Mathematics UN1102
Section 3, Fall 2017
Practice Midterm 2
November 15, 2017
Time Limit: 75 Minutes

Name: ____________________

UNI: ____________________

Instructions: This exam contains 6 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the blank pages stapled to the end of the exam. Please **do not write on the back side of pages**.

The use of outside material including books, notes, calculators, and electronic devices is not allowed.

Question	1	2	3	4	5	6	Total
Points	15	10	20	15	20	20	100
Score	15	10	20	15	20	20	100

Formulas

Maclaurin series:

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ $\quad R = 1$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $\quad R = \infty$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\quad R = \infty$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $\quad R = 1$
- $(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \cdots$ $\quad R = 1$

Problem 1 (15 points) In each of the following questions you will be asked to give examples of sequences or series that satisfy certain properties, or explain why no such examples exist.

(a) (5 points) Write down an example of a sequence $\{a_n\}$ such that

- $\{a_n\}$ is divergent;
- $a_n < 5$ for all n, and;
- $a_n < a_{n+1}$ for all n;

or explain why no such divergent sequence exists.

Answer: $\boxed{\text{No such sequence exists.}}$

The sequence $\{a_n\}$ is monotone increasing and bounded above, hence converges to a limit.

(b) (5 points) Write down two divergent series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ such that the series $\sum_{n=1}^{\infty} a_n b_n$ is convergent, or explain why no such pair of series exists.

Answer: $\boxed{a_n = (-1)^n, b_n = \frac{1}{n}}$.

Notice that $\sum_{n=1}^{\infty} a_n b_n$ converges by the alternating series test.

(c) (5 points) Write down an example of a convergent series $\sum_{n=1}^{\infty} a_n$ such that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = 1,$$

or explain why no such convergent series exists.

Answer: $\boxed{a_n = \frac{1}{n^2}}$.

Notice that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{n^2}{n^2(1+1/n)^2} = 1.$$

Problem 2 (10 points) Determine whether the following sequences are convergent or divergent. If the sequence is convergent, determine its limit as $n \to \infty$. Justify your answer.

(a) (5 points) The sequence $\{a_n\}$, where $a_n = \ln\left(\frac{n^2+3n}{7n^2+4}\right)$.

Answer: $\boxed{\text{Converges to } \ln(1/7)}$.

Notice that

$$\lim_{n\to\infty} a_n = \ln\left(\lim_{n\to\infty} \frac{n^2+3n}{7n^2+4}\right) = \ln\left(\lim_{n\to\infty} \frac{n^2(1+3/n)}{7n^2(1+4/(7n^2))}\right) = \ln(1/7).$$

(b) (5 points) The sequence $\{b_n\}$, where $b_n = \frac{4^n}{n!}$.

Answer: $\boxed{\text{Converges to } 0}$.

Notice that $b_n \geq 0$ and for $n \geq 6$, we have

$$b_n \leq \frac{4^4}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \left(\frac{4}{5} \cdots \frac{4}{n-1}\right) \cdot \frac{4}{n} \leq \frac{4^4}{24} \cdot \frac{4}{n},$$

where $\lim_{n\to\infty} \frac{4^4}{24} \cdot \frac{4}{n} = 0$, so $\lim_{n\to\infty} b_n = 0$ by the squeeze theorem.

Problem 3 (20 points) Determine whether the following series are convergent or divergent. If the series is convergent, determine its sum.

(a) (10 points) The series $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n+1}}-\frac{1}{\sqrt{n}}\right)$.

Answer: $\boxed{\text{Converges to } -1}$.

Note that the n^{th} partial sum is

$$s_n = \frac{1}{\sqrt{n+1}} - 1,$$

so $\lim_{n\to\infty} s_n = -1$.

(b) (10 points) The series $\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^3+10}}$.

Answer: $\boxed{\text{Diverges}}$.

Notice that for $n > 10$ we have

$$\frac{n}{\sqrt{n^3+10}} = \frac{1}{\sqrt{n+10/n^2}} > \frac{1}{2\sqrt{n}}.$$

Since $\sum_{n=1}^{\infty}\frac{1}{2\sqrt{n}}$ diverges, so does the given series by the comparison test.

Problem 4 (15 points) Consider the series $\sum_{n=1}^{\infty} ne^{-n^2}$.

(a) (5 points) Check that this series satisfies the hypotheses of the integral test.

Answer: $\boxed{f(x) = xe^{-x^2} \text{ is non-negative and non-increasing for large } x}$

Non-negativity is clear. To check $f(x)$ is non-increasing for large x, notice that $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = (1-2x^2)e^{-x^2} < 0$ for $x > \sqrt{2}/2$.

(b) (10 points) Use the integral test to determine whether this series is convergent or divergent. You **do not** need to determine the sum if convergent.

Answer: $\boxed{\text{Convergent}}$.

Notice that

$$\int_0^{\infty} xe^{-x^2}\,dx = \left[-\frac{1}{2}e^{-x^2}\right]_0^{\infty} = \frac{1}{2} < \infty,$$

so the series converges by the integral test.

Problem 5 (20 points) Write down a power series $\sum_{n=0}^{\infty} c_n x^n$ whose interval of convergence is $(-1, 1]$, and show why the power series you wrote down has interval of convergence $(-1, 1]$.

Answer: $\boxed{\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^n}$

Notice that

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{n+1} \cdot \frac{n}{|x|^n} = |x|,$$

meaning by the ratio test that the series is absolutely convergent on $(-1, 1)$ and divergent on $(-\infty, -1) \cup (1, \infty)$. For $x = -1$, the series coincides with the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. For $x = 1$, the series converges by the alternating series test. Thus the interval of convergence is $(-1, 1]$.

Problem 6 (20 points)

(a) (10 points) Find the Maclaurin series for the function

$$f(x) = \frac{1}{\sqrt{1-x^2}}.$$

Answer: $\boxed{\sum_{n=0}^{\infty} \frac{(-1/2)(-1/2-1)\cdots(-1/2-n+1)}{n!}(-1)^n x^{2n}}$.

We notice that $f(x) = h(-x^2)$ for $h(y) = (1+y)^{-1/2}$. Applying the formula for the Maclaurin series of $h(y)$, we obtain

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1/2)(-1/2-1)\cdots(-1/2-n+1)}{n!}(-1)^n x^{2n}.$$

(b) (10 points) Using your answer above, find the Maclaurin series for the function

$$g(x) = \sin^{-1}(x).$$

Hint: Recall that $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1}(x) + C$.

Answer: $\boxed{\sum_{n=0}^{\infty} \frac{(-1/2)(-1/2-1)\cdots(-1/2-n+1)}{(2n+1)n!}(-1)^n x^{2n+1}}$.

By the hint, the Maclaurin series for $g(x)$ is the term-by-term integral of the Maclaurin series for $f(x)$. We conclude that for some C, we have

$$g(x) = C + \sum_{n=0}^{\infty} \frac{(-1/2)(-1/2-1)\cdots(-1/2-n+1)}{(2n+1)n!}(-1)^n x^{2n+1}.$$

Since $g(0) = 0$, we find that $C = 0$, giving the answer.