Mathematics UN1102 Name: ____________________________
Section 1, Spring 2020
Practice Final B UNI: ____________________________
Time Limit: 170 Minutes

Instructions: This exam contains 10 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the blank pages stapled to the end of the exam. **Do not write on the back side of your test papers.**

You will have 170 minutes to complete this exam. You may choose any 75 minute period during the 12-hour exam period to take the exam. You must scan and upload the completed exam to Gradescope by the end of the 12 hour exam period. Please write the 170 minute period you took the exam below.

Start Time: ____________________________

End Time: ____________________________

The use of outside material including books, notes, calculators, and electronic devices is not allowed. Due to the coronavirus situation, this exam will be take-home, meaning that these rules will be enforced by the honor code. Please sign below to affirm that you have followed these rules.

Signature: ____________________________

Formulas

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta)) \qquad \cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

Taylor series of $f(x)$ at $x = a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Maclaurin series:

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ $R = 1$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $R = \infty$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $R = \infty$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $R = 1$
- $(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!}x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$ $R = 1$

Problem 1 (8 points) Determine whether each of the following statements are true or false. If a statement is true, explain why; if a statement is false, give an example that shows why the statement is false.

(a) (2 points) Suppose $f(x)$ is a continuous function on $(0, \infty)$. If the improper integral $\int_1^\infty f(x)dx$ is convergent, then the improper integral $\int_0^\infty f(x)dx$ must also be convergent.

Answer: False. Consider $f(x) = 1/x^2$.

(b) (2 points) If the infinite series $\sum_{n=1}^\infty a_n$ satisfies $0 \leq a_n < \frac{1}{n}$ for all n, the series must be convergent.

Answer: False. Consider $a_n = \frac{1}{2n}$.

(c) (2 points) The function e^x is equal to its Maclaurin series for all x.

Answer: True. The interval of convergence of the Maclaurin series is $(-\infty, \infty)$.

(d) (2 points) If $y_1(x)$ is a solution to $y' + \sin(x)y = 0$ and $y_2(x)$ is a solution to $y' + \cos(x)y = 0$, then $y_1(x) + y_2(x)$ is a solution to $y' + (\sin(x) + \cos(x))y = 0$.

Answer: False. If $y = y_1 + y_2$, we have that

$$y' = y_1' + y_2' = -\sin(x)y_1 - \cos(x)y_2 \neq -(\sin(x) + \cos(x))y.$$

Problem 2 (12 points) This problem is on two pages. Evaluate the following integrals.

(a) (4 points)

$$\int e^x \cot(e^x) dx.$$

Answer: $\boxed{\ln|\sin e^x| + C}$ Consider $u = e^x$ so that $du = e^x dx$, meaning that $dx = u^{-1}du$, so we find that

$$\int e^x \cot(e^x) dx = \int \cot(u) du = \int \frac{\cos u}{\sin u} du = \ln|\sin u| + C = \ln|\sin e^x| + C.$$

(b) (4 points)

$$\int \frac{x^2 + 3x - 3}{(x+1)(x^2 + 6x + 10)} dx.$$

Answer: $\boxed{-\ln|x+1| + \log|(x+3)^2 + 1| + \arctan(x+3) + C.}$ We compute the partial fraction decomposition

$$\frac{x^2 + 3x - 3}{(x+1)(x^2 + 6x + 10)} = \frac{A}{x+1} + \frac{B(x+3) + C}{(x+3)^2 + 1}.$$

This equation is equivalent to

$$x^2 + 3x - 3 = A(x^2 + 6x + 10) + B(x+3)(x+1) + C(x+1).$$

Setting $x = -1$, we find that $A = -1$, which implies that

$$2x^2 + 9x + 7 = B(x+3)(x+1) + C(x+1),$$

which implies that

$$2x + 7 = B(x+3) + C,$$

which implies that $B = 2$ and $C = 1$. We conclude that

$$\int \frac{x^2 + 3x - 3}{(x+1)(x^2 + 6x + 10)} dx = \int \left(\frac{-1}{x+1} + \frac{2(x+3) + 1}{(x+3)^2 + 1} \right) dx = -\ln|x+1| + \log|(x+3)^2 + 1| + \arctan(x+3) + C.$$

(c) (4 points) Using the trig substitution $x = 2\tan\theta$ or otherwise, find

$$\int \frac{dx}{(4+x^2)^{3/2}}.$$

Answer: $\boxed{\frac{x}{4\sqrt{x^2+4}} + C.}$ We apply the substitution with $dx = 2\sec^2\theta d\theta$ so that

$$\int \frac{dx}{(4+x^2)^{3/2}} = \int \frac{2\sec^2\theta}{(4+4\tan^2\theta)^{3/2}} d\theta = \int \frac{2\sec^2\theta}{8\sec^3\theta} d\theta = \frac{1}{4}\int \cos\theta d\theta = \frac{1}{4}\sin\theta + C.$$

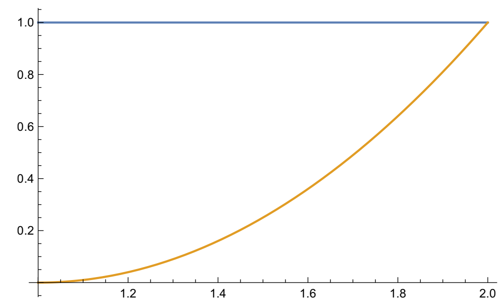
With $\theta = \arctan(x/2)$, we find that $\sin\theta = \frac{x}{\sqrt{x^2+4}}$, hence the answer is

$$\frac{x}{4\sqrt{x^2+4}} + C.$$

Problem 3 (10 points) Consider the region A bounded by the curves $x = 1$, $x = 2$, $y = 1$, and $y = (x-1)^2$.

(a) (2 points) Sketch the region A.

Answer:



(b) (3 points) Find the area of the region A.

Answer: $\boxed{\frac{2}{3}}$. The area is given by

$$A = \int_1^2 (1-(x-1)^2)dx = \int_1^2 (2x - x^2)dx = [x^2 - \frac{1}{3}x^3]_1^2 = 3 - \frac{7}{3} = \frac{2}{3}.$$

(c) (5 points) Find the volume of the solid of revolution obtained by rotating A about the y-axis.

Answer: $\boxed{\frac{11\pi}{6}}$. We apply the cylinder method to obtain

$$V = 2\pi \int_1^2 x(1-(x-1)^2)dx = 2\pi \int_1^2 (2x^2 - x^3)dx = 2\pi[\frac{2}{3}x^3 - \frac{1}{4}x^4]_1^2 = 2\pi(\frac{14}{3} - \frac{15}{4}) = \frac{11\pi}{6}.$$

Problem 4 (10 points) This problem is on two pages. Determine whether each of the following series is convergent or divergent. Justify your answer.

(a) (2 points)

$$\sum_{n=0}^{\infty} \frac{n^n}{n!}.$$

Answer: $\boxed{\text{Divergent.}}$ Since $\frac{n^n}{n!} \geq 1$, the series diverges by comparison with $\sum_{n=0}^{\infty} 1$.

(b) (2 points)

$$\sum_{n=1}^{\infty} \frac{\sqrt{n-1}}{n^2}.$$

Answer: $\boxed{\text{Convergent.}}$ Since $\frac{\sqrt{n-1}}{n^2} < \frac{\sqrt{n}}{n^2} = n^{-3/2}$, this converges by comparison and the p-series test.

(c) (3 points)

$$\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{1}{n}\right).$$

Answer:	Divergent.	Since $\lim_{n\to\infty} (-1)^n \cos(1/n)$ does not exist, the series must diverge.

(d) (3 points)

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n^{3/2}}.$$

Answer: Convergent. By the integral test, it suffices to check that

$$\int_2^{\infty} \frac{\ln(x)}{x^{3/2}}\,dx = [-2\ln(x)x^{-1/2}]_2^{\infty} - \int_2^{\infty} \frac{-2}{x^{3/2}}\,dx$$

converges, which holds by the p-integral test.

Problem 5 (10 points) Consider the power series

$$\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}(x-1)^n$$

(a) (5 points) Determine its radius of convergence.

Answer: $\boxed{1.}$ We apply the ratio test and find

$$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^3}{(n+1)^4+1} \frac{n^4+1}{n^3} |x-1| = |x-1|,$$

so that $L < 1$ if and only if $x \in (0,2)$, meaning that the radius is 1.

(b) (5 points) Determine its interval of convergence.

Answer: $\boxed{[0,2).}$ We must check $x = 0$ and $x = 2$. For $x = 0$, we get

$$\sum_{n=1}^{\infty} \frac{n^3}{n^4+1}(-1)^n,$$

which converges by the alternating series test. For $x = 2$, we get

$$\sum_{n=1}^{\infty} \frac{n^3}{n^4+1},$$

which diverges by comparison with $\frac{n^3}{n^4+1} > \frac{n^3}{2n^4} = 1/(2n)$. So the interval of convergence is $[0,2)$.

Problem 6 (10 points)

(a) (7 points) Find the Maclaurin series of the function

$$f(x) = \frac{1}{(1-x)^2}.$$

What is its interval of convergence?

Answer: $\boxed{\sum_{n=0}^{\infty}(n+1)x^n \text{ with interval of convergence } (-1,1).}$ The Maclaurin series for $1/(1-x)$ is

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$$

with radius of convergence 1. Taking the term by term derivative, we find the Maclaurin series for $\frac{1}{(1-x)^2}$ is

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n$$

with radius of convergence 1. To find the interval of convergence, we note that the series diverges at $x = 1$ and $x = -1$, so the interval of convergence is $(-1, 1)$.

(b) (3 points) Compute the limit

$$\lim_{x \to 0} \frac{f(x) - 1 - 2x}{x^2}.$$

Answer: $\boxed{3.}$ We substitute the Maclaurin series for $f(x)$ to obtain

$$\lim_{x \to 0} \frac{\sum_{n=0}^{\infty}(n+1)x^n - 1 - 2x}{x^2} = \lim_{x \to 0} \sum_{n=2}^{\infty}(n+1)x^{n-2} = 3.$$

Problem 7 (10 points) Consider the parametric curve given by $x = t^3 - 3t$ and $y = 3t^2 - 9$.

(a) (3 points) Find the unique point of self-intersection of the curve.

Answer: $\boxed{(0,0).}$ We find that $y(t) = y(t')$ if and only if $t = -t'$, so we must solve

$$t^3 - 3t = x(t) = x(-t) = -t^3 + 3t,$$

which implies that $2t^3 - 6t = 0$, hence $t = 0, \pm\sqrt{3}$. Only $t = \pm\sqrt{3}$ leads to self-intersection, hence the answer is $(x(\sqrt{3}), y(\sqrt{3})) = (0,0)$.

(b) (7 points) Determine the (x, y)-coordinates of the points where the curve has horizontal or vertical tangents.

Answer: $\boxed{\text{Horizontal: } (0,-9), \text{ Vertical: } (-2,-6) \text{ and } (2,-6).}$ Horizontal tangents occur when $y'(t) = 6t = 0$, meaning that $t = 0$, leading to $(0, -9)$. Vertical tangents occur when $x'(t) = 3t^2 - 3$, meaning that $t = \pm 1$, leading to $(-2, -6)$ and $(2, -6)$.

Problem 8 (10 points) Consider the differential equation

$$\frac{dy}{dx} = -\frac{2x}{y}$$

(a) (5 points) Let $y = f(x)$ be the solution to the equation with initial condition $f(1) = -1$. Write an equation for the line tangent to $f(x)$ at $(1, -1)$ and use it to approximate $f(1.01)$.

Answer: $\boxed{y = 2x - 3 \text{ and } -0.98.}$ This line has slope $\frac{dy}{dx} = 2$ and passes through $(1, -1)$, hence it has equation

$$y = 2x - 3.$$

Using this linear approximation, we get $y(1.01) \approx 2 \cdot 1.01 - 3 = -0.98$.

(b) (5 points) Find the solution $y = f(x)$ with initial condition $f(1) = -1$.

Answer: $\boxed{f(x) = -\sqrt{3 - 2x^2}.}$ This is a separable equation, so we find that

$$\int y\, dy = -\int 2x dx,$$

which implies $\frac{1}{2}y^2 = -x^2 + C$, hence

$$y = \pm\sqrt{2C - 2x^2}.$$

Since $y(1) = -1$, we must choose the negative sign, and we may solve for C by taking

$$-1 = y(1) = -\sqrt{2C - 2},$$

which implies that $C = 3/2$. The answer is therefore $f(x) = -\sqrt{3 - 2x^2}$.

Problem 9 (10 points) Find the solution to the differential equation

$$\frac{dy}{dx} = e^{-y}(2x - 4)$$

satisfying the initial condition $y(5) = 0$.

Answer: $\boxed{y = \ln(x^2 - 4x - 4).}$ We find that

$$\int e^y dy = \int (2x - 4) dx,$$

which implies that $e^y = x^2 - 4x + C$, hence

$$y = \ln(x^2 - 4x + C).$$

Solving for C, we find that

$$0 = y(5) = \ln(5 + C),$$

which implies that $C = -4$. The answer is therefore $y = \ln(x^2 - 4x - 4)$.

Problem 10 (10 points) Find the general solution to the differential equation

$$\cos(x)y' + \sin(x)y = 1.$$

Answer: $\boxed{\sin(x) + C\cos(x).}$ We simplify the equation to

$$y' + \tan(x)y = \sec(x).$$

The integrating factor is

$$I(x) = e^{\int \tan(x)dx} = e^{-\ln\cos(x)} = \sec(x),$$

so we find that

$$(\sec(x)y)' = \sec(x)y' + \sec(x)\tan(x)y = \sec^2(x).$$

Integrating both sides yields

$$\sec(x)y = \tan(x) + C,$$

hence the general solution is

$$y = \sin(x) + \frac{C}{\sec(x)} = \sin(x) + C\cos(x).$$