Mathematics UN1102
Section 1, Spring 2020
Midterm 1 Practice Exam B
Time Limit: 75 Minutes

Name: ____________________________

UNI: ____________________________

Instructions: This exam contains 7 problems. Please make sure you attempt all problems.

Present your solutions in a **legible, coherent** manner. Unless otherwise specified, you should show your work; you will be evaluated on both your reasoning and your answer. Unsupported or illegible solutions may not receive full credit.

Please write your **final answer** for each problem in the provided box. Please show your work in the space below the box. If you need additional space for scratchwork, you may use the back side of the problem sheets or the blank pages stapled to the end of the exam.

The use of outside material including books, notes, calculators, and electronic devices is not allowed.

Formulas

$$\sin^2\theta + \cos^2\theta = 1 \qquad 1 + \tan^2\theta = \sec^2\theta$$

$$\sin(2\theta) = 2\sin\theta\cos\theta \qquad \cos(2\theta) = \cos^2\theta - \sin^2\theta$$

$$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta)) \qquad \cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C \qquad \int \frac{1}{\sqrt{a^2-x^2}}dx = \arcsin\left(\frac{x}{a}\right) + C$$

$$\int \tan x\,dx = \ln|\sec x| + C \qquad \int \sec x\,dx = \ln|\sec x + \tan x| + C$$

Problem 1 (10 points) Evaluate the definite integral

$$\int_0^1 (1-x)^{2017} dx.$$

Answer: $\boxed{\frac{1}{2018}}$

We have for $u = 1 - x$ and $du = -dx$ that

$$\int_0^1 (1-x)^{2017} dx = \int_1^0 -u^{2017} du = -\left[\frac{1}{2018} u^{2018}\right]_0^1 = \frac{1}{2018}.$$

Problem 2 (15 points) Evaluate the indefinite integral

$$\int \frac{1}{(9-x^2)^{3/2}}\,dx.$$

Answer: $\boxed{\frac{x}{9\sqrt{9-x^2}} + C}$

Substitute $x = 3\sin\theta$ with $dx = 3\cos\theta\, d\theta$ to obtain

$$\int \frac{1}{(9-x^2)^{3/2}}dx = \int \frac{1}{(9-9\sin^2\theta)^{3/2}} 3\cos\theta\, d\theta = \int \frac{\cos\theta}{9(\cos^2\theta)^{3/2}} d\theta$$

$$= \int \frac{1}{9}\sec^2\theta d\theta = \frac{1}{9}\tan\theta + C = \frac{1}{9}\tan\arcsin\frac{x}{3} + C = \frac{x}{9\sqrt{9-x^2}} + C.$$

Problem 3 (15 points) Evaluate the indefinite integral

$$\int \frac{1}{1+e^x}\,dx.$$

Answer: $\boxed{x - \ln|1+e^x| + C}$

Substitute $u = e^x$ with $du = e^x dx = u dx$ to obtain

$$\int \frac{1}{1+e^x} dx = \int \frac{1}{1+u}\frac{1}{u} du.$$

We now have the partial fraction expansion

$$\frac{1}{(1+u)u} = \frac{1}{u} - \frac{1}{u+1},$$

which implies that

$$\int \frac{1}{1+u}\frac{1}{u} du = \int \Big(\frac{1}{u} - \frac{1}{u+1}\Big) du = \ln|u| - \ln|u+1| + C = x - \ln|1+e^x| + C.$$

Problem 4 (15 points) Evaluate the indefinite integral

$$\int \frac{x^2+x+1}{x^3-x^2}dx.$$

Hint: It may help to first write a partial fraction decomposition of the integrand.

Answer: $\boxed{-2\ln|x| + 3\ln|x-1| + \frac{1}{x}}$

We solve

$$\frac{x^2+x+1}{x^3-x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

Multiplying through yields

$$x^2+x+1 = Ax(x-1) + B(x-1) + Cx^2.$$

Setting $x = 1$ implies that $C = 3$. Setting $x = 0$ implies that $B = -1$. Setting $x = -1$ implies that $A = -2$. We conclude that

$$\int \frac{x^2+x+1}{x^3-x^2}dx = \int \left(\frac{-2}{x} + \frac{-1}{x^2} + \frac{3}{x-1}\right)dx = -2\ln|x| + 3\ln|x-1| + \frac{1}{x}.$$

Problem 5 (15 points) Consider the improper integral

$$\int_1^{\infty} \frac{\ln x}{x^2}\, dx.$$

(a) (5 points) Write the definition of this improper integral as a limit.

Answer: $\boxed{\lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2} dx}$

(b) (10 points) Determine whether the improper integral converges or diverges and explain why. If the improper integral converges, compute its value.

Answer: $\boxed{\text{Converges and equals 1}}$

The integral converges, since $\frac{\ln x}{x^2} < \frac{1}{x^{3/2}}$ for x large. We compute its value by applying integration by parts with $u = \ln x$ and $v = -\frac{1}{x}$, yielding

$$\int_1^{\infty} \frac{\ln x}{x^2} dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2} dx = \lim_{t\to\infty} \left[-\frac{\ln x}{x}\right]_1^t + \lim_{t\to\infty} \int_1^t \frac{1}{x^2} dx = -\lim_{t\to\infty} \frac{\ln t}{t} + \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = 1 - \lim_{t\to\infty} \frac{\ln t}{t}.$$

By l'Hôpital's rule, we see that

$$\lim_{t\to\infty} \frac{\ln t}{t} = \lim_{t\to\infty} \frac{1/t}{1} = 0,$$

hence the limit from (a) exists and equals 1.

Problem 6 (20 points) Let A be the area enclosed by the graphs of $y = (x+1)^2$, $y = 1$, and $x = 1$.

(a) (10 points) Sketch A, and set up a definite integral to compute its area. You **do not** need to evaluate the integral.

Answer: Picture omitted; area is given by $\int_0^1 (x+1)^2 - 1dx$.

(b) (10 points) Compute the volume of the solid of revolution obtained by rotating A about the y-axis. State whether you are using the method of disks/washers or the method of cylindrical shells.

Answer: $\frac{11\pi}{6}$

Using cylindrical shells, we have

$$V = \int_0^1 2\pi x \cdot \Big((x+1)^2 - 1\Big)dx = 2\pi \int_0^1 (x^3 + 2x^2)dx = 2\pi\Big(\frac{1}{4} + \frac{2}{3}\Big) = \frac{11\pi}{6}.$$

Using disks or washers, we have

$$V = \int_1^4 \pi\Big(1 - (\sqrt{y} - 1)^2\Big)dy = \pi \int_1^4 (2\sqrt{y} - y)dy = \pi\left[\frac{4}{3}y^{3/2} - \frac{1}{2}y^2\right]_1^4 = \pi\Big(\frac{32}{3} - 8 - \frac{4}{3} + \frac{1}{2}\Big) = \frac{11\pi}{6}.$$

Problem 7 (15 points) Compute the length of the graph of

$$f(x) = +\sqrt{1-x^2}$$

between $x = 1/2$ and $x = 1$.

Answer: $\boxed{\frac{\pi}{3}}$

Notice that

$$f'(x) = \frac{-2x}{2\sqrt{1-x^2}} = -\frac{x}{\sqrt{1-x^2}},$$

so we may compute

$$L = \int_{1/2}^{1} \sqrt{1+f'(x)^2}dx = \int_{1/2}^{1} \sqrt{1+\frac{x^2}{1-x^2}}dx = \int_{1/2}^{1} \frac{1}{\sqrt{1-x^2}}dx = [\arcsin(x)]_{1/2}^{1}$$

$$= \arcsin(1) - \arcsin(1/2) = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}.$$