Mathematics UN1102
Section 3, Fall 2017
Practice Final 2
December 18, 2017
Time Limit: 170 Minutes

Name: ______________________________

UNI: ______________________________

Instructions: This exam contains 10 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the blank pages stapled to the end of the exam. **Do not write on the back side of your test papers.**

The use of outside material including books, notes, calculators, and electronic devices is not allowed.

Question	1	2	3	4	5	6	7	8	9	10	Total
Points	15	8	15	8	10	10	10	8	8	8	100
Score	15	8	15	8	10	10	10	8	8	8	100

Formulas

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta)) \qquad \cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \tan x\, dx = \ln|\sec x| + C \qquad \int \sec x\, dx = \ln|\sec x + \tan x| + C$$

Taylor series of $f(x)$ at $x = a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Maclaurin series:

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ $\quad R = 1$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $\quad R = \infty$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\quad R = \infty$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $\quad R = 1$
- $(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$ $\quad R = 1$

Problem 1 This problem is spread over two pages.

(a) (5 points) Evaluate

$$\int (3x+8)^{241} dx.$$

Answer: $\boxed{\frac{1}{726}(3x+8)^{242}+C.}$ Set $u = 3x+8$ so that $dx = \frac{1}{3}du$. The integral is then

$$\int (3x+8)^{241} dx = \frac{1}{3}\int u^{241} du = \frac{1}{726}u^{242} + C = \frac{1}{726}(3x+8)^{242} + C.$$

(b) (5 points) Evaluate

$$\int \sin^3(\theta)\cos^{14}(\theta) d\theta.$$

Answer: $\boxed{-\frac{1}{15}\cos^{15}(\theta)+\frac{1}{17}\cos^{17}(\theta)+C.}$ Set $x = \cos(\theta)$ so that $dx = -\sin(\theta)d\theta$, meaning that

$$\int \sin^3(\theta)\cos^{14}(\theta) d\theta = \int -(1-x^2)x^{14} dx = -\frac{1}{15}x^{15} + \frac{1}{17}x^{17} + C = -\frac{1}{15}\cos^{15}(\theta) + \frac{1}{17}\cos^{17}(\theta) + C.$$

(c) (5 points) Evaluate

$$\int \frac{6x^2 - x + 18}{x(x^2+9)} dx.$$

Answer: $\boxed{2\ln|x| + 2\ln|x^2+9| - \frac{1}{3}\arctan(x/3) + C.}$ We express this in partial fractions by solving

$$\frac{6x^2 - x + 18}{x(x^2+9)} = \frac{A}{x} + \frac{Bx + C}{x^2+9}.$$

This yields

$$6x^2 - x + 18 = A(x^2+9) + x(Bx + C) = (A+B)x^2 + Cx + 9A.$$

Equating coefficients yields $A = 2$, $C = -1$, and $B = 4$. We thus obtain

$$\int \frac{6x^2 - x + 19}{x(x^2+9)} dx = \int \left(\frac{2}{x} + \frac{4x-1}{x^2+9}\right) dx = 2\ln|x| + 2\ln|x^2+9| - \frac{1}{3}\arctan(x/3) + C.$$

Problem 2 (8 points) Consider the region A bounded by the curves $y = x^4$ and $y = x$. Note that A lies entirely in the region $x \geq 0$.

(a) (2 points) Sketch the region A and find the coordinates of the two points where the curves intersect.

> **Answer:** $\boxed{(0,0) \text{ and } (1,1).}$ See Wolfram Alpha for a plot. The curves intersect when $x^4 = x$, which implies that $x = 0$ or $x = 1$.

(b) (6 points) Set up and evaluate an integral to find the volume of the solid of revolution obtained by rotating the region A about the x-axis.

> **Answer:** $\boxed{\frac{2}{9}\pi.}$ We apply the washer method and get
>
> $$V = \int_0^1 \pi(x^2 - x^8)dx = \pi[\frac{1}{3}x^3 - \frac{1}{9}x^9]_0^1 = \frac{2}{9}\pi.$$

Problem 3 (15 points) Determine whether each of the following series is convergent or divergent. You do not need to include your work.

(a) (3 points) $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n}$.

Answer: $\boxed{\text{Convergent.}}$ Apply the alternating series test.

(b) (3 points) $\sum_{n=1}^{\infty} \left(\frac{2n^2+n}{4n^2-3}\right)^n$.

Answer: $\boxed{\text{Convergent.}}$ By comparison to $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$.

(c) (3 points) $\sum_{n=1}^{\infty} \frac{2n^2+n}{4n^2-3}$.

Answer: $\boxed{\text{Divergent.}}$ Since the limit of the summand is $\frac{1}{2}$.

(d) (3 points) $\sum_{n=3}^{\infty} \frac{\ln(n)}{n}$.

Answer: $\boxed{\text{Divergent.}}$ By comparison to $\sum_{n=3}^{\infty} \frac{1}{n}$.

(e) (3 points) $\sum_{n=1}^{\infty} \ln(n) - \ln(n+1)$.

Answer: $\boxed{\text{Divergent.}}$ Since the k^{th} partial sum is $-\ln(k+1)$.

Problem 4 (8 points) Consider the power series

$$\sum_{n=1}^{\infty} \frac{1}{5^n \cdot n^4} x^n.$$

(a) (4 points) Determine the radius of convergence of this power series.

Answer: $\boxed{5.}$ By the ratio test we see that

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{|x|}{5} \frac{n^4}{(n+1)^4} = \frac{|x|}{5},$$

so that $L < 1$ if and only if $|x| < 5$, meaning the radius of convergence is 5.

(b) (4 points) Determine the interval of convergence of this power series.

Answer: $\boxed{[-5, 5].}$ We need to check $x = 5$ and $x = -5$. At $x = 5$, the series converges by the p-series test, and at $x = -5$ it converges by the alternating series test. So the interval of convergence is $[-5, 5]$.

Problem 5 (10 points)

(a) (4 points) Using either the Maclaurin series for $\sin(x)$ on the cover page or the definition of Maclaurin series, determine the Maclaurin series for $f(x) = \cos(x)$.

Answer: $\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.}$ Taking the term by term derivative, we see that

$$\cos(x) = \frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

(b) (3 points) Using your answer to (a), find the Maclaurin series for $g(x) = \cos(2x^2)$.

Answer: $\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{4n}.}$ We substitute $2x^2$ in the argument to get

$$\sum_{n=0}^{\infty} \frac{(-1)^n 4^n}{(2n)!} x^{4n}.$$

(c) (3 points) Using your answer to (b), find the value of $g^{(12)}(0)$ (the twelfth derivative of g at 0).

Answer: $\boxed{-\frac{4^3 \cdot 12!}{6!}.}$ The coefficient of x^{12} is given by $\frac{1}{12!} g^{(12)}(0)$ and is equal to $-\frac{4^3}{6!}$, hence $g^{(12)}(0)$ is given by $-\frac{4^3 \cdot 12!}{6!}$.

Problem 6 (10 points)

(a) (3 points) Sketch the graph of the polar curve $r = 2\cos(5\theta)$.

> **Answer:** See Wolfram Alpha for plot.

(b) (7 points) Set up and evaluate an integral to find the area enclosed by this curve.

> **Answer:** $\boxed{\pi.}$ The area is
>
> $$\int_0^{\pi} \frac{1}{2} r^2 d\theta = \int_0^{\pi} 2\cos^2(5\theta) d\theta = \int_0^{\pi} 1 + \cos(10\theta) d\theta = \pi.$$

Problem 7 (10 points)

(a) (3 points) Use the Maclaurin series of $\sin(x)$ to find the Maclaurin series of $f(x) = x\sin(3x)$.

Answer: $\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} x^{2n+2}.}$ We plug in the formula to get

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}}{(2n+1)!} x^{2n+2}.$$

(b) (3 points) Use the binomial theorem to find the Maclaurin series of $g(x) = 1/\sqrt{1+x^2}$.

Answer: $\boxed{\sum_{n=0}^{\infty} \frac{-1/2(-1/2-1)\cdots(-1/2-n+1)}{n!} x^{2n}.}$ This is given by $(1+x^2)^{-1/2}$, so we see that the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{-1/2(-1/2-1)\cdots(-1/2-n+1)}{n!} x^{2n}.$$

(c) (4 points) Use your answers from (a) and (b) to evaluate

$$\lim_{x\to 0} \frac{3x^2 - x\sin(3x)}{-1 + \frac{x^2}{2} + \frac{1}{\sqrt{1+x^2}}}.$$

Answer: $\boxed{12.}$ We substitute the series to get

$$\lim_{x\to 0} \frac{3x^2 - 3x^2 + \frac{3^3}{3!}x^4 + \cdots}{-1 + x^2/2 + 1 - x^2/2 + \frac{3}{8}x^4 + \cdots} = \frac{3^3}{6} \cdot \frac{8}{3} = 12.$$

Problem 8 (8 points) Use Euler's method with stepsize 0.1 to approximate $y(0.2)$, where y is the solution to

$$y' = x + y$$

with initial condition $y(0) = -1$.

Answer: $\boxed{-1.2.}$ We start at $x_0 = 0$ and $y_0 = -1$ so $y_0' = -1$. We get that

$$y_1 = y_0 + 0.1 \cdot y_0' = -1.1$$

so that $y_1' = -1$. Repeating, we see that

$$y_2 = y_1 + 0.1 \cdot y_1' = -1.2.$$

Problem 9 (8 points) Find the solution to the differential equation

$$y' = e^y + x^2 e^y$$

satisfying the initial condition $y(0) = 0$.

Answer: $\boxed{y = -\ln(-x - \frac{1}{3}x^3 - 1).}$ This is separable, so we get

$$\int e^{-y} dy = \int (1 + x^2) dx,$$

hence

$$-e^{-y} = x + \frac{1}{3}x^3 + C,$$

so that

$$y = -\ln(-x - \frac{1}{3}x^3 - C).$$

We obtain that

$$0 = y(0) = -\ln(-C),$$

so that $C = -1$, yielding

$$y = -\ln(-x - \frac{1}{3}x^3 - 1).$$

Problem 10 (8 points) Find the general solution to the following differential equation

$$xy' + 3y = \frac{\sin(x)}{x}.$$

Answer: $\boxed{y = -\frac{\cos(x)}{x^2} - \frac{\sin(x)}{x^3} + \frac{C}{x^3}.}$ This simplifies to

$$y' + \frac{3}{x}y = \frac{\sin(x)}{x^2}.$$

We use the integrating factor $I(x) = e^{\int 3/x dx} = x^3$, so we get that

$$x^3 y = \int \sin(x) x dx = -x\cos(x) - \int \cos(x) dx = -x\cos(x) - \sin(x) + C.$$

We conclude the general solution is

$$y = -\frac{\cos(x)}{x^2} - \frac{\sin(x)}{x^3} + \frac{C}{x^3}.$$