Mathematics UN1102
Section 1, Fall 2019
Practice Midterm 2A
November 13, 2019
Time Limit: 75 Minutes

Name: Solution Key

UNI: ______________________

Instructions: This exam contains 6 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the blank pages stapled to the end of the exam. **Do not write on the back side of your test papers.**

The use of outside material including books, notes, calculators, and electronic devices is not allowed.

Question	1	2	3	4	5	6	Total
Points	15	10	20	20	15	20	100
Score	15	10	20	20	15	20	100

Formulas

Taylor series of $f(x)$ at $x = a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Maclaurin series:

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ $\quad R = 1$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $\quad R = \infty$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\quad R = \infty$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $\quad R = 1$
- $(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$ $\quad R = 1$

Problem 1 (15 points) Determine whether each of the following statements are true or false. If a statement is true, explain why; if a statement is false, give an example that shows why the statement is false.

(a) (5 points) If $\{a_n\}$ satisfies $a_n \leq \frac{1}{n^2}$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges.

Answer: $\boxed{\text{False}}$. If $a_n = -\frac{1}{n}$ for all n, then $\sum_{n=1}^{\infty} a_n$ diverges.

(b) (5 points) If $\sum_{n=0}^{\infty} a_n$ is a series with $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 1/2$, then $\lim_{n\to\infty} a_n = 0$.

Answer: $\boxed{\text{True}}$. The series is absolutely convergent by the ratio test, so the summands must have limit 0.

(c) (5 points) If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = 2$, then it converges at $x = -2$.

Answer: $\boxed{\text{False}}$. Consider $a_n = (-1/2)^n \frac{1}{n}$.

Problem 2 (10 points) Determine whether the following sequences are convergent or divergent. If the sequence is convergent, determine its limit as $n \to \infty$. Justify your answer.

(a) (5 points) The sequence $\{a_n\}$, where $a_n = \frac{3e^{2n}+1}{2e^{2n}+e^n+1}$.

Answer: $\boxed{\text{Converges to } \frac{3}{2}}$.

We have that

$$\lim_{n\to\infty} \frac{3e^{2n}+1}{2e^{2n}+e^n+1} = \lim_{n\to\infty} \frac{3}{2}\frac{e^{2n}(1+1/(3e^{2n})}{e^{2n}(1+1/2e^{-n}+1/2e^{-2n})} = \frac{3}{2}.$$

(b) (5 points) The sequence $\{b_n\}$, where $b_n = \frac{n!}{e^n}$.

Answer: $\boxed{\text{Diverges}}$.

Since $e < 3$, for $n \geq 3$ we have that

$$b_n = \frac{n(n-1)\cdots 1}{e^n} = \left(\frac{n}{e}\right)\left(\frac{n-1}{e}\right)\cdots\left(\frac{3}{e}\right)\left(\frac{2}{e}\right)\left(\frac{1}{e}\right) \geq \frac{n}{e}\frac{2}{e^2},$$

which diverges.

Problem 3 (20 points) Determine whether the following series are convergent or divergent. If the series is convergent, you **do not need to** determine its sum.

(a) (10 points) The series $\sum_{n=1}^{\infty} \frac{\sin(n)}{2^n}$.

Answer: $\boxed{\text{Converges}}$.

Notice that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges and that

$$0 \leq \frac{1 + \sin(n)}{2^n} \leq \frac{2}{2^n},$$

where $\sum_{n=1}^{\infty} \frac{2}{2^n}$ converges. By comparison, this means that $\sum_{n=1}^{\infty} \frac{1+\sin(n)}{2^n}$ converges, hence $\sum_{n=1}^{\infty} \frac{\sin(n)}{2^n}$ converges.

(b) (10 points) The series $\sum_{n=1}^{\infty} \frac{\ln(e^n+1)}{n^2}$.

Answer: $\boxed{\text{Diverges}}$.

Notice that $\frac{\ln(e^n+1)}{n^2} \geq \frac{n}{n^2} = \frac{1}{n}$, so the series diverges by comparison with $\sum_{n=1}^{\infty} \frac{1}{n}$.

Problem 4 (20 points) Consider the series $\sum_{n=2}^{\infty}(-1)^n \frac{1}{n(\ln n)^2}$.

(a) (10 points) Is the series convergent or divergent? You **do not** need to determine the sum if convergent.

Answer: $\boxed{\text{Convergent}}$.

The series is alternating with $\frac{1}{n(\ln n)^2}$ decreasing and $\lim_{n\to\infty} \frac{1}{n(\ln n)^2} = 0$, so it converges.

(b) (10 points) Is the series absolutely convergent? You **do not** need to determine the sum if convergent. Hint: Try applying the integral test.

Answer: $\boxed{\text{Absolutely convergent}}$.

By the integral test, the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$ converges or diverges simultaneously with the improper integral

$$\int_2^{\infty} \frac{1}{x\ln(x)^2}dx = \lim_{t\to\infty}\int_2^t \frac{1}{x\ln(x)^2}dx = \lim_{t\to\infty}\int_{\ln(2)}^{\ln(t)} \frac{1}{y^2}dy = \lim_{t\to\infty}\left[-\frac{1}{y}\right]_{\ln(2)}^{\ln(t)} = \lim_{t\to\infty}\left(\frac{1}{\ln(2)} - \frac{1}{\ln(t)}\right) = \frac{1}{\ln(2)},$$

which converges.

Problem 5 (15 points) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{3^n}{n!}(x-2)^n.$$

Answer: $\boxed{(-\infty, \infty)}$.

We apply the ratio test. Notice that $a_n = \frac{3^n}{n!}(x-2)^n$, so

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{3}{n}|x-2| = 0$$

for all x, meaning that the series is absolutely convergent for all x.

Problem 6 (20 points)

(a) (10 points) Find the Maclaurin series for the function

$$f(x) = \ln(1 - x^2).$$

What is its radius of convergence?

Answer: $\boxed{-\sum_{n=1}^{\infty} \frac{x^{2n}}{n} = -x^2 - \frac{x^4}{2} - \frac{x^6}{3} - \cdots}$ with radius of convergence 1.

We know from the table that $f(x) = \ln(1 + x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$ with radius of convergence 1. Therefore, the Maclaurin series for $\ln(1 - x^2) = f(-x^2)$ is given by substitution. It converges for

$$|-x^2| < 1 \iff |x| < 1,$$

so the radius of convergence is 1.

(b) (10 points) Using your answer above, compute the value of the limit

$$\lim_{x \to 0} \frac{-x^2 - \ln(1 - x^2)}{x^4}.$$

Answer: $\boxed{\frac{1}{2}}$.

Substituting the Maclaurin series from (a), we find that

$$\lim_{x \to 0} \frac{x^2 - \ln(1 - x^2)}{x^4} = \lim_{x \to 0} \frac{\sum_{n=2}^{\infty} x^{2n} n}{x^4} = \lim_{x \to 0} \sum_{n=2}^{\infty} x^{2n-4} \frac{1}{n} = \frac{1}{2}.$$