Mathematics UN1102 Name: ____________________________
Section 1, Spring 2020
Practice Final B UNI: ____________________________
Time Limit: 170 Minutes

Instructions: This exam contains 10 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the blank pages stapled to the end of the exam. **Do not write on the back side of your test papers.**

You will have 170 minutes to complete this exam. You may choose any 75 minute period during the 12-hour exam period to take the exam. You must scan and upload the completed exam to Gradescope by the end of the 12 hour exam period. Please write the 170 minute period you took the exam below.

Start Time: ____________________________

End Time: ____________________________

The use of outside material including books, notes, calculators, and electronic devices is not allowed. Due to the coronavirus situation, this exam will be take-home, meaning that these rules will be enforced by the honor code. Please sign below to affirm that you have followed these rules.

Signature: ____________________________

Formulas

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta)) \qquad \cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \tan x\, dx = \ln|\sec x| + C \qquad \int \sec x\, dx = \ln|\sec x + \tan x| + C$$

Taylor series of $f(x)$ at $x = a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Maclaurin series:

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ $R = 1$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $R = \infty$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $R = \infty$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $R = 1$
- $(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$ $R = 1$

Problem 1 (8 points) Determine whether each of the following statements are true or false. If a statement is true, explain why; if a statement is false, give an example that shows why the statement is false.

(a) (2 points) Suppose $f(x)$ is a continuous function on $(0, \infty)$. If the improper integral $\int_1^\infty f(x)dx$ is convergent, then the improper integral $\int_0^\infty f(x)dx$ must also be convergent.

Answer:

(b) (2 points) If the infinite series $\sum_{n=1}^\infty a_n$ satisfies $0 \leq a_n < \frac{1}{n}$ for all n, the series must be convergent.

Answer:

(c) (2 points) The function e^x is equal to its Maclaurin series for all x.

Answer:

(d) (2 points) If $y_1(x)$ is a solution to $y' + \sin(x)y = 0$ and $y_2(x)$ is a solution to $y' + \cos(x)y = 0$, then $y_1(x) + y_2(x)$ is a solution to $y' + (\sin(x) + \cos(x))y = 0$.

Answer:

Problem 2 (12 points) This problem is on two pages. Evaluate the following integrals.

(a) (4 points)

$$\int e^x \cot(e^x) dx.$$

Answer:

(b) (4 points)

$$\int \frac{x^2 + 3x - 3}{(x+1)(x^2 + 6x + 10)} dx.$$

Answer:

(c) (4 points) Using the trig substitution $x = 2\tan\theta$ or otherwise, find

$$\int \frac{dx}{(4+x^2)^{3/2}}.$$

Answer:

Problem 3 (10 points) Consider the region A bounded by the curves $x = 1$, $x = 2$, $y = 1$, and $y = (x - 1)^2$.

(a) (2 points) Sketch the region A.

Answer:

(b) (3 points) Find the area of the region A.

Answer:

(c) (5 points) Find the volume of the solid of revolution obtained by rotating A about the y-axis.

Answer:

Problem 4 (10 points) This problem is on two pages. Determine whether each of the following series is convergent or divergent. Justify your answer.

(a) (2 points)

$$\sum_{n=0}^{\infty} \frac{n^n}{n!}.$$

Answer:

(b) (2 points)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n-1}}{n^2}.$$

Answer:

(c) (3 points)

$$\sum_{n=1}^{\infty}(-1)^n \cos\left(\frac{1}{n}\right).$$

Answer:

(d) (3 points)

$$\sum_{n=1}^{\infty}\frac{\ln(n)}{n^{3/2}}.$$

Answer:

Problem 5 (10 points) Consider the power series

$$\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}(x-1)^n$$

(a) (5 points) Determine its radius of convergence.

Answer:

(b) (5 points) Determine its interval of convergence.

Answer:

Problem 6 (10 points)

(a) (7 points) Find the Maclaurin series of the function

$$f(x) = \frac{1}{(1-x)^2}.$$

What is its interval of convergence?

Answer:

(b) (3 points) Compute the limit

$$\lim_{x \to 0} \frac{f(x) - 1 - 2x}{x^2}.$$

Answer:

Problem 7 (10 points) Consider the parametric curve given by $x = t^3 - 3t$ and $y = 3t^2 - 9$.

(a) (3 points) Find the unique point of self-intersection of the curve.

Answer:

(b) (7 points) Determine the (x, y)-coordinates of the points where the curve has horizontal or vertical tangents.

Answer:

Problem 8 (10 points) Consider the differential equation

$$\frac{dy}{dx} = -\frac{2x}{y}$$

(a) (5 points) Let $y = f(x)$ be the solution to the equation with initial condition $f(1) = -1$. Write an equation for the line tangent to $f(x)$ at $(1, -1)$ and use it to approximate $f(1.01)$.

Answer:

(b) (5 points) Find the solution $y = f(x)$ with initial condition $f(1) = -1$.

Answer:

Problem 9 (10 points) Find the solution to the differential equation

$$\frac{dy}{dx} = e^{-y}(2x - 4)$$

satisfying the initial condition $y(5) = 0$.

Answer:

Problem 10 (10 points) Find the general solution to the differential equation

$$\cos(x)y' + \sin(x)y = 1.$$

Answer: