Mathematics UN1102
Section 1, Spring 2020
Midterm 2
April 13, 2020
Time Limit: 75 Minutes

Name: ____________________________

UNI: ____________________________

Instructions: This exam contains 6 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the blank pages stapled to the end of the exam. Please **do not write on the back side of pages**.

You will have 75 minutes to complete this exam. You may choose any 75 minute period between 12 noon and 12 midnight Eastern time on April 13, 2020 to take the exam. You must scan and upload the completed exam to Gradescope by **12 midnight Eastern time on April 13, 2020**. Please write the 75 minute period you took the exam below.

Start Time: ____________________________

End Time: ____________________________

The use of outside material including books, notes, calculators, and electronic devices is not allowed. Due to the coronavirus situation, this exam will be take-home, meaning that these rules will be enforced by the honor code. Please sign below to affirm that you have followed these rules.

Signature: ____________________________

Formulas

Taylor series of $f(x)$ at $x = a$:

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Maclaurin series:

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ $\quad R = 1$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ $\quad R = \infty$
- $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ $\quad R = \infty$
- $\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ $\quad R = \infty$
- $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ $\quad R = 1$
- $(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\cdots(k-n+1)}{n!} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$ $\quad R = 1$

Problem 1 (15 points) Determine whether each of the following statements are true or false. If a statement is true, explain why; if a statement is false, give an example that shows why the statement is false.

(a) (5 points) If $\{a_n\}$ satisfies $a_n > \frac{1}{n^2}$ for all n, then $\sum_{n=1}^{\infty} a_n$ diverges.

Answer: $\boxed{\text{False}}$. Consider $a_n = \frac{2}{n^2}$.

(b) (5 points) If $\sum_{n=0}^{\infty} a_n$ diverges, then $\sum_{n=10}^{\infty} a_n$ diverges.

Answer: $\boxed{\text{True}}$, since removing a finite number of terms does not affect convergence or divergence.

(c) (5 points) If the power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence 2, then the power series $\sum_{n=0}^{\infty} n a_n x^{n-1}$ converges at $x = 1$.

Answer: $\boxed{\text{True}}$, since the power series $\sum_{n=0}^{\infty} n a_n x^{n-1}$ is the derivative of the first power series, it has radius of convergence 2 and hence converges at $x = 1$.

Problem 2 (10 points) Determine whether the following sequences are convergent or divergent. If the sequence is convergent, determine its limit as $n \to \infty$. Justify your answer.

(a) (5 points) The sequence $\{a_n\}$, where $a_n = \frac{n^n}{n!}$.

Answer: $\boxed{\text{Divergent}}$. Notice that $\frac{n^n}{n!} \geq n$, which diverges.

(b) (5 points) The sequence $\{b_n\}$, where $b_n = \frac{n^2+n}{3n^2+\sqrt{n}}$.

Answer: $\boxed{\text{Converges to } \frac{1}{3}}$. We have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1+1/n}{3+n^{-\frac{3}{2}}} = \frac{1}{3}.$$

Problem 3 (20 points) Determine whether the following series are convergent or divergent and **explain your reasoning**. If the series is convergent, you **do not need to** determine its sum.

(a) (10 points) The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+3n}}$.

Answer: $\boxed{\text{Converges}}$, by comparison to $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$.

(b) (10 points) The series $\sum_{n=1}^{\infty} ne^{-n}$.

Answer: $\boxed{\text{Converges}}$, by applying the integral test and noting that

$$\int_1^{\infty} xe^{-x}dx = \lim_{a\to\infty} \left([-xe^{-x}]_1^a + \int_1^a e^{-x}dx \right) = \frac{2}{e}$$

converges.

Problem 4 (20 points) Consider the series $X = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2^n+1}$.

(a) (5 points) Show that the series converges. You **do not** need to determine the sum.

Answer: $\boxed{\text{Converges}}$ by the alternating series test, since $\frac{1}{2^n+1}$ is decreasing and limits to 0.

(b) (5 points) Is the series absolutely convergent?

Answer: $\boxed{\text{Yes}}$, by comparison to $\sum_{n=0}^{\infty} \frac{1}{2^n}$.

(c) (10 points) We wish to approximate X by the partial sum $s_k := \sum_{n=0}^{k}(-1)^n \frac{1}{2^n+1}$. What is the minimum value of k for which $|s_k - X| < \frac{1}{30}$?

Answer: $\boxed{k=4}$. We have that

$$|s_k - X| \leq \frac{1}{2^{k+1}+1},$$

so we want to find the minimum k for which

$$\frac{1}{2^{k+1}+1} < \frac{1}{30} \iff 30 < 2^{k+1}+1 \iff k \geq 4.$$

Problem 5 (15 points) Find the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \ln(n) \cdot 2^n (x-1)^n.$$

Answer: $\boxed{(1/2, 3/2)}$

We first apply the ratio test, finding that

$$L = \lim_{n\to\infty} \frac{\ln(n+1)2^{n+1}|x-1|^{n+1}}{\ln(n)2^n|x-1|^n} = 2|x-1|,$$

which means that the power series is absolutely convergent on $(1/2, 3/2)$ and divergent on $(-\infty, 1/2) \cup (3/2, \infty)$. At $x = \frac{1}{2}$, we obtain the series

$$\sum_{n=1}^{\infty} (-1)^n \ln(n),$$

which diverges because the summand has non-zero limit. At $x = \frac{3}{2}$, we obtain the series

$$\sum_{n=1}^{\infty} \ln(n),$$

which again diverges because the summand has non-zero limit.

Problem 6 (20 points)

(a) (10 points) Find the Taylor series about $x = 0$ for the function

$$f(x) = \frac{1}{1 + 4x^2}.$$

What is its radius of convergence?

Answer: $\boxed{\sum_{n=0}^{\infty}(-1)^n 4^n x^{2n} \text{ with radius of convergence } \frac{1}{2}.}$

Applying the geometric series expansion for $\frac{1}{1-y}$ with $y = -4x^2$ yields the desired series, which converges for $|y| < 1 \iff |4x^2| < 1 \iff |x| < \frac{1}{2}$.

(b) (10 points) Using your answer above, compute

$$\lim_{x \to 0} \frac{\frac{1}{1+4x^2} - 1 + 4x^2}{x^4}.$$

Answer: $\boxed{16}$

We find that

$$\lim_{x \to 0} \frac{\frac{1}{1+4x^2} - 1 + 4x^2}{x^4} = \lim_{x \to 0} \frac{16x^4 + \sum_{n=3}^{\infty}(-1)^n 4^n x^{2n}}{x^4} = \lim_{x \to 0} 16 + \sum_{n=3}^{\infty}(-1)^n 4^n x^{2n-4} = 16.$$