Mathematics UN1102
Section 1, Spring 2020
Final Exam
April 11, 2020
Time Limit: 170 Minutes

Name: ______________________________

UNI: ______________________________

Instructions: This exam contains 10 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the blank pages stapled to the end of the exam. **Do not write on the back side of your test papers.**

You will have 170 minutes to complete this exam. You may choose any 170 minute period during the 12-hour exam period to take the exam. You must scan and upload the completed exam to Gradescope by the end of the 12 hour exam period. Please write the 170 minute period you took the exam below.

Start Time: ______________________________

End Time: ______________________________

The use of outside material including books, notes, calculators, and electronic devices is not allowed. Due to the coronavirus situation, this exam will be take-home, meaning that these rules will be enforced by the honor code. Please sign below to affirm that you have followed these rules.

Signature: ______________________________

Formulas

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta)) \qquad \cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

Taylor series of $f(x)$ at $x = a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Maclaurin series:

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ $\quad R = 1$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $\quad R = \infty$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\quad R = \infty$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $\quad R = 1$
- $(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$ $\quad R = 1$

Problem 1 (8 points) Determine whether each of the following statements are true or false. If a statement is true, explain why; if a statement is false, give an example that shows why the statement is false.

(a) (2 points) Suppose $f(x)$ is a continuous function on $(0, \infty)$. If $0 < f(x) < \frac{1}{x}$ for all x, then the integral $\int_0^1 f(x)dx$ converges.

Answer:

(b) (2 points) Suppose $a_n > 0$ for all n. If $\sum_{n=1}^{\infty} a_n$ converges, so does $\sum_{n=1}^{\infty} (-1)^n a_n$.

Answer:

(c) (2 points) If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges at $x = 1$, then it converges at $x = -3$.

Answer:

(d) (2 points) The differential equation $y' + 2xy = e^{-x^2}$ has a unique solution satisfying $y(0) = 1$.

Answer:

Problem 2 (12 points) This problem is on two pages. Evaluate the following integrals.

(a) (4 points)

$$\int \cos(x)e^{\sin(x)}dx.$$

Answer:

(b) (4 points)

$$\int \frac{x^2}{\sqrt{1-x^2}}dx.$$

Answer:

(c) (4 points)

$$\int x^2 \ln(x) dx.$$

Answer:

Problem 3 (10 points) Consider the region A bounded by the curve $y = x^2$ and the line $y = 4$.

(a) (2 points) Sketch the region A.

Answer:

(b) (3 points) Find the area of the region A.

Answer:

(c) (5 points) Find the volume of the solid of revolution obtained by rotating A about the line $y = -1$.

Answer:

Problem 4 (10 points) This problem is on two pages. Determine whether each of the following series is convergent or divergent. Justify your answer.

(a) (2 points)

$$\sum_{n=0}^{\infty} \frac{\sqrt{n+1}}{\sqrt{n^4+1}}.$$

Answer:

(b) (2 points)

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{n^n}.$$

Answer:

(c) (3 points)

$$\sum_{n=1}^{\infty} \frac{e^{\sin(n)}}{n^2+1}.$$

Answer:

(d) (3 points)

$$\sum_{n=2}^{\infty} \frac{1}{n\ln(n)}.$$

Answer:

Problem 5 (10 points) Consider the power series

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}(2x-1)^n.$$

Determine its interval and radius of convergence.

Answer:

Problem 6 (10 points)

(a) (6 points) Find the Maclaurin series of the function

$$f(x) = 1 + x + e^{2x^2}.$$

What is its interval of convergence?

Answer:

(b) (4 points) Write down a series expression for $f'(x)$.

Answer:

Problem 7 (10 points) Consider the curve given by $r = \sin(\theta) + \cos(\theta)$ in polar coordinates for $\theta \in [0, 2\pi]$.

(a) (5 points) Determine the θ-values of the points where the curve has horizontal or vertical tangents.

Answer:

(b) (5 points) Determine the area contained within the curve.

Answer:

Problem 8 (10 points) Consider the differential equation

$$\frac{dy}{dx} = 2y\Big(1 - y\Big).$$

(a) (5 points) Find all values of y for which $\frac{dy}{dx} = 0$.

Answer:

(b) (5 points) Let $y = f(x)$ be the solution with $f(0) = \frac{1}{2}$. Find the approximation for $f(0.2)$ given by Euler's method with step size 0.1.

Answer:

Problem 9 (10 points) Solve the differential equation

$$\frac{dy}{dx} = x^2 \cos^2(y)$$

with initial condition $y(0) = \frac{\pi}{4}$.

Answer:

Problem 10 (10 points) Find the general solution to the differential equation

$$y' + \cos(x)y = e^{-\sin(x)}.$$

Answer: