Mathematics UN1102
Section 1, Spring 2020
Midterm 1 Solutions
February 24, 2020
Time Limit: 75 Minutes

Name: ______________________________

UNI: ______________________________

Instructions: This exam contains 7 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the back side of the problem sheets or the blank pages stapled to the end of the exam.

The use of outside material including books, notes, calculators, and electronic devices is not allowed.

Formulas

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta)) \qquad \cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$$

$$\int \frac{1}{a^2 + x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2 - x^2}}dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \tan x\, dx = \ln|\sec x| + C \qquad \int \sec x\, dx = \ln|\sec x + \tan x| + C$$

Problem 1 (10 points) Evaluate the definite integral

$$\int_1^2 \frac{\ln(x)}{x^2}\,dx$$

Answer: $\boxed{-\frac{\ln(2)}{2} + \frac{1}{2}}$

By integration by parts we have

$$\int_1^2 \frac{\ln(x)}{x^2} dx = -\left[\frac{\ln(x)}{x}\right]_1^2 - \int_1^2 \frac{1}{x^2} dx = -\frac{\ln(2)}{2} + \left[\frac{1}{x}\right]_1^2 = -\frac{\ln(2)}{2} + \frac{1}{2}.$$

Problem 2 (15 points) Evaluate the indefinite integral

$$\int \frac{x^2}{\sqrt{4-x^2}}\,dx.$$

Answer: $\boxed{2\arcsin(x/2) - \frac{1}{2}x\sqrt{4-x^2} + C}$

We substitute $x = 2\sin\theta$ so that $dx = 2\cos\theta d\theta$, meaning that

$$\int \frac{x^2}{\sqrt{4-x^2}}dx = \int 4\sin^2\theta\, d\theta = \int (2 - 2\cos(2\theta))d\theta = -\sin(2\theta) + 2\theta + C = 2\arcsin(x/2) - \frac{1}{2}x\sqrt{4-x^2} + C.$$

Problem 3 (15 points) Evaluate the indefinite integral

$$\int \cos^4(x)\, dx.$$

Answer: $\boxed{\frac{1}{32}\sin(4x) + \frac{1}{4}\sin(2x) + \frac{3}{8}x + C}$

We notice that $\cos^2(x) = \frac{1}{2}(\cos(2x) + 1)$ so that

$$\cos^4(x) = \frac{1}{4}(\cos^2(2x) + 2\cos(2x) + 1) = \frac{1}{8}(\cos(4x) + 1) + \frac{1}{2}\cos(2x) + \frac{1}{4},$$

so term-by-term integration yields the result.

Problem 4 (15 points) Evaluate the indefinite integral

$$\int \frac{1}{x^3 + 2x^2} dx.$$

Hint: It may help to first write a partial fraction decomposition of the integrand.

Answer: $\boxed{-\frac{1}{2}\frac{1}{x} - \frac{1}{4}\ln|x| + \frac{1}{4}\ln|x+2| + C}$

We find the partial fraction decomposition

$$\frac{1}{x^3 + 2x^2} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x+2},$$

which yields

$$1 = A(x+2) + Bx(x+2) + Cx^2 = (B+C)x^2 + (2B+A)x + 2A,$$

which implies that $2A = 1$, $2B + A = 0$, and $B + C = 0$ by matching coefficients. This means that $A = \frac{1}{2}$, $B = -\frac{1}{4}$, and $C = \frac{1}{4}$. We then find that

$$\int \frac{1}{x^3 + 2x^2} dx = \int \frac{1/2}{x^2} + \frac{-1/4}{x} + \frac{1/4}{x+2} dx = -\frac{1}{2}\frac{1}{x} - \frac{1}{4}\ln|x| + \frac{1}{4}\ln|x+2| + C.$$

Problem 5 (15 points) Consider the improper integral

$$\int_1^\infty \frac{x}{1+x^2}\,dx.$$

(a) (5 points) Write the definition of this improper integral as a limit.

Answer: $\boxed{\lim_{a\to\infty} \int_1^a \frac{x}{1+x^2}\,dx}$

(b) (10 points) Determine whether the improper integral converges or diverges and explain why. If the improper integral converges, compute its value.

Answer: $\boxed{\text{Diverges by comparison to } \int_1^\infty \frac{1}{x}\,dx.}$

Problem 6 (20 points) Let A be the area enclosed by the graphs $x = 0$, $y = 1$, and $y = \sqrt{x}$.

(a) (10 points) Sketch A and set up a definite integral to compute its area. You **do not** need to evaluate the integral.

Answer: $\boxed{\int_0^1 (1 - \sqrt{x})dx}$

(b) (10 points) Compute the volume of the solid of revolution obtained by rotating A about the line $y = 1$. State whether you are using the method of disks/washers or the method of cylindrical shells.

Answer: $\boxed{\frac{\pi}{6}}$

By cylindrical shells, we find

$$\int_0^1 2\pi(y-1)^2 y dy = \int_0^1 2\pi(y^3 - 2y^2 + y)dy = 2\pi\Big(\frac{1}{4} - \frac{2}{3} + \frac{1}{2}\Big) = \frac{\pi}{6}.$$

Problem 7 (15 points) Compute the length of the graph of

$$f(x) = \sqrt{x}$$

between $x = 0$ and $x = 1$.

Answer: $\boxed{\frac{1}{8}\ln\frac{\sqrt{5/4}+1}{\sqrt{5/4}-1} + \frac{\sqrt{5}}{2}}$

We find the length is

$$\int_0^1 \sqrt{1 + \frac{1}{4x}}dx.$$

To evaluate this integral, we set $u = \sqrt{1 + \frac{1}{4x}}$ so that $u^2 = 1 + \frac{1}{4x}$ and therefore

$$x = \frac{1}{4(u^2-1)},$$

which means that

$$dx = -\frac{u}{2(u^2-1)^2}du.$$

We therefore want to find

$$-\int_{\infty}^{\sqrt{5/4}} \frac{u^2}{2(u^2-1)^2}du = \int_{\sqrt{5/4}}^{\infty} \frac{u^2}{2(u^2-1)^2}du.$$

We apply partial fractions to do this integral. We must solve

$$\frac{u^2}{2(u^2-1)^2} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{C}{u+1} + \frac{D}{(u+1)^2}.$$

Multiplying both sides by $(u^2-1)^2$, we find that

$$\frac{1}{2}u^2 = A(u-1)(u+1)^2 + B(u+1)^2 + C(u+1)(u-1)^2 + D(u-1)^2.$$

Setting $u = 1$ implies that $B = \frac{1}{8}$. Setting $u = -1$ implies that $D = \frac{1}{8}$. Setting $u = 0$ implies that $0 = -A + B + C + D = -A + \frac{1}{4} + C$. Setting $u = 2$ implies that $2 = 9A + 9B + 3C + D = 9A + 3C + \frac{5}{4}$. Solving, this implies that $A = C + \frac{1}{4}$, so that

$$2 = 12C + \frac{7}{2}$$

and hence $C = -\frac{1}{8}$ and $A = \frac{1}{8}$. Together, these imply that

$$\frac{u^2}{2(u^2-1)^2} = \frac{1/8}{u-1} + \frac{1/8}{(u-1)^2} - \frac{1/8}{u+1} + \frac{1/8}{(u+1)^2}.$$

We therefore find that

$$\int_{\sqrt{5/4}}^{\infty} \frac{u^2}{2(u^2-1)^2}du = \left[\frac{1}{8}\ln\left|\frac{u-1}{u+1}\right| - \frac{1}{8}\frac{1}{u-1} - \frac{1}{8}\frac{1}{u+1}\right]_{\sqrt{5/4}}^{\infty} = \frac{1}{8}\ln\frac{\sqrt{5/4}+1}{\sqrt{5/4}-1} + \frac{\sqrt{5}}{2}.$$