Mathematics UN1102
Section 3, Fall 2017
Practice Final
December 18, 2017
Time Limit: 170 Minutes

Name: ______________________________

UNI: ______________________________

Instructions: This exam contains 10 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the blank pages stapled to the end of the exam. **Do not write on the back side of your test papers.**

The use of outside material including books, notes, calculators, and electronic devices is not allowed.

Question	1	2	3	4	5	6	7	8	9	10	Total
Points	8	12	10	10	8	12	10	10	10	10	100
Score	8	12	10	10	8	12	10	10	10	10	100

Formulas

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta)) \qquad \cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \tan x\, dx = \ln|\sec x| + C \qquad \int \sec x\, dx = \ln|\sec x + \tan x| + C$$

Taylor series of $f(x)$ at $x = a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Maclaurin series:

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ $\qquad R = 1$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $\qquad R = \infty$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\qquad R = \infty$
- $\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $\qquad R = 1$
- $(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$ $\qquad R = 1$

Problem 1 (8 points) Determine whether each of the following statements are true or false. If a statement is true, explain why; if a statement is false, give an example that shows why the statement is false.

(a) (2 points) Suppose $f(x)$ is a continuous function on $[1, \infty)$. If the improper integral $\int_5^\infty f(x)dx$ is convergent, then the improper integral $\int_2^\infty f(x)dx$ must also be convergent.

Answer: True. If $f(x)$ is continuous on $[1, \infty)$, then $\int_2^5 f(x)dx$ is convergent, hence

$$\lim_{t\to\infty} \int_2^t f(x)dx = \int_2^5 f(x)dx + \lim_{t\to\infty} \int_5^t f(x)dx$$

exists and $\int_2^\infty f(x)dx$ is also convergent.

(b) (2 points) If the infinite series $\sum_{n=1}^\infty a_n$ is convergent, then the infinite series

$$\sum_{n=1}^\infty (a_{n+1} - a_n)$$

must also be convergent.

Answer: True. The partial sums of the series are given by $s_k = \sum_{n=1}^k (a_{n+1} - a_n) = a_{k+1} - a_1$. If $\sum_{n=1}^\infty a_n$ is convergent, we see that $\lim_{k\to\infty} a_k = 0$, hence $\lim_{k\to\infty} s_k = -a_1$, so $\sum_{n=1}^\infty (a_{n+1} - a_n)$ converges.

(c) (2 points) If $z(x)$ is a solution to the differential equation $z' = 0.2 \cdot z(1 - z/10)$, then the limit $\lim_{x\to\infty} z(x)$ must be equal to 10.

Answer: False. This is not true for the solution $z(x) = 0$.

(d) (2 points) If $y_1(x), y_2(x)$ are solutions to the differential equation $y' + y = \sin(x)$, then $y_1(x) + y_2(x)$ must also be a solution.

Answer: False. In general $f(x) = y_1(x) + y_2(x)$ solves $f'(x) + f(x) = 2\sin(x)$.

Problem 2 (12 points) This problem is on two pages. Evaluate the following integrals.

(a) (4 points)

$$\int e^{\sqrt{x}}dx.$$

Answer: $\boxed{2(\sqrt{x}-1)e^{\sqrt{x}}+C}$ Set $u=\sqrt{x}$ so that $du=\frac{1}{2\sqrt{x}}dx$, hence $dx=2udu$. We conclude that

$$\int e^{\sqrt{x}}dx=\int 2ue^u du=2ue^u-\int 2e^u du=(2u-2)e^u+C=2(\sqrt{x}-1)e^{\sqrt{x}}+C.$$

(b) (4 points)

$$\int \frac{3x^2+5x+8}{(x+1)^2(x-1)}dx.$$

Answer: $\boxed{4\ln|x-1|-\ln|x+1|+\frac{3}{x+1}+C.}$

Write the partial fraction decomposition

$$\frac{3x^2+5x+8}{(x+1)^2(x-1)}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{C}{(x+1)^2}.$$

We solve to find that

$$3x^2+5x+8=A(x+1)^2+B(x+1)(x-1)+C(x-1).$$

Plugging in $x=1$, we find that $A=4$ and therefore that

$$-x^2-3x+4=-(x+4)(x-1)=B(x+1)(x-1)+C(x-1).$$

This implies that $-x-4=B(x+1)+C$, hence $B=-1$ and $C=-3$. We conclude that

$$\int \frac{3x^2+5x+8}{(x+1)^2(x-1)}dx=\int \left(\frac{4}{x-1}-\frac{1}{x+1}-\frac{3}{(x+1)^2}\right)dx=4\ln|x-1|-\ln|x+1|+\frac{3}{x+1}+C.$$

(c) (4 points) Using the trig substitution $x = 3\cos\theta$ or otherwise, find

$$\int \frac{dx}{x\sqrt{9-x^2}}.$$

Answer: $\boxed{-\frac{1}{3}\ln\left|\frac{3+\sqrt{9-x^2}}{x}\right| + C.}$

Making the substitution, we find that $dx = -3\sin\theta\, d\theta$, hence

$$\int \frac{dx}{x\sqrt{9-x^2}} = \int \frac{-3\sin\theta}{3\cos\theta \cdot 3\sin\theta} d\theta = \int -\frac{1}{3}\sec\theta d\theta = -\frac{1}{3}\ln|\sec\theta + \tan\theta| + C.$$

Since $\theta = \arccos(x/3)$, we see that $\sec\theta = \frac{3}{x}$ and $\tan\theta = \frac{\sqrt{9-x^2}}{x}$, so the answer is $-\frac{1}{3}\ln\left|\frac{3+\sqrt{9-x^2}}{x}\right| + C$.

Problem 3 (10 points) Consider the region A bounded by the curves $y = \sin(x)$ and $y = -\sin(x)$ between $x = 0$ and $x = \pi$.

(a) (2 points) Sketch the region A.

Answer: See Wolfram Alpha for a plot.

(b) (3 points) Find the area of the region A.

Answer: $\boxed{4.}$ The area is

$$A = \int_0^\pi \Big(\sin(x) - (-\sin(x))\Big)dx = \int_0^\pi 2\sin(x)dx = [-2\cos(x)]_0^\pi = 4.$$

(c) (5 points) Find the volume of the solid of revolution obtained by rotating A about the y-axis.

Answer: $\boxed{4\pi^2.}$ We apply the cylinder method to get

$$V = 2\pi \int_0^\pi x\Big(\sin(x) - (-\sin(x))\Big)dx = 4\pi \int_0^\pi x\sin(x)dx$$

$$= 4\pi[-x\cos(x)]_0^\pi - 4\pi \int_0^\pi (-\cos(x))dx = 4\pi^2 - 4\pi[-\sin(x)]_0^\pi = 4\pi^2.$$

Problem 4 (10 points) This problem is on two pages. Determine whether each of the following series is convergent or divergent. Justify your answer.

(a) (2 points)

$$\sum_{n=0}^{\infty} \frac{5^n}{4^n}.$$

Answer: $\boxed{\text{Divergent.}}$ The sequence $a_n = \frac{5^n}{4^n}$ has $\lim_{n\to\infty} a_n = \infty$ and diverges, so the series diverges.

(b) (2 points)

$$\sum_{n=0}^{\infty} \frac{n^3 + 3n + \sin(n)}{n^4 - 2}.$$

Answer: $\boxed{\text{Divergent.}}$ For $n \geq 1$, we have

$$\frac{n^3 + 3n + \sin(n)}{n^4 - 2} > \frac{n^3}{n^4} = \frac{1}{n},$$

so the series diverges by comparison with $\sum_{n\geq 1} \frac{1}{n}$.

(c) (3 points)

$$\sum_{n=1}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \dots \cdot (3n-1)}.$$

Answer: Convergent. Notice that

$$\frac{n!}{2 \cdot 5 \cdot \dots \cdot (3n-1)} = \frac{1}{2} \cdot \frac{2}{5} \cdot \dots \cdot \frac{n}{3n-1} < \left(\frac{1}{2}\right)^n,$$

so the series converges by comparison with $\sum_{n=1}^{\infty} \frac{1}{2^n}$.

(d) (3 points)

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}.$$

Answer: Convergent. Notice that $e^{1/n} \leq e$, so

$$\frac{e^{1/n}}{n^2} \leq \frac{e}{n^2},$$

so the series converges by comparison with $\sum_{n=1}^{\infty} \frac{e}{n^2}$.

Problem 5 (8 points) Consider the power series

$$\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}(x-5)^n$$

(a) (4 points) Determine its radius of convergence.

Answer: $\boxed{1/2.}$ By the ratio test, we see that

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2\sqrt{n}}{\sqrt{n+1}}|x-5| = 2|x-5|,$$

so that $L < 1$ if and only if $x \in (5-1/2, 5+1/2)$ and the radius of convergence is $1/2$.

(b) (4 points) Determine its interval of convergence.

Answer: $\boxed{[5-1/2, 5+1/2).}$ We need to check $x = 5+1/2$ and $x = 5-1/2$. At $x = 5+1/2$, the series is

$$\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}} \frac{1}{2^n} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}},$$

which diverges by the p-series test. At $x = 5-1/2$, the series is

$$\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}(-1)^n \frac{1}{2^n} = \sum_{n=1}^{\infty}(-1)^n \frac{1}{\sqrt{n}},$$

which converges by the alternating series test. We conclude the interval of convergence is $[5-1/2, 5+1/2)$.

Problem 6 (12 points)

(a) (4 points) Find the Maclaurin series of the function

$$f(x) = \frac{1}{1+9x^2}.$$

Answer: $\boxed{\sum_{n=0}^{\infty}(-1)^n 9^n x^{2n}.}$ We substitute $y = -9x^2$ into the series $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$.

(b) (2 points) What is the radius of convergence of the Maclaurin series of $f(x)$?

Answer: $\boxed{\frac{1}{3}.}$ The series $\frac{1}{1-y} = \sum_{n=0}^{\infty} y^n$ has radius of convergence 1, so this converges if $|9x^2| < 1$, which is equivalent to $|x| < 1/3$.

(c) (4 points) Find the Maclaurin series of the function $g(x) = \tan^{-1}(3x)$.

Answer: $\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3 \cdot 9^n}{2n+1} x^{2n+1}.}$ Notice that $3\tan^{-1}(3x) = \int \frac{9}{1+9x^2}dx$, hence

$$\tan^{-1}(3x) = 3\int \frac{1}{1+9x^2}dx = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3 \cdot 9^n}{2n+1} x^{2n+1},$$

where we apply term-by-term integration.

(d) (2 points) What is the radius of convergence of the Maclaurin series of $g(x)$? Hint: Use your answer to (b).

Answer: $\boxed{1/3.}$ The radius of convergence of a term-by-term integrated series is the same as that of the original series, so the answer is the same as that of (b).

Problem 7 (10 points) Consider the polar curve $r = 2 + \cos(3\theta)$.

(a) (3 points) Find two values r_1, r_2 such that the points with polar coordinates $(r_1, \pi/9)$, $(r_2, \pi/9)$ lie on the polar curve.

Answer: $\boxed{5/2 \text{ and } -3/2.}$ This can occur if $\theta = \pi/9$ and $\theta = 10\pi/9$, which correspond to $(5/2, \pi/9)$ and $(3/2, 10\pi/9) = (-3/2, \pi/9)$

(b) (7 points) Set up and evaluate an integral to find the area enclosed by the polar curve $r = 2 + \cos(3\theta)$.

Answer: $\boxed{\frac{9\pi}{2}.}$ The area is given by

$$A = \int_0^{2\pi} \frac{1}{2}(2 + \cos(3\theta))^2 d\theta = \int_0^{2\pi} \Big(2 + 2\cos(3\theta) + \frac{1}{2}\cos^2(3\theta)\Big) d\theta.$$

Recall the identity $\cos(6\theta) = 2\cos^2(3\theta) - 1$, so $\cos^2(3\theta) = \frac{1+\cos(6\theta)}{2}$. We see that

$$A = 4\pi + [\frac{2}{3}\sin(3\theta)]_0^{2\pi} + \frac{\pi}{2} + [\frac{1}{24}\sin(6\theta)]_0^{2\pi} = \frac{9\pi}{2}.$$

Problem 8 (10 points) Consider the differential equation

$$y' = x - y + 2.$$

(a) (4 points) There is exactly one choice of numbers m and b for which $y(x) = mx + b$ satisfies the differential equation. Find these values of m and b.

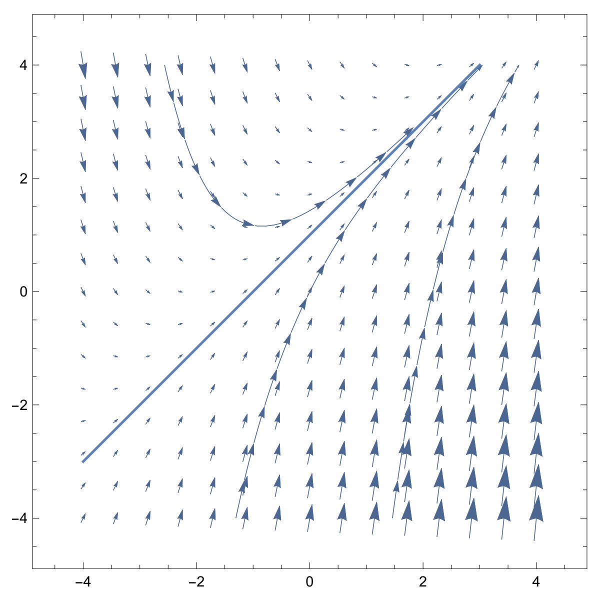
Answer: $\boxed{(m, b) = (1, 1).}$ To solve for m and b, we find

$$m = x - (mx + b) + 2 = (1 - m)x + (2 - b),$$

meaning that $m = 1$ and $b = 1$.

(b) (6 points) Draw the direction field for the differential equation in the region $-4 \leq x \leq 4$, $-4 \leq y \leq 4$. On the direction field, draw the solution curve you found in (a), along with two other solution curves.

Answer:



Problem 9 (10 points) Find the solution to the differential equation

$$\sec^2(x)y' - y^3 = 0$$

satisfying the initial condition $y(\pi/2) = 1$.

Answer: $\boxed{y(x) = \sqrt{\frac{1}{1+\pi/2-x-\frac{1}{2}\sin(2x)}}.}$ This equation is separable, so we may write

$$\int \frac{1}{y^3} dy = \int \cos^2(x) dx.$$

The LHS is given by $-\frac{1}{2}y^{-2}$, and the RHS is given by

$$\int \cos^2(x) dx = \int \frac{1+\cos(2x)}{2} dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C,$$

from which we conclude that

$$y = \sqrt{\frac{-1}{x + \frac{1}{2}\sin(2x) + C}}.$$

Evaluating at $\pi/2$, we see that

$$1 = y(\pi/2) = \sqrt{\frac{-1}{\pi/2 + C}}$$

so that $C = -\pi/2 - 1$. Putting everything together, we see that

$$y(x) = \sqrt{\frac{1}{1 + \pi/2 - x - \frac{1}{2}\sin(2x)}}.$$

Problem 10 (10 points) Find the general solution to the linear differential equation

$$(x^2 + x)y' + 2(2x + 1)y = 2x.$$

Answer: $\boxed{\frac{\frac{1}{2}x^2+\frac{2}{3}x+C}{(x+1)^2}}.$

We may rewrite this as

$$y' + \frac{2(2x+1)}{x(x+1)}y = \frac{2}{x+1}.$$

Notice that

$$\int \frac{2(2x+1)}{x(x+1)}dx = \int \Big(\frac{2}{x} + \frac{2}{x+1}\Big)dx = 2\ln|x| + 2\ln|x+1|,$$

so we have the integrating factor

$$I(x) = e^{\int \frac{2(2x+1)}{x(x+1)}dx} = x^2(x+1)^2.$$

Multiplying both sides by the integrating factor gives the equation

$$(x^2(x+1)^2y)' = x^2(x+1)^2y' + 2(2x+1)x(x+1)y = 2x^2(x+1).$$

Integrating both sides yields

$$x^2(x+1)^2y = \int 2x^2(x+1)dx = \int (2x^3 + 2x^2)dx = \frac{1}{2}x^4 + \frac{2}{3}x^3 + C,$$

from which we conclude that

$$y = \frac{\frac{1}{2}x^2 + \frac{2}{3}x + C}{(x+1)^2}.$$