Mathematics UN1102
Section 1, Fall 2019
Midterm 2
November 13, 2019
Time Limit: 75 Minutes

Name: ______________________________

UNI: ______________________________

Instructions: This exam contains 6 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the blank pages stapled to the end of the exam. **Do not write on the back side of your test papers.**

The use of outside material including books, notes, calculators, and electronic devices is not allowed.

Question	1	2	3	4	5	6	Total
Points	15	10	20	20	15	20	100
Score							

Formulas

Taylor series of $f(x)$ at $x = a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Maclaurin series:

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ $\qquad R = 1$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $\qquad R = \infty$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\qquad R = \infty$
- $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ $\qquad R = \infty$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $\qquad R = 1$
- $(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$ $\qquad R = 1$

Problem 1 (15 points) Determine whether each of the following statements are true or false. If a statement is true, explain why; if a statement is false, give an example that shows why the statement is false.

(a) (5 points) If $\{a_n\}$ satisfies $0 \le a_n < \frac{1}{n}$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges.

Answer: $\boxed{\text{False; for } a_n = \frac{1}{2n}\text{, the series diverges.}}$

(b) (5 points) If $\sum_{n=10}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.

Answer: $\boxed{\text{True}}$ We have that $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{9} a_n + \sum_{n=10}^{\infty} a_n$.

(c) (5 points) If the power series $\sum_{n=0}^{\infty} a_n(x-1)^n$ has radius of convergence 1, then it diverges at $x = 2$.

Answer: $\boxed{\text{False}}$ Consider $a_n = (-1)^n \frac{1}{n}$.

Problem 2 (10 points) Determine whether the following sequences are convergent or divergent. If the sequence is convergent, determine its limit as $n \to \infty$. Justify your answer.

(a) (5 points) The sequence $\{a_n\}$, where $a_n = \frac{\sin(n^2)\ln(n)}{n}$.

Answer: $\boxed{\text{Converges to 0.}}$

Notice that $-\frac{\ln(n)}{n} \leq a_n \leq \frac{\ln(n)}{n}$, where $\lim_{n\to\infty} \frac{\ln(n)}{n} = 0$. So $\lim_{n\to\infty} a_n = 0$ by the squeeze theorem.

(b) (5 points) The sequence $\{b_n\}$, where $b_n = \ln\left(\frac{n!}{n^n}\right)$.

Answer: $\boxed{\text{Diverges.}}$

Notice that

$$\frac{n!}{n^n} = \frac{n \cdot (n-1) \cdot \cdots \cdot 1}{n \cdot n \cdot \cdots \cdot n} \leq \frac{1}{n},$$

which implies that

$$b_n \leq \ln \frac{1}{n} = -\ln(n).$$

By the comparison theorem, because $\{-\ln(n)\}$ diverges and is negative, $\{b_n\}$ diverges.

Problem 3 (20 points) Determine whether the following series are convergent or divergent and **explain your reasoning**. If the series is convergent, you **do not need to** determine its sum.

(a) (10 points) The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{e^{2n}+1}}$.

Answer: $\boxed{\text{Converges.}}$

Notice that $0 \leq \frac{1}{\sqrt{e^{2n}+1}} \leq \frac{1}{e^n}$. Therefore, the series converges by comparison to $\sum_{n=1}^{\infty} \frac{1}{e^n}$.

(b) (10 points) The series $\sum_{n=1}^{\infty} \frac{\ln(n)}{n^2}$.

Answer: $\boxed{\text{Converges.}}$

By integration by parts, the integral

$$\int_1^{\infty} \frac{\ln(x)}{x^2} dx = [-\ln(x)x^{-1}]_1^{\infty} + \int_1^{\infty} \frac{1}{x^2} dx < \infty$$

converges. Since $f(x) = \frac{\ln(x)}{x^2}$ is positive and decreasing, the series converges by the integral test.

Problem 4 (20 points) Consider the series $X = \sum_{n=0}^{\infty}(-1)^n \frac{1}{\sqrt{n^2+1}}$.

(a) (5 points) Show that the series converges. You **do not** need to determine the sum.

Answer: $\boxed{\text{By the alternating series test, since } \frac{1}{\sqrt{n^2+1}} \text{ is positive and decreasing.}}$

(b) (5 points) Is the series absolutely convergent?

Answer: $\boxed{\text{No.}}$

Notice that $\frac{1}{\sqrt{n^2+1}} \geq \frac{1}{2n}$, so $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges by comparison to $\sum_{n=1}^{\infty} \frac{1}{2n}$.

(c) (10 points) We wish to approximate X by the partial sum $s_k := \sum_{n=0}^{k}(-1)^n \frac{1}{\sqrt{n^2+1}}$. What is the minimum value of k for which $|s_k - X| < \frac{1}{100}$?

Answer: $\boxed{k = 99.}$

Let $a_n = \frac{1}{\sqrt{n^2+1}}$. For alternating series, we know that $|s_k - X| < a_{k+1}$, so we want to find the minimum k so that

$$a_{k+1} = \frac{1}{\sqrt{(k+1)^2+1}} < \frac{1}{100} \iff 100^2 < (k+1)^2 + 1 \iff k \geq 99.$$

Problem 5 (15 points) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n}{2^n} x^n.$$

Answer: $\boxed{(-2, 2)}$

By the ratio test, for $a_n = \frac{n}{2^n} x^n$, we find that

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{n+1}{n} \frac{|x|}{2} = \frac{1}{2}|x| =: L.$$

Therefore, we see that $L < 1$ when $|x| < 2$, meaning the radius of convergence is 2. For $x = 2$, we get $\sum_{n=1}^{\infty} n$, which diverges. For $x = -2$, we get $\sum_{n=1}^{\infty} (-1)^n n$, which also diverges. So the interval of convergence is $(-2, 2)$.

Problem 6 (20 points)

(a) (10 points) Find the Taylor series about $x = 0$ for the function

$$f(x) = x^2 e^{2x^2}.$$

What is its radius of convergence?

Answer: $\boxed{\sum_{n=0}^{\infty} \frac{2^n x^{2n+2}}{n!} \text{ with } R = \infty}$

First, the Taylor series at $x = 0$ for e^{2x^2} is given by

$$\sum_{n=0}^{\infty} \frac{y^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^{2n}}{n!}$$

for $y = 2x^2$. Multiplying by x^2 yields

$$\sum_{n=0}^{\infty} \frac{2^n x^{2n+2}}{n!}.$$

By the ratio test, we find that for $a_n = \frac{2^n x^{2n+2}}{n!}$, we have

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2|x^2|}{n+1} = 0,$$

meaning that the radius of convergence is $R = \infty$.

(b) (10 points) Using your answer above, write down a series for

$$\int_0^1 x^2 e^{2x^2}\, dx.$$

Answer: $\boxed{\sum_{n=0}^{\infty} \frac{2^n}{n!(2n+3)}}$

This follows by integrating the Taylor series term by term.