Mathematics UN1102 Name: ____________________________
Section 1, Spring 2020
Practice Final A UNI: ____________________________
Time Limit: 170 Minutes

Instructions: This exam contains 10 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the blank pages stapled to the end of the exam. **Do not write on the back side of your test papers.**

You will have 170 minutes to complete this exam. You may choose any 75 minute period during the 12-hour exam period to take the exam. You must scan and upload the completed exam to Gradescope by the end of the 12 hour exam period. Please write the 170 minute period you took the exam below.

Start Time: ____________________________

End Time: ____________________________

The use of outside material including books, notes, calculators, and electronic devices is not allowed. Due to the coronavirus situation, this exam will be take-home, meaning that these rules will be enforced by the honor code. Please sign below to affirm that you have followed these rules.

Signature: ____________________________

Formulas

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta)) \qquad \cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

Taylor series of $f(x)$ at $x = a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Maclaurin series:

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ $R = 1$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $R = \infty$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $R = \infty$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $R = 1$
- $(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$ $R = 1$

Problem 1 (8 points) Determine whether each of the following statements are true or false. If a statement is true, explain why; if a statement is false, give an example that shows why the statement is false.

(a) (2 points) Suppose $f(x)$ is a continuous function on $[1, \infty)$. If the improper integral $\int_5^\infty f(x)dx$ is convergent, then the improper integral $\int_2^\infty f(x)dx$ must also be convergent.

Answer:

(b) (2 points) If the infinite series $\sum_{n=1}^\infty a_n$ is convergent, then the infinite series

$$\sum_{n=1}^{\infty}(a_{n+1} - a_n)$$

must also be convergent.

Answer:

(c) (2 points) If $z(x)$ is a solution to the differential equation $z' = 0.2 \cdot z(1 - z/10)$, then the limit $\lim_{x\to\infty} z(x)$ must be equal to 10.

Answer:

(d) (2 points) If $y_1(x), y_2(x)$ are solutions to the differential equation $y' + y = \sin(x)$, then $y_1(x) + y_2(x)$ must also be a solution.

Answer:

Problem 2 (12 points) This problem is on two pages. Evaluate the following integrals.

(a) (4 points)

$$\int e^{\sqrt{x}} dx.$$

Answer:

(b) (4 points)

$$\int \frac{3x^2 + 5x + 8}{(x+1)^2(x-1)} dx.$$

Answer:

(c) (4 points) Using the trig substitution $x = 3\cos\theta$ or otherwise, find

$$\int \frac{dx}{x\sqrt{9-x^2}}.$$

Answer:

Problem 3 (10 points) Consider the region A bounded by the curves $y = \sin(x)$ and $y = -\sin(x)$ between $x = 0$ and $x = \pi$.

(a) (2 points) Sketch the region A.

Answer:

(b) (3 points) Find the area of the region A.

Answer:

(c) (5 points) Find the volume of the solid of revolution obtained by rotating A about the y-axis.

Answer:

Problem 4 (10 points) This problem is on two pages. Determine whether each of the following series is convergent or divergent. Justify your answer.

(a) (2 points)

$$\sum_{n=0}^{\infty} \frac{5^n}{4^n}.$$

Answer:

(b) (2 points)

$$\sum_{n=0}^{\infty} \frac{n^3 + 3n + \sin(n)}{n^4 - 2}.$$

Answer:

(c) (3 points)

$$\sum_{n=1}^{\infty} \frac{n!}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n-1)}.$$

Answer:

(d) (3 points)

$$\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}.$$

Answer:

Problem 5 (8 points) Consider the power series

$$\sum_{n=1}^{\infty} \frac{2^n}{\sqrt{n}}(x-5)^n$$

(a) (4 points) Determine its radius of convergence.

Answer:

(b) (4 points) Determine its interval of convergence.

Answer:

Problem 6 (12 points)

(a) (4 points) Find the Maclaurin series of the function

$$f(x) = \frac{1}{1+9x^2}.$$

Answer:

(b) (2 points) What is the radius of convergence of the Maclaurin series of $f(x)$?

Answer:

(c) (4 points) Find the Maclaurin series of the function $g(x) = \tan^{-1}(3x)$.

Answer:

(d) (2 points) What is the radius of convergence of the Maclaurin series of $g(x)$? Hint: Use your answer to (b).

Answer:

Problem 7 (10 points) Consider the polar curve $r = 2 + \cos(3\theta)$.

(a) (3 points) Find two values r_1, r_2 such that the points with polar coordinates $(r_1, \pi/9)$, $(r_2, \pi/9)$ lie on the polar curve.

Answer:

(b) (7 points) Set up and evaluate an integral to find the are enclosed by the polar curve $r = 2 + \cos(3\theta)$.

Answer:

Problem 8 (10 points) Consider the differential equation

$$y' = x - y + 2.$$

(a) (4 points) There is exactly one choice of numbers m and b for which $y(x) = mx + b$ satisfies the differential equation. Find these values of m and b.

Answer:

(b) (6 points) Draw the direction field for the differential equation in the region $-4 \leq x \leq 4$, $-4 \leq y \leq 4$. On the direction field, draw the solution curve you found in (a), along with two other solution curves.

Answer:

Problem 9 (10 points) Find the solution to the differential equation

$$\sec^2(x)y' - y^3 = 0$$

satisfying the initial condition $y(\pi/2) = 1$.

Problem 10 (10 points) Find the general solution to the linear differential equation

$$(x^2 + x)y' + 2(2x + 1)y = 2x.$$