Mathematics UN1102
Section 3, Fall 2017
Practice Midterm 1
October 9, 2017
Time Limit: 75 Minutes

Name: Solution Key

UNI: sk1337

Instructions: This exam contains 7 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the back side of the problem sheets or the blank pages stapled to the end of the exam.

The use of outside material including books, notes, calculators, and electronic devices is not allowed.

Question	1	2	3	4	5	6	7	Total
Points	10	15	15	20	15	20	5	100
Score	10	15	15	20	15	20	5	100

Formulas

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta)) \qquad \cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$$

$$\int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2 - x^2}}dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

Problem 1 (10 points) Evaluate the definite integral

$$\int_0^1 xe^x dx.$$

Answer: $\boxed{1}$

By integration by parts with $u = x$ and $v = e^x$, we find

$$\int_0^1 xe^x dx = [xe^x]_0^1 - \int_0^1 e^x dx = e - [e^x]_0^1 = 1.$$

Problem 2 (15 points) Using an appropriate trigonometric substitution, evaluate the indefinite integral

$$\int \frac{\sqrt{x^2-9}}{x^4} dx.$$

Answer: $\boxed{\frac{(x^2-9)^{3/2}}{27x^3} + C}$

We substitute $x = 3\sec\theta$ so that $dx = 3\sec\theta\tan\theta\, d\theta$, obtaining

$$\int \frac{\sqrt{x^2-9}}{x^4} dx = \int \frac{\sqrt{9\sec^2\theta - 9}}{81\sec^4\theta} 3\sec\theta\tan\theta\, d\theta = \int \frac{\sqrt{\sec^2\theta - 1}}{9\sec^3\theta}\tan\theta\, d\theta$$

$$= \int \frac{\tan^2\theta}{9\sec^3\theta} d\theta = \frac{1}{9}\int \sin^2\theta\,\cos\theta d\theta = \frac{1}{9}\int u^2 du = \frac{1}{27}u^3 + C = \frac{1}{27}\sin^3\theta + C = \frac{(x^2-9)^{3/2}}{27x^3} + C,$$

where we substitute $u = \sin\theta$ and use that

$$\sin\theta = \sin(\operatorname{arc\,sec}(x/3)) = \frac{\sqrt{x^2-9}}{x}.$$

Problem 3 (15 points) Evaluate the indefinite integral

$$\int \theta \cos^2 \theta \, d\theta.$$

Answer: $\boxed{\frac{1}{4}\theta^2 + \frac{1}{4}\theta \sin(2\theta) + \frac{1}{8}\cos(2\theta) + C.}$

Notice that

$$\int \cos^2 \theta \, d\theta = \int \frac{1}{2}(1 + \cos(2\theta))d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C.$$

Therefore, we may apply integration by parts with $u = \theta$ and $v = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)$ to obtain

$$\begin{aligned}\int \theta \cos^2 \theta \, d\theta &= \frac{1}{2}\theta^2 + \frac{1}{4}\theta \sin(2\theta) - \int \Big(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\Big) d\theta \\ &= \frac{1}{2}\theta^2 + \frac{1}{4}\theta \sin(2\theta) - \frac{1}{4}\theta^2 + \frac{1}{8}\cos(2\theta) + C = \frac{1}{4}\theta^2 + \frac{1}{4}\theta \sin(2\theta) + \frac{1}{8}\cos(2\theta) + C.\end{aligned}$$

Problem 4 (20 points) Evaluate the indefinite integral

$$\int \frac{x^2+4x-3}{x^3+x^2+x+1}dx$$

in the following two steps.

(a) (10 points) Write a partial fraction decomposition for

$$\frac{x^2+4x-3}{x^3+x^2+x+1}.$$

Answer: $\boxed{\frac{x^2+4x-3}{x^3+x^2+x+1} = \frac{-3}{x+1} + \frac{4x}{x^2+1}}$

We make the factorization

$$x^3+x^2+x+1 = (x+1)(x^2+1)$$

and search for A, B, C so that

$$\frac{x^2+4x-3}{x^3+x^2+x+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \iff x^2+4x-3 = A(x^2+1) + (Bx+C)(x+1).$$

Plugging $x = -1$ implies that $A = -3$. Plugging in $x = 0$ then implies that $C = 0$. Plugging in $x = 1$ then implies that $B = 4$. Therefore, our final decomposition is

$$\frac{x^2+4x-3}{x^3+x^2+x+1} = \frac{-3}{x+1} + \frac{4x}{x^2+1}.$$

(b) (10 points) Evaluate the resulting integral.

Answer: $\boxed{-3\ln|x+1| + 2\ln|x^2+1| + C}$

Integrating we get

$$\int \frac{x^2+4x-3}{x^3+x^2+x+1}dx = \int \left(\frac{-3}{x+1} + \frac{4x}{x^2+1}\right)dx = -3\ln|x+1| + 2\ln|x^2+1| + C.$$

Problem 5 (15 points) Does the improper integral

$$\int_1^\infty \frac{1}{x^2+1}\,dx$$

converge or diverge? If it converges, compute its value.

Answer: $\boxed{\text{Converges}}$ to $\boxed{\frac{\pi}{4}}$

Because we have $\frac{1}{x^2+1} < \frac{1}{x^2}$ for $x \geq 1$ and $\int_1^\infty \frac{1}{x^2}dx$ converges, this integral converges. Its value is

$$\int_1^\infty \frac{1}{x^2+1}dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2+1}dx = \lim_{t\to\infty}[\arctan(x)]_1^t = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Problem 6 (20 points) Consider the region A between the line $y = x$ and the curve $y = x^4$.

(a) (5 points) Find the two points where the two graphs intersect, and sketch the region between the two graphs.

Answer: Intersect at $\boxed{(0,0) \text{ and } (1,1)}$.

(b) (15 points) Compute the volume of the solid of revolution obtained by rotating A about the y-axis. State whether you are using the method of disks/washers or the method of cylindrical shells.

Answer: $\boxed{\frac{1}{3}\pi}$

With washers, we integrate along the y-axis. The cross-sectional area is given by

$$A(y) = \pi\Big((y^{1/4})^2 - y^2\Big),$$

so the volume is

$$V = \int_0^1 A(y)dy = \int_0^1 \pi\Big((y^{1/4})^2 - y^2\Big)dy = \pi\int_0^1 (y^{1/2} - y^2)dy = \pi\Big(\frac{2}{3} - \frac{1}{3}\Big) = \frac{1}{3}\pi.$$

With cylindrical shelles, we integrate along the x-axis, where the circumference of the shell is $C(x) = 2\pi x$. We obtain

$$V = \int_0^1 C(x)(x - x^4)dx = 2\pi\int_0^1 (x^2 - x^5)dx = 2\pi\Big(\frac{1}{3} - \frac{1}{6}\Big) = \frac{1}{3}\pi.$$

Problem 7 (5 points) Set up a definite integral to compute the arc length of the curve

$$y = \sin(x), \qquad \frac{\pi}{2} \leq x \leq \pi.$$

You **do not** need to evaluate the integral.

Answer: We apply the formula to obtain

$$L = \int_{\pi/2}^{\pi} \sqrt{1 + \cos^2 x}dx.$$