables with parameters (s, λ) and (t, λ), then $X + Y$ is a gamma random variable with parameters $(s + t, \lambda)$.
3. [10pts] The weekly sales at a certain restaurant is a normal random variable with mean \$2200 and standard deviation \$230. The sales for each week are independent. What is the probability that
   a. the total sales over the next 2 weeks exceeds \$5000?
   b. weekly sales exceed \$2000 in at least 2 of the next 3 weeks?
4. [10pts] The joint density of X and Y is
$$f(x, y) = c\left(x^2 - y^2\right)e^{-x} \qquad 0 \leq x < \infty, -x \leq y \leq x.$$
Find the conditional distribution of Y given $X = x$.
5. [10pts] Let X and Y denote the coordinates of a point uniformly chosen in the circle of radius 1 centered at the origin. That is, their joint density is
$$f(x, y) = \frac{1}{\pi} \qquad x^2 + y^2 \leq 1$$
Find the joint density function of the polar coordinates $R = \sqrt{X^2 + Y^2}$ and $\Theta = \tan^{-1}(Y/X)$.
6. [10pts] Let X and Y have joint density function
$$f(x, y) = \frac{1}{x^2 y^2} \qquad x \geq 1, y \geq 1.$$
Compute the joint density function of $U = XY$ and $V = X/Y$.
7. [10pts] A set of 1000 cards numbered 1 through 1000 is randomly distributed among 1000 people with each receiving one card. Compute the expected sum of the numbers on cards that are given to people whose number matches the number on the card.
8. [10pts] Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed continuous random variables. Let $N \geq 2$ be such that
$$X_1 \geq X_2 \geq \cdots \geq X_{N-1} < X_N$$
That is, N is the point at which the sequence stops decreasing. Calculate $\mathbb{E}[N]$. (Hint: recall the summation formula in Lecture 19.)