Statistics 251, Autumn 2020 — Homework 8

Due date: 11:30am on Monday, November 30, 2020 on Gradescope.

Instructions: Please present your solutions in a legible, coherent manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Points may be deducted for unclear or messy solutions.

Collaboration and Academic Integrity: You are encouraged to collaborate on homework. However, you must write your solutions alone and **understand what you write**. When submitting your homework, list in the space below any sources you used (in print, online, or human) other than the textbook or the teaching staff.

1. [10pts] Prove the conditional variance formula
$$\mathrm{Var}(X) = \mathbb{E}[\mathrm{Var}(X \mid Y)] + \mathrm{Var}(\mathbb{E}[X \mid Y]).$$
2. [10pts] If X_1, X_2, X_3, and X_4 are pairwise uncorrelated random variables, each having mean 0 and variance 1, compute the correlations of
   a. $X_1 + X_2$ and $X_2 + X_3$
   b. $X_1 + X_2$ and $X_3 + X_4$
3. [10pts] The joint density function of X and Y is given by
$$f(x, y) = \frac{1}{y} e^{-(y+x/y)} \qquad x > 0, y > 0.$$
   Compute the values of
   a. $\mathbb{E}[X], \mathbb{E}[Y]$,
   b. $\mathrm{Cov}(X, Y)$
   c. $\mathbb{E}\left[X^2 \mid Y = y\right]$
4. [10pts] The joint density of X and Y is given by
$$f(x, y) = \frac{e^{-y}}{y} \qquad 0 < x < y, \quad 0 < y < \infty.$$
   Compute $\mathbb{E}\left[X^4 \mid Y = y\right]$.
5. [10pts] There are two misshapen coins in a box; the probabilities they land heads when flipped are 0.4 and 0.7. One of the coins is to be randomly chosen and flipped 10 times. Given that exactly two of the first three flips landed heads, what is the conditional expected number of heads in the 10 flips?
6. [10pts] Show that $\mathrm{Cov}(X, \mathbb{E}[Y \mid X]) = \mathrm{Cov}(X, Y)$.
7. [10pts] Suppose that X is a random variable with mean and variance both equal to 20. What can be said about $\mathbb{P}(0 < X < 40)$?
8. [10pts] Ten numbers are rounded to the nearest integer and then summed. If the individual rounding errors are uniformly distributed over $(-0.5, 0.5)$ approximate the probability that the resulting sum differs from the exact sum by more than 3.